Climate games:
Who's on first? What's on second?*

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Abstract

We study three different climate change games and compare with the outcome of choices by a Social Planner. In a dynamic setting, two players choose levels of carbon emissions. Rising atmospheric carbon stocks increase average global temperature which damages player utilities. Temperature is modeled as a stochastic differential equation. We contrast the results of a Stackelberg game with a game in which both players as leaders (a Leader-Leader or Trumpian game). We also examine a game, called an Interleaved game, where there is a significant time interval between player decisions. One or both players may be better off in these alternative games compared to the Stackelberg game, depending on state variables. We conclude that it is important to consider alternate game structures in examining strategic interactions in pollution games. We also demonstrate that the Stackelberg game is the limit of the Interleaved game as the time between decisions goes to zero.

*The authors gratefully acknowledge funding from the Global Risk Institute, *globalriskinstitute.org*. 'Who's on first? What's on second?' is a reference to the famous comedy routine by Americans Bud Abbott and Lou Costello.

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1 Introduction

Many of the world's serious environmental problems can be described in terms of a tragedy of the commons whereby individual agents ignore the effect of their own actions on the state of particular natural assets, whether fish or forest stocks or the resilience of the world's ecosystems. The tragedy of the commons can only be alleviated by some sort of collective actions, whether through formal government action or through informal activities such as moral suasion at the community level. The effectiveness of actions to thwart the tragedy of the commons will depend on individual circumstances of each situation, and in particular on the strength of the incentives for individual agents to act strategically to further their own interests at the expense of the common good.

Strategic incentives related to the tragedy of the commons have long been studied in the literature using models of differential games, mostly in a deterministic setting. Long (2010) and Dockner et al. (2000) provide surveys of this large literature. Some notable contributions include Dockner & Long (1993), Zagonari (1998), Wirl (2011), List & Mason (2001). Papers tacking pollution games in a stochastic setting include Xepapadeas (1998), Nkuiya (2015), Wirl (2006). Key questions addressed are conditions for the existence of Nash equilibria, whether players are better off with cooperative behaviour, and the steady state level of pollution under cooperative versus non-cooperative games. Linear quadratic games in which utility is a quadratic function of the state variable and the state variable is linear in the control, have been used extensively as these permit an analytic solution for certain types of problems. A leading edge of the literature studies problems which include a more robust characterization of uncertainty and game characteristics such that optimal player controls which depend on state variables and are not restricted in terms of permitted strategies.

Economic models of climate change have been sharply criticized in recent years for their arbitrary assumptions regarding the costs of climate change and inadequate accounting of the uncertainly over how quickly the earth's climate will change and how human society might adapt. Pindyck (2013) is a good example of this critique. In the earlier literature, uncertainty was typically been addressed through sensitivity analysis or Monte Carlo simulation. A developing literature uses more sophisticated approaches, in particular by depicting optimal choices in fully dynamic models with explicit characterization of uncertainty in key state variables. Chesney, Lasserre & Troja (2017) examine optimal climate policies when temperature is stochastic and there is a known temperature threshold which will cause disastrous consequences if exceeded for a prolonged period of time. Other recent papers which incorporate stochasticity into one or more state variables include Crost & Traeger (2014), Ackerman, Stanton & Bueno (2013), Traeger (2014), Hambel, Kraft & Schwartz (2017).

Bressan (2011) provides an excellent summary of the specification and solution of non-cooperative differential games. He shows that in cases where the state variables evolve according to an Ito process with drift depending on player controls, value functions can be found by solving a Cauchy problem for a system of parabolic equations. The Cauchy problem is well posed if the diffusion tensor has full rank. We note that in the model studied in this paper, the diffusion tensor is not of full rank, and hence we cannot necessarily expect Nash equilibria to exist.

Insley, Snoddon & Forsyth (2018) develop a pollution game model to address the specific circumstances of climate change. The model depicts two players, each being a large contributor to global carbon emissions. Players emit carbon in order to generate income, thereby increasing the atmospheric stock of carbon. Rising carbon stocks increase the average global temperature, which is modelled as an Ito process to reflect the inherent uncertainty associated with temperature. Players choose emissions in a repeated Stackelberg game. The game occurs every two years, at which time the leader and follower choose their optimal emission level, with the follower choosing immediately after the leader. There is no analytical solution to this game. A numerical approach is presented, based on the solution of a Hamilton-Jacobi-Bellman (HJB) equation.

The results of Insley, Snoddon & Forsyth (2018) indicated a classic tragedy of the commons whereby player utility is lower than would be achieved by a Social Planner seeking to maximize the sum of player utilities. Players in the game choose emission levels that are too high relative the levels chosen by aSocial Planner. The paper also demonstrates the importance of temperature volatility, and asymmetric damages and preferences on optimal choices. Insley, Snoddon & Forsyth (2018) do not impose the requirement that optimal strategies represent Nash equilibria. However it is possible to check for the existence of Nash equilibrium at every time step for all possible values of the state variables. This is done in the numerical example, and is reported in the paper.

The Stackelberg game has the advantage that, unlike a Nash equilibrium, a solution will always exist. However it is reasonable to ask whether the Stackelberg game is the most appropriate for modelling climate change and other pollution games. The purpose of this paper is to examine two other types of games that might be of interest in studying a pollution game. First we consider a case where both players act as leaders. In a normal Stackelberg game the leader chooses optimal emissions with the knowledge of how the follower will respond (via the follower's best response function). However it seems reasonable to ask what would happen if each player acts as a leader, mistakenly assuming the other player will respond in a rational fashion to the leader's choice. We call this game the Leader-Leader or Trumpian scenario. To preview results, we find that in the Trumpian game, true leader (i.e. the one choosing first at time zero)is worse off than the leader in the Stackelberg game. The true follower (the player choosing second at time zero)in the Trump game is worse off than in the Stackelberg over most values of the state variables, but for certain low values of the carbon stock state variable, the follower can be better off in a Trumpian game.

In our second game variation, we focus on the time lag between the leader and follower decisions. In a case we refer to as the Interleaved game, we assume that players take turns choosing their optimal control, and there is a significant time interval between decisions. This reflects the reality that in the real world, policy decisions to change carbon emissions may take time. Again to preview our results, we find that for a medium size gap between decisions, total utility improves compared to the Stackelberg game. However, when the gap between decisions gets too large, all players are worse off.

While limited to only two additional game types, our results imply that if players could

Table 1: List of Model Variables

Variable	Description
$E_p(t)$	Emissions in region p
e_1, e_2	Particular realizations of $E_p(t)$
$S(t)$	Stock of pollution at time t, a state variable
s	A realization of $S(t)$
$\bar{S}$	preindustrial level of carbon
$\rho(t)$	Rate of natural removal of the pollution stock
$X(t)$	Average global temperature, a state variable
x	A realization of $X(t)$
$\bar{X}$	long run equilibrium level of carbon
$B_p(t)$	Benefits from emissions
$C_p(t)$	Damages from pollution
π_p	Flow of net benefits to region p
r	Discount rate
$\rho(X, S, t)$	removal rate of atmospheric carbon
σ	temperature volatility
$\eta(t)$	speed of mean reversion in temperature equation

choose other games rather than the simple Stackelberg games, it may be in their interests to do so. We hope these results will lead to further research on decision timing and game type which will inform our understanding of strategic interactions in real world pollution games.

2 Problem Formulation

This section provides an broad overview of the climate change game, which will be modelled using three different depictions of the strategic interactions of decision makers. Details of the specific games are provided in Section 3. Details of functional forms and parameter values are provided in Section 4. A summary of variable names is given in Table 1. The problem formulation is similar to that described in Insley, Snoddon & Forsyth (2018), but is repeated here for completeness of the paper.

The climate change game comprises two players each of which generate income by emitting carbon. Carbon emissions contribute to the global atmospheric stock of green house gases, which causes rising average global temperatures. Each player experiences damages from rising temperature which reduces income. Players seek to maximize their own utility through the optimal choice of per period carbon emissions, balancing the benefits from emissions with the costs that come from rising carbon stocks. And of course, the rate at which carbon stocks increase depends in part on the actions of the other player.

For simplicity we assume that there is a one to one relation between emissions and a player's income. The two players are indexed by $p = 1, 2$ and E_p refers to carbon emissions from player p. The stock of atmospheric carbon is increased by emissions, but is also reduced by a natural cycle depicted by the function $\rho(X, S, t)$ and referred to as the removal rate, where X refers to average global temperature, measured in °C above preindustrial levels and t represents time. As described in Section 4, we will drop the dependence on X and S, and assume that ρ is a function only of time. Carbon stock over time is described by the deterministic differential equation:

$$\frac{dS(t)}{dt} = E_1 + E_2 + (\bar{S} - S(t))\rho(X, S, t);\ S(0) = S_0 \quad S \in [s_{min},\ s_{max}]\,. \tag{1}$$

$\bar{S}$ is the pre-industrial equilibrium level of atmospheric carbon.

We capture uncertainty in the evolution of the earth's average temperature by modelling temperature as an Ornstein Uhlenbeck process:

$$dX(t) = \eta(t)\left[\bar{X}(S, t) - X(t)\right]dt + \sigma dZ. \tag{2}$$

where $\eta(t)$ represents the speed of mean reversion, $\bar{X}$ represents the long run mean of global average temperature, σ is the volatility parameter, and dZ is the increment of a Wiener process.

The net benefits from carbon emissions, represented by π_p are composed of the benefits from emissions, $B(E_p, t)$ and the damages from increasing temperature, $C_p(X, t)$:

$$\pi_p = B_p(E_p, t) - C_p(X, t) \quad p = 1, 2; \tag{3}$$

The detailed specification of benefits and damages is left to Section 4

It is assumed that the control (choice of emissions) is adjusted at discrete decision times denoted by:

$$\mathcal{T} = \{t_0 = 0 < t_1 < ...t_m... < t_M = T\}. \tag{4}$$

Let t_m^- and t_m^+ denote instants just before and after t_m, with $t_m^- = t_m - \epsilon$ and $t_m^+ = t_m + \epsilon$, $\epsilon \to 0^+$, and where T is the time horizon of interest.

$e_1^+(E_1, E_2, X, S, t_m)$ and $e_2^+(E_1, E_2, X, S, t_m)$ denote the controls implemented by the players 1 and 2 respectively, which are contained within the set of admissible controls: $e_1^+ \in Z_1$ and $e_2^+ \in Z_2$. K denotes a control set of the optimal controls for all t_m.

$$K = \left\{(e_1^+, e_2^+)_{t_0=0},\ (e_1^+, e_2^+)_{t_1=1},\ \dots, (e_1^+, e_2^+)_{t_M=T}\right\}. \tag{5}$$

In this paper we will consider four possibilities for selection of the controls (e_1^+, e_2^+) at $t \in \mathcal{T}$: which are referred to as Stackelberg, Social Planner, Trumpian (leader-leader), and Interleaved. We delay the precise specification of how these controls are determined until Section 3.2.

Regardless of the control strategy, the value function for player p, $V_p(e_1, e_2, x, s, t)$ is defined as:

$$\begin{aligned} V_p(e_1, e_2, x, s, t) &= \mathcal{E}_K \left[\int_{t'=t}^{T} e^{-rt'} \pi_p(E_1(t'), E_2(t'), X(t'), S(t'))\ dt' + e^{-r(T-t)} V(0, 0, X(T), S(T), T) \right. \\ &\qquad \left. \Big| E_1(t) = e_1, E_2(t) = e_2, X(t) = x,\ S(t) = s \right], \end{aligned} \tag{6}$$

where $\mathcal{E}_K[\cdot]$ is the expectation under control set K. As per convention, lower case letters e_1, e_2, x, s are used to denote realizations of the state variables E_1, E_2, X, S. The value in the final time period, T, is assumed to be the present value of a perpetual stream of expected net benefits at given carbon stock, S, and temperature levels, X, with emissions set to zero. This is reflected in the term $V(0,0,x,s,T)$ and is described in Section 3.1 as a boundary condition.

3 Dynamic Programming Solution

Using dynamic programming,the problem represented by Equation (6) is solved backwards in time, breaking the solution phases up into two components for $t \in (t_m^-, t_m^+)$ and (t_m^+, t_{m+1}^-). In the interval (t_m^-, t_m^+), we determine the optimal controls, while in the interval (t_m^+, t_{m+1}^-), we solve a system of PDEs. As a visual aid, Equation (7) shows the noted time intervals going forward in time,

$$t_m^- \ \rightarrow \ t_m^+ \ \rightarrow \ t_{m+1}^- \ \rightarrow \ t_{m+1}^+ \ . \tag{7}$$

3.1 Advancing the solution from $t_{m+1}^- \rightarrow t_m^+$

The solution proceeds going backward in time from $t_{m+1}^- \rightarrow t_m^+$. Define the differential operator, $\mathcal{L}$ for player p, in Equation (8). The arguments in the V_p function have been suppressed when there is no ambiguity.

$$\mathcal{L}V_p \equiv \frac{(\sigma)^2}{2}\frac{\partial^2 V_p}{\partial x^2} + \eta(\bar{X} - x)\frac{\partial V_p}{\partial x} + [(e_1 + e_2) + \rho(\bar{S} - s)]\frac{\partial V_p}{\partial s} - rV_p; \quad p = 1, 2 \ . \tag{8}$$

where r is the discount rate. Then using standard techniques (Dixit & Pindyck 1994), the equation satisfied by the value function, V_p is expressed as:

$$\frac{\partial V_p}{\partial t} + \pi_p(e_1, e_2, x, s, t) + \mathcal{L}V_p = 0, \quad p = 1, 2 \ . \tag{9}$$

The domain of Equation (9) is $(e_1, e_2, x, s, t) \in \Omega^\infty$, where $\Omega^\infty \equiv Z_1 \times Z_2 \times [x^0, \infty] \times [\bar{S}, \infty] \times [0, \infty]$. In principle, x^0 would be zero degrees Kelvin in our units. For computational purposes, we truncate the domain Ω^∞ to Ω, where $\Omega \equiv Z_1 \times Z_2 \times [x_{min}, x_{\max}] \times [s_{\min}, s_{\max}] \times$

$[0, T]$. T, $s_{\min}$, $s_{\max}$, Z_1, Z_2, x_{min}, and $x_{\max}$ are specified based on reasonable values for the climate change problem, and are given in Section 4.

Remark 1 (Admissible sets Z_1, Z_2)**.** *We will assume in the following that Z_1, Z_2 are compact discrete sets, which would be the only realistic situation.*

Boundary conditions for the PDEs are specified below.

$$\frac{\partial^2 V_p(e_1, e_2, x_{\max} s, t)}{\partial x^2} = 0 \tag{10a}$$

$$\sigma \to 0 \ ; \ x \to x_{\min} \tag{10b}$$

$$\frac{\partial V_P}{\partial S}(e_1 + e_2) \to 0 \ ; \ s \to s_{\max} \tag{10c}$$

$$s \to s_{min} \ ; \ \text{No boundary condition needed, outgoing characteristics} \tag{10d}$$

At $t = T$ it is assumed that V_p is equal to the present value of the infinite stream of benefits associated with a given temperature when emissions are set to zero. Further details regarding these boundary conditions can be found in Insley, Snoddon & Forsyth (2018).

More details of the numerical solution of the system of PDEs are provided in Appendix A. Suppose that the value function is decreasing in temperature at t_{m+1}^-, and that the benefits from emissions are always decreasing as a function of the temperature, then the exact value function (i.e. solution of Equation (9)) must be non-increasing at t_m^+. However, in some of our tests with extreme damage functions, this property was violated in the finite difference solution. In order to ensure this property holds for the finite difference solution, we require a mild timestep condition, as described in Appendix B.

3.2 Advancing the solution from $t_m^+ \to t_m^-$

Going backward in time, the optimal control, is determined between $t_m^+ \to t_m^-$. We consider several possibilities for selection of the controls (e_1^+, e_2^+) at $t \in \mathcal{T}$:

- Stackelberg;
- Social Planner;

- Leader-Leader (Trumpian);
- Interleave.

For reference, we also include the definition of a Nash equilibrium, although we observe that a Nash equilibrium frequently does not always exist. We remind the reader that our controls are assumed to be feedback, i.e. a function of state. However, to avoid notational clutter in the following, we will fix $(e_1^-, e_2^-, s, x, t_m^-)$, so that, if there is no ambiguity, we will write (e_1^+, e_2^+) which will be understood to mean $(e_1^+(e_1^-, e_2^-, s, x, t_m^-), e_2^+(e_1^-, e_2^-, s, x, t_m^-))$, where e_1^- and e_2^- are the state values at t_m^- before the control is applied.

Given the optimal controls (e_1^+, e_2^+) at a point in the state space $(e_1^-, e_2^-, s, x, t_m^-)$, the dynamic programming principle implies

$$\begin{aligned} V_1(e_1^-, e_2^-, s, x, t_m^-) &= V_1(e_1^+(\cdot), e_2^+(\cdot), s, x, t_m^+) \ , \\ V_2(e_1^-, e_2^-, s, x, t_m^-) &= V_2(e_1^+(\cdot), e_2^+(\cdot), s, x, t_m^+) \ . \end{aligned} \tag{11}$$

Equation (11) is used to advance the solution backwards in time $t_m^+ \to t_m^-$, for all types of games. We describe the specific rule for determining the optimal control pair (e_1^+, e_2^+) for each type of game in the following.

3.2.1 Stackelberg Game

In the case of a Stackelberg game, suppose that, in forward time, player 1 goes first, and then player 2. Conceptually, we can then think of the time intervals (in forward time) as $(t_m^-, t_m]$, (t_m, t_m^+). Player 1 chooses control e_1^+ in $(t_m^-, t_m]$, then player 2 chooses control e_2^+ in (t_m, t_m^+).

We suppose at t_m^+, we have the value functions $V_1(e_1, e_2, s, x, t_m^+)$ and $V_2(e_1, e_2, s, x, t_m^+)$.

Definition 1 (Response set of player 2)**.** *The best response set of player 2, $R_2(\omega_1; e_2; s, x, t_m)$ is defined to be the best response of player 2 to a control ω_1 of player 1.*

$$R_2(\omega_1; e_2; s, x, t_m) = \operatorname*{argmax}_{e_2' \in Z_2} V_2(\omega_1, e_2', s, x, t_m^+) \ ; \ \omega_1 \in Z_1 \ . \tag{12}$$

Remark 2 (Tie breaking)**.** *We break ties by (i) staying at the current emission level if possible, or (ii) choosing the lowest emission level. Rule (i) has priority over rule (ii). The notation* $R_2(\cdot; e_2; \cdot)$ *shows dependence on the state* e_2 *due to the tie breaking rule.*

Similarly, we define the best response set of player 1.

Definition 2 (Response set of player 1)**.** *The best response set of player 1,* $R_1(\omega_2; e_1; s, x, t_m)$ *is defined to be the best response of player 1 to a control* ω_2 *of player 2.*

$$R_1(\omega_2; e_1; s, x, t_m) \quad = \quad \underset{e_1' \in Z_1}{\operatorname{argmax}} V_1(e_1', \omega_2, s, x, t_m^+) \ ; \ \omega_2 \in Z_2 \ . \tag{13}$$

Ties are broken as in Remark 2. Again, to avoid notational clutter, we will fix (e_1, e_2, s, x, t_m) so that we can usually write without ambiguity $R_1(\omega_2; e_1) = R_1(\omega_2; e_1; s, x, t_m)$ and $R_2(\omega_1; e_2) = R_2(\omega_1; e_2; s, x, t_m)$.

Definition 3 (Stackelberg Game: Player 1 first)**.** *The optimal controls* (e_1^+, e_2^+) *assuming player 1 goes first are given by*

$$\begin{aligned} e_1^+ &= \underset{\omega_1' \in Z_1}{\operatorname{argmax}} V_1(\omega_1', R_2(\omega_1'; e_2^-), s, x, t_m^+)\Big|_{break\ ties\ e_1^-} , \\ e_2^+ &= R_2(e_1^+; e_2^-) \ . \end{aligned} \tag{14}$$

3.2.2 Leader-Leader (Trumpian) Game

A leader-leader game is determined by assuming that each player (mistakenly) assumes that they are the leader. Somewhat tongue-in-cheek, we refer to this as a *Trumpian* game. The Trumpian controls are determined from

$$\begin{aligned} e_1^+ &= \underset{\omega_1' \in Z_1}{\operatorname{argmax}} V_1(\omega_1', R_2(\omega_1'; e_2^-), s, x, t_m^+)\Big|_{break\ ties\ e_1^-} , \\ e_2^+ &= \underset{\omega_2' \in Z_2}{\operatorname{argmax}} V_2(R_1(\omega_2'; e_1^-), \omega_2', s, x, t_m^+)\Big|_{break\ ties\ e_2^-} . \end{aligned} \tag{15}$$

3.2.3 Interleave Game

Suppose that at decision times $t_{2m}; m = 0, 1, \ldots$ player one chooses an optimal control, while player two's control is fixed. At decision times $t_{2m+1}; m = 0, 1, \ldots$ player two chooses an optimal control, while player one's control is fixed. More precisely, at t_{2m}

$$
\begin{aligned}
e_1^{(2m)+} &= \text{ optimal control for player 1 },\\
e_2^{(2m)+} &= e_2^{(2m)-} \ ; \ \text{ player 2 control fixed }.
\end{aligned} \tag{16}
$$

At time $t_{(2m+1)}$, we have

$$
\begin{aligned}
e_1^{(2m+1)+} &= e_1^{(2m+1)-} \ ; \ \text{ player 1 control fixed },\\
e_2^{(2m+1)+} &= \text{ optimal control for player 2 }.
\end{aligned} \tag{17}
$$

More details for the Interleaved game are given in Appendix D. Suppose we hold player one's decision times t_{2m} fixed, and move player two's decision times t_{2m+1} to be just after t_{2m}. More precisely,

$$
t_{2m} = \text{ fixed } \ ; \ (t_{2m+1} - t_{2m}) \to 0^+ \ . \tag{18}
$$

In this case, intuitively, we would expect that the result of this limiting process is a Stackelberg game at times t_{2m}, with player one being the leader, and player two the follower. We confirm this intuition in Proposition 3, Appendix D.

3.2.4 Social Planner

For the Social Planner case, we have that an optimal pair (e_1^+, e_2^+) is given by

$$
(e_1^+, e_2^+) = \underset{\substack{\omega_1 \in Z_1 \\ \omega_2 \in Z_2}}{\text{argmax}} \left\{ V_1(\omega_1, \omega_2, s, x, t_m^+) + V_2(\omega_1, \omega_2, s, x, t_m^+) \right\} . \tag{19}
$$

Ties are broken by (i) minimizing $|V_1(e_1^+, e_2^+, s, x, t_m^+) - V_2(e_1^+, e_2^+, s, x, t_m^+)|$, (ii) choosing the lowest emission level. Rule (i) has priority over rule (ii). In other words, the Social Planner picks the emissions choices which give the most equal distribution of welfare across the two players.

3.2.5 Nash Equilibrium

In Appendix C we describe the necessary and sufficient conditions for a Nash equilibrium to exist. However, in general, we have no reason to believe that Nash equilibria exist at all points in the state space, since the system of PDEs depicted in Equation (8) is degenerate (i.e. there is no diffusion in the S direction). This observation is confirmed in our numerical tests, i.e. for information purposes only, we check to see if a Nash equilibrium exists at each point in the discretized state space.

4 Detailed model specification and parameter values

The functional forms and parameter values used in this paper are the same as in Insley, Snoddon & Forsyth (2018). For the convenience of the reader a brief review is provided in this section. Assumed parameter values are summarized in Table 2.

4.1 Carbon stock details

The evolution of the carbon stock is described in Equation (1). In our numerical example, we use a simplified specification of the path of carbon stock, based on Traeger (2014). We denote the rate at which carbon is removed from the atmosphere by $\rho(t)$ and assume it is a deterministic function of time which approximates the removal rates in the DICE 2016 model.

$$\rho(t) = \bar{\rho} + (\rho_0 - \bar{\rho})e^{-\rho^* t}$$

Table 2: Base Case Parameter Values

Parameter	Description	Equation Reference	Assigned Value
$\bar{S}$	Pre-industrial atmospheric carbon stock	(1)	588 Gt carbon
s_{min}	Minimum carbon stock	(1)	588 Gt carbon
s_{max}	Maximum carbon stock	(1)	10000 Gt carbon
$\bar{\rho}$, ρ_0, ρ^*	Parameters for carbon removal Equation	(20)	0.0003, 0.01, 0.01
ϕ_1, ϕ_2, ϕ_3	Parameters of temperature Equation	(20)	0.02, 1.1817, 0.088
ϕ_4	Forcings at CO2 doubling	(22)	3.681
$F_{EX}(0)$ $F_{EX}(100)$	Parameters from forcing Equation	(22)	0.5 1
α_1, α_2	Ratio of the deep ocean to surface temp, $\alpha(t) = \alpha_1 + \alpha_2 \times t$, t is time in years with 2015 set as year 0	(20)	0.008, 0.0021
σ	Temperature volatility	(20)	0.1
x_{min}, x_{max}	Upper and lower limits on average temperature, °C	(20)	-3, 20
a_1, a_2	Parameter in benefit function, player p	(24)	10
Z_1, Z_2	Admissible controls	(5)	0, 3, 7, 10
b_1,b_2	Cost scaling parameter, players 1 & 2 respectively	(25)	15, 15
κ_1	Linear parameter in cost function for both players	(25)	0.05
κ_3	Term in exponential cost function for both players	(25)	1
T	terminal time		150 years
r	risk free rate	(8)	0.01

ρ_0 is the initial removal rate per year of atmospheric carbon, $\bar{\rho}$ is a long run equilibrium rate of removal, and ρ^* is the rate of change in the removal rate. Specific parameter assumptions for this Equation are given in Table 2. The resulting removal rate starts at 0.01 per year and falls to 0.0003 per year within 100 years.

The pre-industrial equilibrium level of carbon, $\bar{S}$ in Equation (1), is assumed to be 588 gigatonnes (Gt) based on estimates used in the DICE (2016)[1] model for the year 1750. The allowable range of carbon stock is given by $s_{min} = 588$ and $s_{max} = 10000$. s_{max} is set well above the 6000 Gt carbon in Nordhaus (2013) and will not be a binding constraint in the numerical examples. A 2014 estimate of the atmospheric carbon level is 840 Gt.[2]

4.2 Stochastic process temperature: details

Equation (2) specifies the stochastic differential equation which describes temperature X and includes the parameters $\eta(t)$ and $\bar{X}(t)$. To relate Equation (2) to common forms used in the climate change literature, we rewrite it in the following format:

$$dX = \phi_1 \Big[F(S,t) - \phi_2 X(t) - \phi_3 [1 - \alpha(t)] X(t) \Big] dt + \sigma dZ \tag{20}$$

where ϕ_1, ϕ_2, ϕ_3 and σ are constant parameters.[3] The drift term in Equation (20) is a simplified version of temperature models typical in Integrated Assessment Models, based on Lemoine & Traeger (2014). $\alpha(t)$ represents the ratio of the deep ocean temperature to the mean surface temperature and, for simplicity, is specified as a deterministic function of

[1] The 2013 version of the DICE model is described in Nordhaus & Sztorc (2013). GAMS and Excel versions for the updated 2016 version are available from William Nordhaus's website: http://www.econ.yale.edu/ nordhaus/homepage/.

[2] According to the Global Carbon Project, 2014 global atmospheric CO2 concentration was 397.15 ± 0.10 ppm on average over 2014. At 2.21 Gt carbon per 1 ppm CO2, this amounts to 840 Gt carbon.(www.globalcarbonproject.org)

[3] ϕ_1, ϕ_2, ϕ_3 are denoted as ξ_1, ξ_2, and ξ_3 in Nordhaus (2013).

time.[4] Equation (20) is equivalent to Equation (2) with:

$$\eta(t) \equiv \phi_1 \Big(\phi_2 + \phi_3(1 - \alpha(t)) \Big) \tag{21}$$
$$\bar{X}(t) \equiv \tfrac{F(S,t)}{(\phi_2+\phi_3(1-\alpha(t))}.$$

$F(S,t)$ refers to radiative forcing, where

$$F(S,t) = \phi_4 \left(\frac{ln(S(t)/\bar{S})}{ln(2)} \right) + F_{EX}(t) \ . \tag{22}$$

ϕ_4 indicates the forcing from doubling atmospheric carbon.[5] $F_{EX}(t)$ is forcing from causes other than carbon and is modelled as an exogenous function of time as specified in Lemoine & Traeger (2014) as follows:

$$F_{EX}(t) = F_{EX}(0) + 0.01\big(F_{EX}(100) - F_{EX}(0)\big) \min\{t, 100\} \tag{23}$$

The values for the parameters in Equation (20) are taken from the DICE (2016) model. Note that $\phi_1 = 0.02$ which is the value reported in Dice (2016) divided by five to convert to an annual basis from the five year time steps used in the DICE (2016) model. $F_{EX}(0)$ and $F_{EX}(100)$ (Equation (22)) are also from the DICE (2016) model. The ratio of the deep ocean temperature to surface temperature, $\alpha(t)$, is modelled as a linear function of time.

4.3 Benefits and Damages

4.3.1 Benefits

Following the norm in the pollution game literature, benefits of emissions are assumed to be quadratic according the following utility function:

$$B_p(E_p) = a_p E_p(t) - E_p^2(t)/2, \ p = 1, 2 \tag{24}$$

[4] We are able to get a good match to the DICE2016 results using a simple linear function of time.
[5] ϕ_4 translates to Nordhaus's η (Nordhaus & Sztorc 2013).

a_p is a constant parameter which may be different for different players. $E_p \in [0, a_p]$ so that the marginal benefit from emissions is always positive. In the numerical example, there are four possible emissions levels for each player $E_p \in \{0, 3, 7, 10\}$ in gigatonnes (Gt) of carbon and we set $a_1 = a_2 = 10$.

4.3.2 Damages

As is discussed in Pindyck (2013),the modelling of damages from climate change is highly controversial. We choose a functional form whereby damages depend on average global temperature through an exponential function:

$$C_p(t) = \kappa_1 e^{\kappa_3 X(t)} \quad p = 1, 2., \tag{25}$$

where κ_2 and κ_3 are a constant and $p = 1, 2$ refers to the two players. As discussed in Insley, Snoddon & Forsyth (2018), this function implies that damages become very large, dwarfing any benefits from emissions, for temperatures exceeding 3 °C. We view this exponential specification of damages as an alternative approach to capturing disastrous consequences, compared to adopting a Poisson jump process which is sometimes used in the literature.

5 Numerical Results

5.1 Base case: the Stackelberg game

This section summarizes the results for the Stackelberg game which is used as the base case for comparison with other games. In this case, the leader and follower play a series of Stackelberg games at fixed decision times, set to be every two years, with the first game occurring at time zero. It is challenging to get a good sense of the results due to the numerous state variables including carbon stock, temperature, and current emission levels of each player. For the Stackelberg game, as noted in Section 3.2.1, the optimal control depends on current levels of emissions e_1 and e_2 only in the event of a tie. However, in the Interleaved case, discussed below, current emissions levels have an impact on results. We have chosen to present results for state variables close to current levels (1 °C for temperature and and 800 Gt for the atmospheric stock of carbon). We also comment on and graph results for other values of state variables. All results are presented for time zero. For clarity when comparisons are made with other games, we will consistently refer to the leader in the Stackelberg game as Player 1 and the Follower as Player 2.

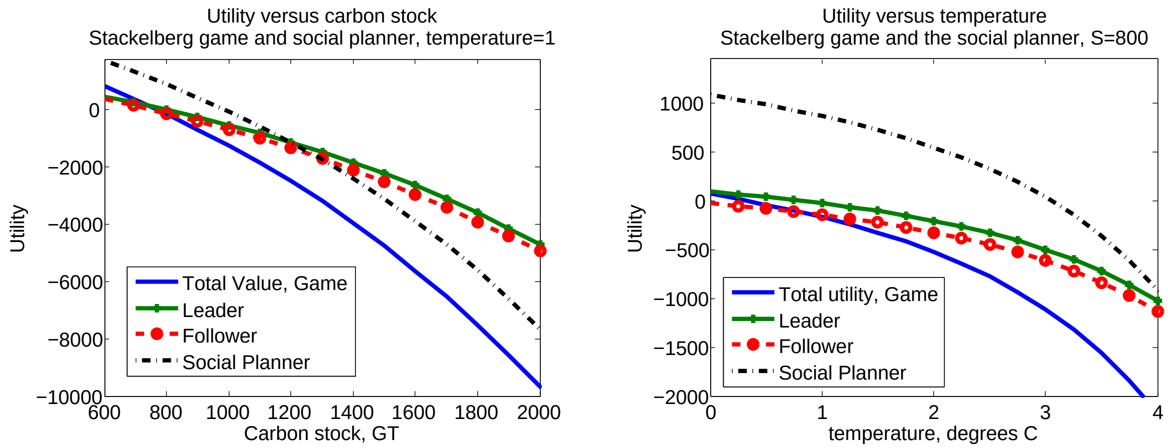
Figure 1 shows utilities for the base case game versus the Social Planner. These represent expected utility at time zero if optimal controls are followed from time zero to time T, given the dependence of the stock of carbon on the choice of emissions and given the evolution of temperature, which depends on the the carbon stock as well as a random component. Figure 1(a) plots utility versus carbon stock for a temperature of 1 °C, and for fixed state variables e_1 and e_2 both set at 10 Gt. We observe, as expected, that utility declines with carbon stock. The Social Planner case yields significantly higher utility, confirming a tragedy of the commons as an important feature of the Stackelberg game. Individual player utilities are also depicted. The leader achieves higher utility than the follower, showing that there is a benefit to being the first mover in this repeated game.

Figure 1(b) depicts how utility changes with temperature, this time with the state variable carbon stock set at 800 Gt. (e_1 and e_2 are again set at 10 Gt, but this is immaterial in the Stackelberg case.) As expected, utility declines with increasing temperature.

Figure 2 compares emissions optimal choices at time zero over a range of carbon stock levels when the temperature is fixed at 1 °C. In the left diagram the Social Planner picks lower emissions than the total that results from the Stackelberg game. The diagram on the right shows that the players have largely the same strategy at time zero, but there is a window of carbon stock levels over which their strategies diverge.

5.2 A Trumpian Game

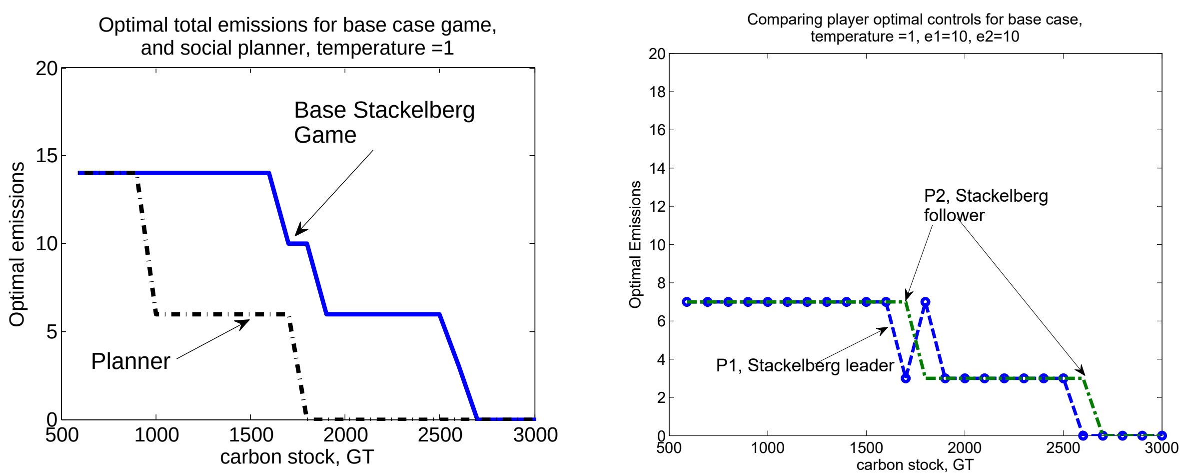
We now contrast the Stackelberg game with the Leader-Leader (*Trumpian*) game, in which both players consider themselves to be the leaders in the game. Each chooses her actions



(a) Utility versus stock

(b) Utility versus temperature

Figure 1: Utilities versus carbon stock and temperature for base Stackelberg game and Social Planner, time = 0, state variables $E1 = 10$, $E2 = 10$. Temperature is in °C above preindustrial levels.



(a) Total optimal emissions, base game and Social Planner

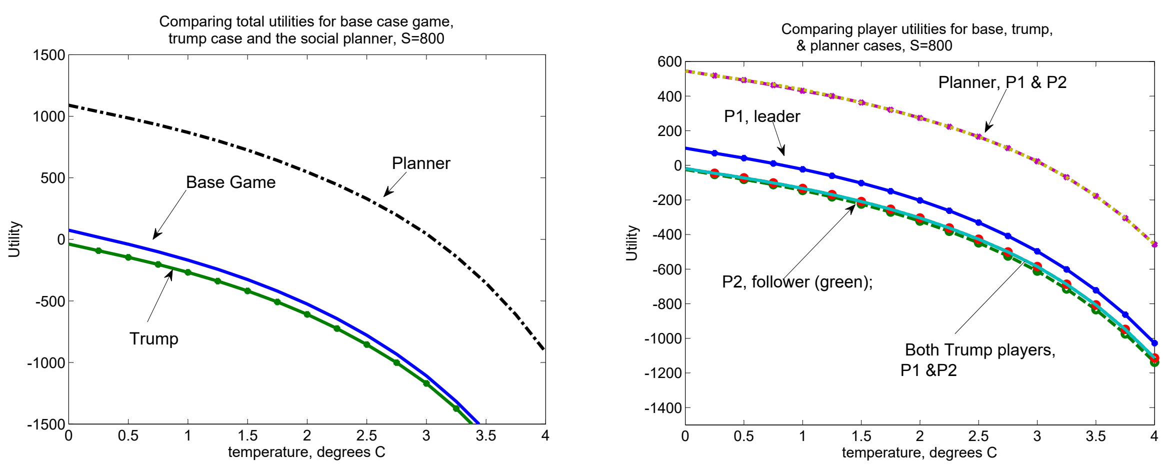
(b) Individual player optimal controls, base game

Figure 2: Comparing optimal controls for the base Stackelberg game and the Social Planner, time = 0. State variables $e1 = e2 = 10$Gt. Temperature is at 1°C above preindustrial levels. P1 refers to player 1, P2 refers to player 2.

assuming incorrectly that the other player will respond according to a rational best response function. (See Section 3.2.2.) In the Trump game both Player 1 and Player 2 act as leaders.

A comparison of utilities of the Trumpian and Stackelberg (base) games, and the Social Planner is given in Figure 3. The comparison shows utility versus carbon stock at time zero, with temperature at 1 °C. We observe in Figure 3(a) that the Trump game yields lower total utility than the base case Stackelberg game. Figure 3(b) presents the results for individual players. Since players are identical and both are playing as leaders, both receive the same utilities in the Trump game. We observe Player 1 loses in this game, experiencing a significant reduction in utility compared to the Stackelberg game. Player 2 in the Trump game has a utility level that is fairly close to what she received in the Stackelberg game. While it is not possible to see given the scale of the graph, Player 2 gets slightly higher utility in the Trump game for the case shown with $S = 800$. At higher levels of the carbon stock, both players are worse off in the Trump game. Under the Social Planner case both players receive higher utilities.

It may seem counter-intuitive that over some state variables Player 2 is better off in the Trump game. This can be explained by the fact the leader is making a mistake at each decision point by assuming Player 2 will act as a follower. This hurts the leader and in some instances can help the follower.

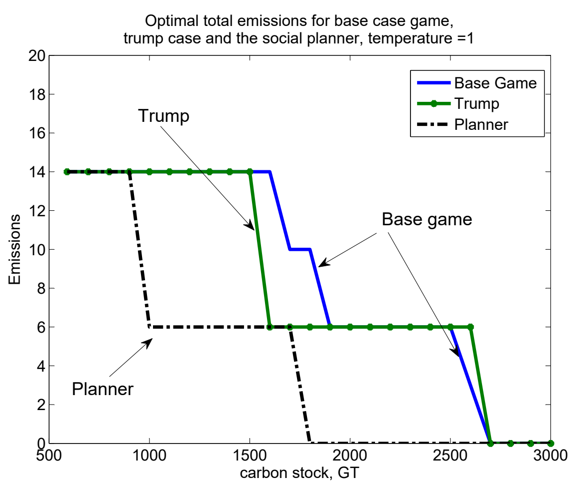


(a) Total utility

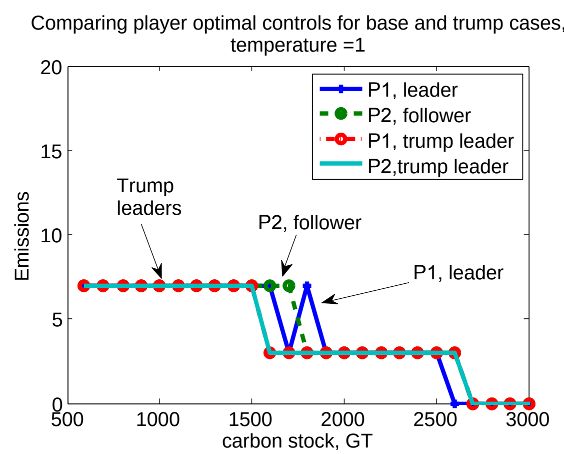
(b) Individual player utilities

Figure 3: Comparing utilities for base Stackelberg game,Trump game, and Social Planner, time = 0.

Figure 4 compares the optimal controls for the Trump case with the Stackelberg game and the planner. Recall that these are optimal controls hold only $t = 0$. Future optimal controls depend on the evolution of the state variables. In Figure 4(a), we observe that in the Trump game total optimal emissions are lower than the base Stackelberg game for a window of carbon stock, s, between 1600 and 1800 Gt. This is reversed over a window of high carbon stock levels (2600 - 2800 Gt) where emissions under the Trump game are higher than under the Stackelberg game. While we have not included graphs of other temperature levels, a similar pattern is observed for temperatures ranging up to 4 degrees, although the range of carbon stocks over which the Trump game has lower emissions is reduced. Figure 4(b) displays individual player optimal controls. Optimal controls for both players in the Trump game are identical. In the Stackelberg game we observe some oscillation of controls at mid carbon stock levels.



(a) Total optimal emissions



(b) Individual player optimal controls

Figure 4: Comparing optimal controls for base Stackelberg game,Trump game, and Social Planner, time = 0.

We conclude that when players are symmetric over most levels of the state variables, it is not worthwhile for players to be part of a Trump game. One might expect that total emissions would be higher under a Trump game, and but we can draw no such conclusion. In fact we observe that the optimal choice of emissions under the Trump game is lower than for the Stackelberg game for certain levels of the carbon stock.

5.3 Contrasting constraints on player decision times - An Interleaved Game

In the Stackelberg game, the follower makes a choice immediately after the leader. In reality, national policies to change emissions take time to implement. This section examines a case in which there are two years between the decisions of leader and follower. This implies that each player must wait four years before choosing a new optimal control. For example, the leader makes a decision at time zero, the follower makes a decision at two years later (t=2 years), and the leader makes its next decision at two years after that (t=4 years). As is demonstrated in Section 3.2.3 and Appendix D, the Stackelberg game is the limit of the Interleaved game as the time between the leader and follower decisions goes to zero.

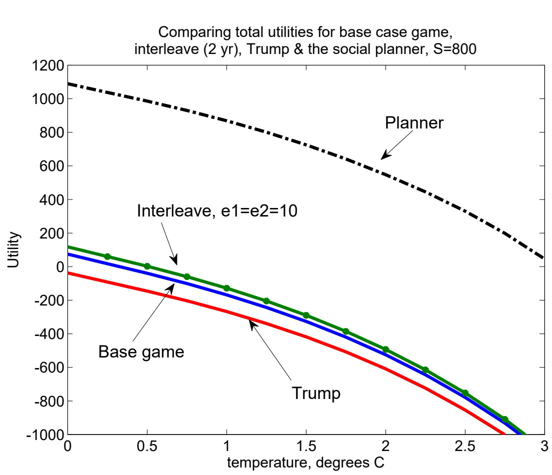
Figure 5(a) plots utility versus temperature for four different cases: the base Stackelberg game, the Trump game, the Interleaved game ($e_1 = e_2 = 10$ Gt), and the Social Planner. Interestingly the Interleaved case shows higher total utility than either the Trump case or the base game. It appears that constraining each player to wait two years following the opposing player's decision before making their own choice has reduced the effect of the tragedy of the commons. Intuitively this enforced delay implies that any individual player's actions will have a more lasting effect. As an extreme, suppose player 1 is able to make decisions every two years, but player 2 is never able to take action to reduce emissions. The entire burden for reducing emissions will fall to player one. Since player two has no control available, there is by definition no tragedy of the commons.

As noted earlier, in the Interleaved game, the state variable representing current emissions affects utility. This is because there is a significant time interval before the follower is able to respond to the leader's optimal choices. At time zero, the leader goes immediately to its optimal choice, but the follower must maintain her current emissions level until two years have passed. Figure 5(b) contrasts total utility showing two different levels for player 2's current emissions, $e_2 = 0$ and $e_2 = 10$. (Player 1's current emissions are immaterial as she immediately goes to her optimal choice.) The state variable at $e_2 = 0$ gives a slightly higher total utility than when $e_2 = 10$. Note that the optimal choice of emissions for both leader and follower over this range of temperatures, and given $s = 800$ Gt, is 7 Gt.

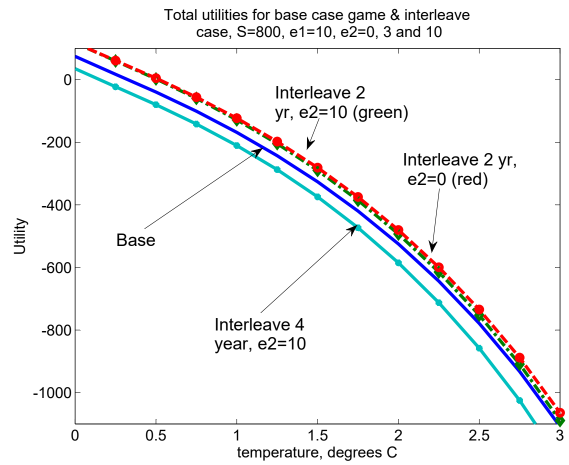
For contrast we also include a curve labelled 'Interleave 4 year'. In this case, the time between decisions is increased to four years, so that each player can only make a choice every eight years. We see that in the four year Interleaved case, total utility is now lower than in the base game. The 'Interleave 4 year' case also has slightly lower utility than a Stackelberg game played every four years. (The 'Stackelberg 4 year' game is not shown on the graph to avoid clutter.) It is interesting that the 2 year Interleaved case (4 years between an individual player's decisions) increased utility relative to the base Stackelberg game, whereas the 4 year Interleaved case (8 years between an individual player's decisions) causes a reduction. There appears to be two countervailing effects going on. The shorter delay between decisions reduces the tragedy of the commons and increases utility, but with a longer delay this beneficial effect is overwhelmed by the negative effects of not being able to respond promptly to changes in the key state variables, temperature and carbon stock.

Figures 5(c) and 5(d) show the results for individual player utilities. There is some variation depending on the starting value for Player 2. The graph on the left (Figure 5(c)) shows the state variable $e_2 = 10$. Here we see Player 2 (the follower) gains from the Interleaved case relative to the base Stackelberg case, while Player 1 (the leader) is worse off. The graph on the right (Figure 5(d)) shows the state variable $e_2 = 0$. In this case, the both Player 1 and Player 2 are better off. It makes sense that the leader benefits if the follower starts the game with a very low level of emissions, which cannot be changed until 2 years later in this case.

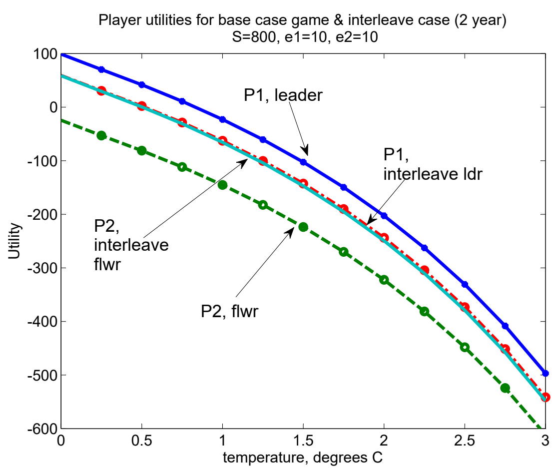
The optimal controls for the Interleaved and base cases are shown in Figure 6. Optimal choice of total emissions at time zero (Figure 6(c)) are lower for the Interleaved case over a range of carbon stock levels around $S = 1800$ and $S = 2600$ Gt. Both leader and follower show different choices compared to the Stackelberg case. Compared to the Social Planner the initial choice of emissions in both games is significantly larger over a wide range of carbon



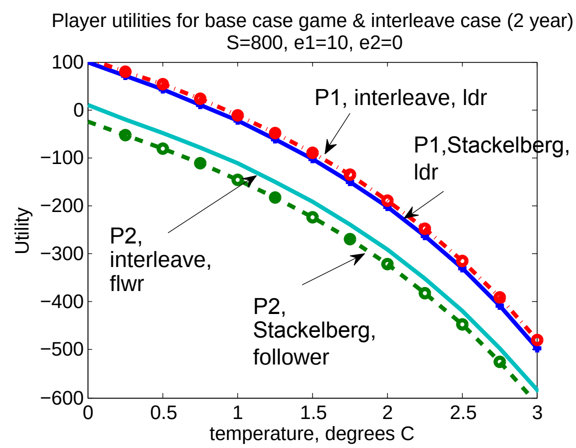
(a) Interleaved, Base, Planner, and Trump



(b) Base and Interleaved, e1 = 10; e2 = 0 and 10



(c) Individual player utilities, e1=10, e2 = 10



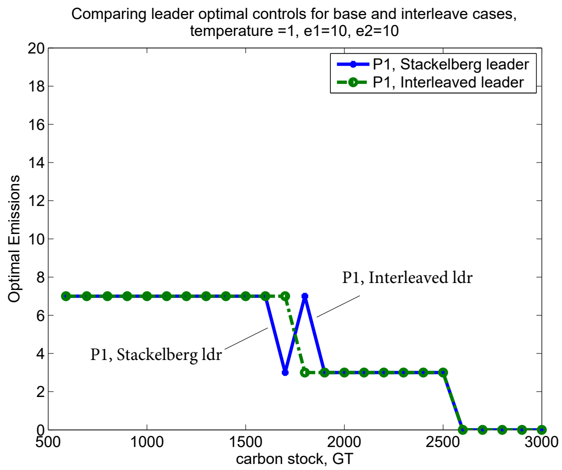
(d) Individual player utilities, e1=10, e2 = 0

Figure 5: Comparing utilities for base Stackelberg game and Interleaved game, time = 0.

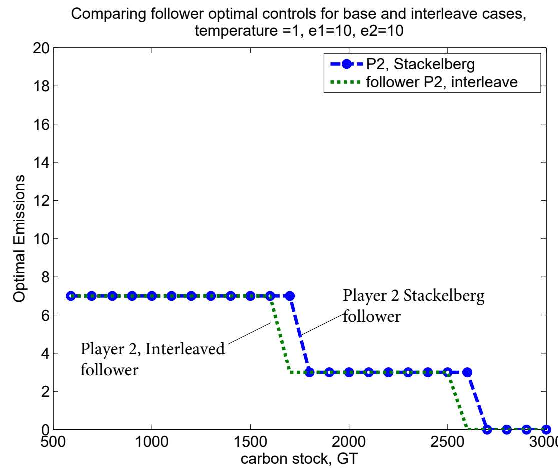
stock levels.

5.4 Existence of Nash Equilibria

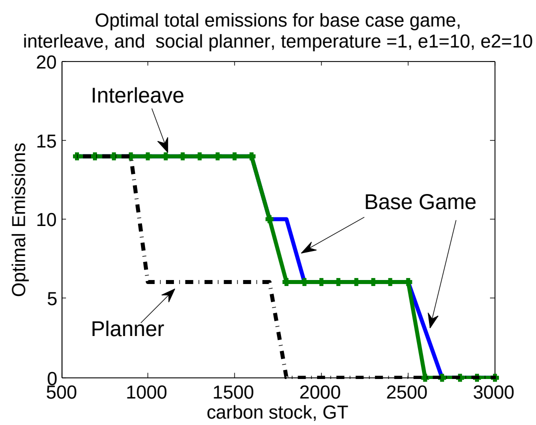
Our numerical tests show that Nash equilibria exist at about 60% of possible values for state variables over all time steps for the Stackelberg case.



(a) US optimal control



(b) CN optimal controls



(c) Total optimal emissions

Figure 6: Comparing optimal controls for base Stackelberg game, Interleaved game, and Social Planner, time zero.

6 Concluding Comments

Strategic actions by decision makers are a key factor in our ability to confront the causes of global warming. Economic models based on game theory approaches have deepened our understanding of the consequences of strategic behaviour for the tragedy of the commons. This paper extends the pollution game literature by examining several different types of games not previously considered. We take as a starting point the differential game model of Insley, Snoddon & Forsyth (2018) which determines the closed loop optimal controls of two players choosing emission levels in a repeated Stackelberg game, while facing damages caused by rising temperatures in response to the build up of the atmospheric carbon stock. In the current paper we consider two alternative specification of the games which we call the Trump game and the Interleaved game. Both of these variations provide some interesting insights into the climate change game.

In the Trump game, both players act as leaders, mistakenly assuming the other player will respond rationally as a follower. Not surprisingly, total utility is lower in this game. However it is Player 1 (the leader in the Stackelberg base game) who suffers the most. At lower levels of carbon stock, Player 2 (the follower in the Stackelberg base game) actually gains slightly from the Trump game. As the carbon stock increases both players are worse off in the Trump game, but relatively speaking the leader experiences the largest reduction in utility. We conclude that in the Stackelberg game the follower might as well play like a leader, as she will be no worse off and may be better off at lower levels of the carbon stock. However the Trump game is not good for the environment as total utility or welfare suffers in this game, particularly at higher carbon stock levels.

In the Interleaved game, unlike the Stackelberg game, Player 2 does not make a decision immediately after Player 1 makes her choice. Rather there is a gap of several years between player decisions. This element is intended to add some reality to the game, in that policy changes to reduce emissions do not happen instantaneously in the real world. We prove that in the limit as the time interval between player decisions goes to zero, the Interleaved game converges to the Stackelberg game.

We examined an Interleaved game of two years with a decision made by one of the players every two years, implying each player must wait four years between their own decisions. In this Interleaved game, we found that total utility increased compared to the basic Stackelberg game in which both players make optimal choices at two year intervals, with the follower choosing instantaneously after the leader. We found the follower does better in this Interleaved game compared to the Stackelberg game. The repercussions for the leader are dependent on the starting level of emissions for the follower. For low starting values for the follower, the leader also does better in the Interleaved game. However if the follower starts at high emissions levels, the leader is worse of in this Interleaved game. We interpret this result to mean that there is a benefit to a player in not reacting immediately to the actions of the other player. The follower, in particular, benefits from being able to commit to a level of emissions which cannot be changed for two years. If the follower starts with a high level of emissions, the leader is forced to react.

The relative benefits of the Interleaved game depend on the time interval between decisions. If the time between decisions is increased, eventually both players will be worse off in the Interleaved game as the extended wait between decisions does not allow the players to adequately respond to the environmental problem. We found this to be the case with an Interleaved game of four years, when individual player make decisions every eight years..

From our analysis, we conclude that one should be wary of relying solely on the Stackelberg game to draw inferences about the strategy of players in real world strategic situations, such as decisions about climate change policies. We have demonstrated two other types of games which result in improved welfare for one or both players, implying that if given the choice the players would rather be part of these alternative games. While this paper is limited to examining only these two alternatives, we do demonstrate that the timing of the leader and follower decision has a crucial impact on the outcome of the game for the players, as well as for total welfare.

Appendices

A Numerical methods

A.1 Advancing the solution from $t_{m+1}^{-} \rightarrow t_m^{+}$

Since we solve the PDEs backwards in time, it is convenient to define $\tau = T - t$ and use the definition

$$\begin{aligned} \hat{V}_p(e_1, e_2, x_i, s, \tau) &= V_p(e_1, e_2, x_i, s, T - \tau) \\ \hat{\pi}_p(e_1, e_2, x_i, s, \tau) &= \pi_p(e_1, e_2, x_i, s, T - \tau) \ . \end{aligned} \tag{26}$$

We rewrite Equation (9) in terms of backwards time $\tau = T - t$

$$\begin{aligned} \frac{\partial \hat{V}_p}{\partial \tau} &= \hat{\mathcal{L}} \hat{V}_p + \hat{\pi}_p + [(e_1 + e_2) + \rho(\bar{S} - s)] \frac{\partial \hat{V}_p}{\partial s} \\ \hat{\mathcal{L}} \hat{V}_p &\equiv \frac{(\sigma)^2}{2} \frac{\partial^2 \hat{V}_p}{\partial x^2} + \eta(\bar{X} - x) \frac{\partial \hat{V}_p}{\partial x} - r \hat{V}_p \ . \end{aligned} \tag{27}$$

Defining the Lagrangian derivative

$$\frac{D\hat{V}_p}{D\tau} \equiv \frac{\partial \hat{V}_p}{\partial \tau} + \left(\frac{ds}{d\tau} \right) \frac{\partial \hat{V}_p}{\partial s} \ , \tag{28}$$

then Equation (27) becomes

$$\frac{D\hat{V}_p}{D\tau} = \hat{\mathcal{L}} \hat{V}_p + \pi_p \tag{29}$$

$$\frac{ds}{d\tau} = -[(e_1 + e_2) + \rho(\bar{S} - s)] \ . \tag{30}$$

Integrating Equation (30) from τ to $\tau - \Delta\tau$ gives

$$s_{\tau - \Delta\tau} = s_\tau \exp(-\rho \Delta\tau) + \bar{S}(1 - \exp(-\rho\Delta\tau)) + \big(\frac{e_1 + e_2}{\rho}\big)(1 - \exp(-\rho\Delta\tau)) \ . \tag{31}$$

We now use a semi-Lagrangian timestepping method to discretize Equation (27) in backwards time τ. We use a fully implicit method as described in Chen & Forsyth (2007).

$$\begin{aligned}\hat{V}_p(e_1, e_2, x, s_\tau, \tau) &= (\Delta\tau)\hat{\mathcal{L}}\hat{V}_p(e_1, e_2, x, s_\tau, \tau) \\ &\quad +(\Delta\tau)\pi_p(e_1, e_2, x, s_\tau, \tau) + \hat{V}_p(e_1, e_2, x, s_{\tau-\Delta\tau}, \tau - \Delta\tau) \; . \end{aligned} \tag{32}$$

Equation (32) now represents a set of decoupled one-dimensional PDEs in the variable x, with (e_1, e_2, s) as parameters. We use a finite difference method with forward, backward, central differencing to discretize the $\hat{\mathcal{L}}$ operator, to ensure a positive coefficient method. (See Forsyth & Labahn (2007/2008) for details.) Linear interpolation is used to determine $\hat{V}_p(e_1, e_2, x, s_{\tau-\Delta\tau}, \tau - \Delta\tau)$. We discretize in the x direction using an unequally spaced grid with n_x nodes and in the S direction using n_s nodes. Between the time interval t^-_{m+1}, t^+_m we use n_τ equally spaced time steps. We use a coarse grid with $(n_\tau, n_x, n_s) = (2, 27, 21)$. We repeated the computations with a fine grid doubling the number of nodes in each direction to verify that the results are sufficiently accurate for our purposes.

A.2 Advancing the solution from $t^+_m \rightarrow t^-_m$

We model the possible emission levels as four discrete states for each of e_1, e_2, which gives 16 possible combinations of (e_1, e_2). We then determine the optimal controls using the methods described in Section 3.2.1. We use exhaustive search (among the finite number of possible states for (e_1, e_2)) to determine the optimal policies. This is, of course, guaranteed to obtain the optimal solution. Recall that since we use a tie-breaking rule, the optimal controls are unique.

B Monotonicity of the Numerical Solution

Economic reasoning dictates that if the value function is decreasing as a function of temperature x at $t = t^-_{m+1}$, and if the benefits are decreasing in temperature, then this property should hold at t^+_m. This can be shown to be an exact solution of PDE (9). In our numerical tests with extreme damage functions, which resulted in rapidly changing functions π_p, we sometimes observed numerical solutions which did not have this property. In order to ensure that this desirable property of the solution holds, we require a timestep restriction. To the best of our knowledge, this restriction has not been reported previously. In practice, this restriction is quite mild, but nevertheless important for extreme cases.

We remind the reader that since we solve the PDEs backwards in time, it is convenient to use the definitions

$$\begin{aligned} \hat{V}_p(e_1, e_2, x_i, s, \tau) &= V_p(e_1, e_2, x_i, s, T-\tau) \\ \hat{\pi}_p(e_1, e_2, x_i, s, \tau) &= \pi_p(e_1, e_2, x_i, s, T-\tau) \ . \end{aligned} \tag{33}$$

Assuming that we discretize Equation (32) on a finite difference grid $x_i, i = 1, \ldots, n_x$, we define

$$\begin{aligned} V_i^{n+1} &= \hat{V}_p(e_1, e_2, x_i, s_{\tau^{n+1}}, \tau^{n+1}) \\ c_i \equiv c(x_i) &= \hat{\pi}_p(e_1, e_2, x_i, s_{\tau^{n+1}}, \tau^{n+1})\Delta\tau + \hat{V}_p(e_1, e_2, x_i, s_{\tau^n}, \tau^n) \end{aligned} \tag{34}$$

Using the methods in Forsyth & Labahn (2007/2008), we discretize Equation (32) using the definitions (34) as follows

$$-\alpha_i \Delta\tau V_{i-1}^{n+1} + (1 + (\alpha_i + \beta_i + r)\Delta\tau)V_i^{n+1} - \beta_i \Delta\tau V_{i+1}^{n+1} = c_i \ , \tag{35}$$

for $i = 1, \ldots, n_x$. Note that the boundary conditions used (see Section 3.1) imply that $\alpha_1 = 0$ and that $\beta_{n_x} = 0$, so that Equation (35) is well defined for all $i = 1, \ldots, n_x$. See Forsyth & Labahn (2007/2008) for precise definitions of α_i and β_i.

Note that by construction α_i, β_i satisfy the positive coefficient condition

$$\alpha_i \geq 0 \quad ; \quad \beta_i \geq 0 \quad ; \quad i = 1, \ldots, n_x \ . \tag{36}$$

Assume that

$$\begin{aligned}\hat{V}_p(e_1, e_2, x_{i+1}, s_{\tau^n}, \tau^n) - \hat{V}_p(e_1, e_2, x_i, s_{\tau^n}, \tau^n) &\leq 0 \\ \hat{\pi}_p(e_1, e_2, x_{i+1}, s_{\tau^{n+1}}, \tau^{n+1}) - \hat{\pi}_p(e_1, e_2, x_i, s_{\tau^{n+1}}, \tau^{n+1}) &\leq 0 \ , \end{aligned} \tag{37}$$

which then implies that

$$c_{i+1} - c_i \leq 0 \ . \tag{38}$$

If Equation (38) holds, then we should have that $V_{i+1}^{n+1} - V_i^{n+1} \leq 0$ (this is a property of the exact solution of Equation (32) if $c(y) - c(x) \leq 0$ if $y > x$).

Define $U_i = V_{i+1}^{n+1} - V_i^{n+1}, i = 1, \ldots, n_x - 1$. Writing Equation (35) at node i and node $i+1$ and subtracting, we obtain the following Equation satisfied by U_i,

$$\begin{aligned}[1 + \Delta\tau(r + \alpha_{i+1} + \beta_i)]U_i - \Delta\tau\alpha_i U_{i-1} - \Delta\tau\beta_{i+1}U_{i+1} &= \Delta\tau(c_{i+1} - c_i) \\ & \quad i = 1, \ldots, n_x - 1 \\ & \quad \alpha_1 = 0 \ ; \ \beta_{n_x} = 0 \ . \end{aligned} \tag{39}$$

Let $U = [U_1, U_2, \ldots, U_{nx-1}]'$, $B_i = \Delta\tau(c_{i+1} - c_i)$, $B = [B_1, B_2, \ldots, B_{nx-1}]'$. We can then write Equation (39) in matrix form as

$$QU = B \ , \tag{40}$$

where

$$\big[QU\big]_i = [1 + \Delta\tau(r + \alpha_{i+1} + \beta_i)]U_i - \Delta\tau\alpha_i U_{i-1} - \Delta\tau\beta_{i+1}U_{i+1} \ . \tag{41}$$

Recall the definition of an M matrix (Varga 2009),

Definition 4 (Non-singular M-matrix)**.** *A square matrix Q is a non-singular M matrix if (i) Q has non-positive off-diagonal elements (ii) Q is non-singular and (iii) $Q^{-1} \geq 0$.*

A useful result is the following (Varga 2009)

Theorem 1. *A sufficient condition for a square matrix Q to be a non-singular M matrix is that (i) Q has non-positive off-diagonal elements (ii) Q is strictly row diagonally dominant.*

From Theorem 1, and Equation (41), a sufficient condition for Q to be an M matrix is that

$$1 + \Delta\tau[r + (\alpha_{i+1} - \alpha_i) + (\beta_i - \beta_{i+1})] > 0 \ , \ i = 1, \ \dots \ n_{x-1} \tag{42}$$

which for a fixed temperature grid, can be satisfied for a sufficiently small $\Delta\tau$. If $\min_i(x_{i+1} - x_i) = \Delta x$, then $\alpha_i = O((\Delta x)^{-2})$, $\beta_i = O((\Delta x)^{-2})$. If α_i, β_i are smoothly varying coefficients, then we can assume that

$$|\alpha_{i+1} - \alpha_i| = O\left(\frac{1}{\Delta x}\right) \quad ; \quad |\beta_i - \beta_{i+1}| = O\left(\frac{1}{\Delta x}\right) \ , \tag{43}$$

and hence condition (42) is essentially a condition on $\Delta\tau/\Delta x$. In practice, for smoothly varying coefficients, $|\alpha_{i+1} - \alpha_i|$ and $|\beta_i - \beta_{i+1}|$ are normally small, so the timestep condition (42) is quite mild.

Proposition 1 (Monotonicity result)**.** *Suppose that (i) condition (42) is satisfied and (ii) $B_i = \Delta\tau(c_{i+1} - c_i) \leq 0$, then $U_i = V_{i+1}^{n+1} - V_i^{n+1} \leq 0$.*

Proof. From condition (42), Definition 4, and Theorem 1 we have that $Q^{-1} \geq 0$, hence from Equation (40)

$$U \ = \ Q^{-1} B \leq 0 \ . \tag{44}$$

□

The practical implication of this result is that if conditions (37) hold at $\tau = T - t_{m+1}^-$, then $\hat{V}(\cdot, \tau = T - t_m^+)$ is a non increasing function of temperature. However, this property may be destroyed after application of the optimal control at $\tau = T - t_m^+ \rightarrow T - t_m^-$. In other words, if we observe that the solution is increasing in temperature, this can only be a result of applying the optimal control, and is not a numerical artifact.

C Nash Equilibrium

We again fix (e_1, e_2, s, x, t_m), so that we understand that $e_p^+ = e_p^+(e_1, e_2, s, x, t_m)$, $R_p(\omega; e_1^-) = R_p(\omega; e_p^-; s, x, t_m)$.

Definition 5 (Nash Equilibrium)**.** *Given the best response sets $R_2(\omega_1; e_2^-)$, $R_1(\omega_2; e_1^-)$ defined in Equations (12)-(13), then the pair (e_1^+, e_2^+) is a Nash equilibrium point if and only if*

$$e_1^+ = R_1(e_2^+; e_1^-) \quad ; \quad e_2^+ = R_2(e_1^+; e_2^-) \ . \tag{45}$$

The following proposition is proven in Insley, Snoddon & Forsyth (2018).

Proposition 2 (Sufficient condition for a Nash Equilibrium)**.** *Suppose $(\hat{e}_1^+, \hat{e}_2^+)$ is the Stackelberg control if player 1 goes first and $(\bar{e}_1^+, \bar{e}_2^+)$ is the Stackelberg control if player 2 goes first. A Nash equilibrium exists at a point (e_1, e_2, s, x, t_m) if $(\hat{e}_1^+, \hat{e}_2^+) = (\bar{e}_1^+, \bar{e}_2^+)$.*

Remark 3 (Checking for a Nash equilibrium)**.** *A necessary and sufficient condition for a Nash Equilibrium is given by condition* (45)*. However a sufficient condition for a Nash equilibrium in the Stackelberg game is that optimal control of either player is independent of who goes first.*

D Interleave Game

In this appendix, we consider the situation where each player makes optimal decisions alternatively. These decision times are separated by a finite time interval.

Suppose that player one chooses an optimal control at time t_m, which we denote by e_1^{m+}. Player two's control is fixed at the value e_2^{m-}. At time t_{m+1}, player two chooses a control

$e_2^{(m+1)+}$, while player one's control is fixed at $e_1^{(m+1)-}$. To avoid notational clutter, we will fix the state variables (s, x) in the following, with the dependence on (s, x) understood.

At time t_m, we have, with player two's control fixed at e_2^{m-},

$$
\begin{aligned}
V_1(e_1^{m-}, e_2^{m-}, t_m^-) &= V_1(e_1^{m+}, e_2^{m-}, t_m^+) && (46)\\
V_2(e_1^{m-}, e_2^{m-}, t_m^-) &= V_2(e_1^{m+}, e_2^{m-}, t_m^+)\ . && (47)
\end{aligned}
$$

Player one's control is determined from

$$
\begin{aligned}
V_1(e_1^{m-}, e_2^{m-}, t_m^-) &= \max_{e_1'} V_1(e_1', e_2^{m-}, t_m^+)\big|_{break\ ties:\ e_1^{m-}}\\
&= V_1(e_1^{m+}, e_2^{m-}, t_m^+) && (48)\\
e_1^{m+} &= \operatorname*{argmax}_{e_1'} V_1(e_1', e_2^{m-}, t_m^+)\big|_{break\ ties:\ e_1^{m+}=e_1^{m-}}\ . && (49)
\end{aligned}
$$

We remind the reader that we break ties by staying at the current level (if that is a maxima of equation (49)) or preferring the lowest emission level (if the current state is not a maxima). Consequently, $e_1^{m+} = e_1^{m+}(e_1^{m-}, e_2^{m-}, t_m^+)$ since dependence on e_1^{m-} is induced by the tie-breaking rule.

At time t_{m+1}, player two chooses a control, with player one's control fixed at $e_1^{(m+1)-}$,

$$
\begin{aligned}
V_1(e_1^{(m+1)-}, e_2^{(m+1)-}, t_{m+1}^-) &= V_1(e_1^{(m+1)-}, e_2^{(m+1)+}, t_{m+1}^+) && (50)\\
V_2(e_1^{(m+1)-}, e_2^{(m+1)-}, t_{m+1}^-) &= V_2(e_1^{(m+1)-}, e_2^{(m+1)+}, t_{m+1}^+)\ . && (51)
\end{aligned}
$$

Player two's control is determined from

$$
\begin{aligned}
V_2(e_1^{(m+1)-}, e_2^{(m+1)-}, t_{m+1}^-) &= V_2(e_1^{(m+1)-}, e_2^{(m+1)+}, t_{m+1}^+)\\
&= \max_{e_2'} V_2(e_1^{(m+1)-}, e_2', t_{m+1}^+)\big|_{break\ ties:\ e_2^{(m+1)-}} && (52)\\
e_2^{(m+1)+} &= \operatorname*{argmax}_{e_2'} V_2(e_1^{(m+1)-}, e_2', t_{m+1}^+)\big|_{break\ ties:\ e_2^{(m+1)+}=e_2^{(m+1)-}}\\
&= R_2(e_1^{(m+1)-}; e_2^{(m+1)-}; t_{m+1}^+)\ , && (53)
\end{aligned}
$$

where $R_2(e_1^{(m+1)-}; e_2^{(m+1)-}; t_{m+1}^+)$ is the best response function of player two to player one's control $e_1^{(m+1)-}$. Note that the tie-breaking strategy induces a dependence on the state $e_2^{(m+1)-}$ in $R_2(\cdot)$.

More generally, we can define player two's response function for arbitrary arguments $(\omega_1; \omega_2)$

$$R_2(\omega_1; \omega_2; t_{m+1}^+) \quad = \quad \underset{\omega_2'}{\text{argmax}}\, V_2(\omega_1, \omega_2', t_{m+1}^+)\big|_{break\ ties:\ R_2=\omega_2} \ . \tag{54}$$

Now, consider the limit where $t_{m+1} \to t_m$, so that

$$e_1^{(m+1)-} \to e_1^{m+} \ ; \ e_2^{(m+1)-} \to e_2^{m-} \ ; \ t_{m+1}^- \to t_m^+ \ . \tag{55}$$

Using equation (55) in equation (50) gives

$$V_1(e_1^{m+}, e_2^{m-}, t_m^+) \quad = \quad V_1(e_1^{m+}, e_2^{(m+1)+}, t_{m+1}^+) \ , \tag{56}$$

while equation (55) in equations (52-53) gives

$$\begin{aligned} V_2(e_1^{m+}, e_2^{m-}, t_m^+) &= V_2(e_1^{m+}, e_2^{(m+1)+}, t_{m+1}^+) && (57) \\ e_2^{(m+1)+} &= R_2(e_1^{m+}; e_2^{m-}; t_{m+1}^+) \ . && (58) \end{aligned}$$

From equations (56) and (58) we have

$$V_1(e_1^{m+}, e_2^{m-}, t_m^+) = V_1(e_1^{m+}, R_2(e_1^{m+}; e_2^{m-}; t_{m+1}^+), t_{m+1}^+) \ , \tag{59}$$

and replacing e_1^{m+} by e_1' in equation (59) gives

$$V_1(e_1', e_2^{m-}, t_m^+) = V_1(e_1', R_2(e_1'; e_2^{m-}; t_{m+1}^+), t_{m+1}^+) \ . \tag{60}$$

Recall that (from equation (48))

$$V_1(e_1^{m-}, e_2^{m-}, t_m^-) \;=\; \max_{e_1'} V_1(e_1', e_2^{m-}, t_m^+)\big|_{break\ ties:\ e_1^{m-}} \;, \tag{61}$$

so that substituting equation (60) into equation (61) gives

$$\begin{aligned} V_1(e_1^{m-}, e_2^{m-}, t_m^-) &= \max_{e_1{}'} V_1(e_1', R_2(e_1'; e_2^{m-}; t_{m+1}^+), t_{m+1}^+)\big|_{break\ ties:\ e_1^{m-}} \\ &= V_1(e_1^{m+}, R_2(e_1^{m+}; e_2^{m-}; t_{m+1}^+), t_{m+1}^+) \\ e_1^{m+} &= \operatorname*{argmax}_{e_1'} V_1(e_1', R_2(e_1'; e_2^{m-}; t_{m+1}^+), t_{m+1}^+)\big|_{break\ ties:\ e_1^{m-}} \;. \end{aligned} \tag{62}$$

From equations (47) and (57-58) we also have that

$$\begin{aligned} V_2(e_1^{m-}, e_2^{m-}, t_m^-) &= V_2(e_1^{m+}, e_2^{m-}, t_m^+) \\ &= V_2(e_1^{m+}, e_2^{(m+1)+}, t_{m+1}^+) \\ e_2^{(m+1)+} &= R_2(e_1^{m+}; e_2^{m-}; t_{m+1}^+) \;. \end{aligned} \tag{63}$$

In summary, equations (62-63) give

$$\begin{aligned} V_1(e_1^{m-}, e_2^{m-}, t_m^-) &= V_1(e_1^{m+}, e_2^{(m+1)+}, t_{m+1}^+) \\ V_2(e_1^{m-}, e_2^{m-}, t_m^-) &= V_2(e_1^{m+}, e_2^{(m+1)+}, t_{m+1}^+) \\ & e_1^{m+} = \operatorname*{argmax}_{e_1'} V_1(e_1', R_2(e_1'; e_2^{m-}; t_{m+1}^+), t_{m+1}^+)\big|_{break\ ties:\ e_1^{m-}} \\ & e_2^{(m+1)+} = R_2(e_1^{m+}; e_2^{m-}, t_{m+1}^+) \;, \end{aligned} \tag{64}$$

which, from Definition 3, we recognize as a Stackelberg game if $t_{m+1}^+ \to t_m^+$.

Proposition 3 follows immediately:

Proposition 3 (Limit of Interleaved game)**.** *Suppose we have an Interleaved game at times t_m, given by equations (46-53). Suppose $t_{m+1} - t_m = \Delta t$, and that player one makes decisions for m even, while player two acts optimally for m odd. Consider fixing player one's decision* *times* $t_{2i}, i = 0, 1, \ldots,$ *and moving player two decision times* $t_{2i+1}, i = 0, 1, \ldots,$ *such that*

$$
\begin{aligned}
&(t_{2i+1} - t_{2i}) \to 0^+ \ ; \ \ i = 0, 1, 2, \ldots \\
&t_{2i} - t_{2(i-1)} = 2\Delta t \ ; \ \ i = 1, 2, \ldots
\end{aligned}
\tag{65}
$$

then the Interleaved game becomes a Stackelberg game.

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