Multi-period mean variance asset allocation: Is it bad to win the lottery?

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The Basic Problem

Suppose you are saving for retirement (i.e. 20 years away)

A standard problem is

- What is your portfolio allocation strategy?
  - i.e. how much should you allocate to bonds, and how much to equities (i.e. an index ETF)
- How should this allocation change through time?
  - Typical rule of thumb: fraction of portfolio in stocks $= 100$ *minus your age*.
- Target Date (Lifecycle) funds
  - Automatically adjust the fraction in stocks (risky assets) as time goes on
  - Use a specified *"glide path"* to determine the risky asset proportion as a function of time to go
  - At the end of 2013, over \$600 billion invested in US[1] [2]

[1] Morningstar

[2] Does not include default allocations in DC plans

Risk-reward tradeoff

This problem (and many others) involve a tradeoff between risk and reward.

Intuitive approach: multi-period mean-variance optimization

- When risk is specified by variance, and reward by expected value
  - → Even non-technical managers can understand the tradeoffs and make informed decisions[3]

In this talk, I will determine the optimal asset allocation strategy

- Objective: minimize risk for specified expected gain
- Use tools of *optimal stochastic control*

[3] I am now a member of the University of Waterloo Pension Committee. I have seen this problem first-hand

Multi-period Mean Variance

Criticism: variance as risk measure penalizes upside as well as downside

I hope to convince you that multi-period mean variance optimization

- Can be modified slightly to be (effectively) a downside risk measure
- Has other good properties: small probability of shortfall

Outcome: optimal strategy for a Target Date Fund

- I will show you that most Target Date Funds being sold in the marketplace use a sub-optimal strategy

Example: Target Date (Lifecycle) Fund with two assets

Risk free bond B

$$dB = rB\ dt$$
$$r = \text{ risk-free rate}$$

Amount in risky stock index S

$$dS = (\mu - \lambda\kappa)S\ dt + \sigma S\ dZ + (J-1)S\ dq$$

$$\mu = \mathbb{P} \text{ measure drift} \quad ; \quad \sigma = \text{ volatility}$$
$$dZ = \text{ increment of a Wiener process}$$

$$dq = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt, \end{cases}$$
$$\log J \sim \mathcal{N}(\mu_J, \sigma_J^2). \quad ; \quad \kappa = E[J-1]$$

Optimal Control

Define:

$$\begin{aligned} X &= (S(t), B(t)) = \text{Process} \\ x &= (S(t) = s, B(t) = b) = (s, b) = \text{State} \\ (s+b) &= \text{total wealth} \end{aligned}$$

Let $(s, b) = (S(t^-), B(t^-))$ be the state of the portfolio the instant before applying a control

The control $c(s, b) = (d, B^+)$ generates a new state

$$\begin{aligned} b &\rightarrow B^+ \\ s &\rightarrow S^+ \\ & S^+ = \underbrace{(s+b)}_{\textit{wealth at } t^-} - B^+ - \underbrace{d}_{\textit{withdrawal}} \end{aligned}$$

Note: we allow cash withdrawals of an amount $d \geq 0$ at a rebalancing time

Semi-self financing policy

Since we allow cash withdrawals

$\rightarrow$ The portfolio may not be self-financing

$\rightarrow$ The portfolio may generate a free cash flow

Let $W_a = S(t) + B(t)$ be the **allocated wealth**

- W_a is the wealth available for allocation into $(S(t), B(t))$.

The non-allocated wealth $W_n(t)$ consists of cash withdrawals and accumulated interest

Constraints on the strategy

The investor can continue trading only if solvent

$$\underbrace{W_a(s,b) = s + b > 0}_{Solvency\ \ condition}. \tag{1}$$

In the event of bankruptcy, the investor must liquidate

$$S^+ = 0 \quad ; \quad B^+ = W_a(s,b) \quad ; \quad \text{if } \underbrace{W_a(s,b) \le 0}_{bankruptcy} .$$

Leverage is also constrained

$$\frac{S^+}{W^+} \quad \le \quad q_{\max}$$

$$W^+ = S^+ + B^+ = \text{ Total Wealth}$$

Mean and Variance under control $c(X(t), t)$

Let:

$$\underbrace{E_{t,x}^{c(\cdot)}[W_a(T)]}_{Reward} = \text{Expectation conditional on } (x, t) \text{ under control } c(\cdot)$$

$$\underbrace{\mathrm{Var}_{t,x}^{c(\cdot)}[W_a(T)]}_{Risk} = \text{Variance conditional on } (x, t) \text{ under control } c(\cdot)$$

Important:

- mean and variance of $W_a(T)$ are as observed at time t, initial state x.

Basic problem: find Efficient frontier

We construct the *efficient frontier* by finding the optimal control $c(\cdot)$ which solves (for fixed λ) [4]

$$\sup_{c} \Bigg\{ \underbrace{E_{t,x}^{c(\cdot)}[W_a(T)]}_{Reward} - \lambda \underbrace{Var_{t,x}^{c(\cdot)}[W_a(T)]}_{Risk} \Bigg\} \tag{2}$$

- Varying $\lambda \in [0, \infty)$ traces out the efficient frontier

- $\lambda = 0; \rightarrow$ we seek only maximize cash received, we don't care about risk.
- $\lambda = \infty \rightarrow$ we seek only to minimize risk, we don't care about the expected reward.

[4] We may not find all the Pareto optimal points by this method unless the achievable set in the $(E^c[W_a(T)], Var^c[W_a(T)])$ plane is convex.

Mean Variance: Standard Formulation

Let $c^*_t(x,u), u \geq t$ be the optimal policy for (3)

$$\sup_{c(X(u),u\geq t)} \left\{ \underbrace{E^{c(\cdot)}_{t,x}[W_a(T)]}_{Reward\ \ as\ \ seen\ \ at\ \ t} - \lambda \underbrace{Var^{c(\cdot)}_{t,x}[W_a(T)]}_{Risk\ \ as\ \ seen\ \ at\ \ t} \right\} , \qquad (3)$$

Then $c^*_{t+\Delta t}(x,u), u \geq t + \Delta t$ is the optimal policy which maximizes

$$\sup_{c(X(u),u\geq t+\Delta t))} \left\{ \underbrace{E^{c(\cdot)}_{t+\Delta t, X(t+\Delta t)}[W_a(T)]}_{Reward\ \ as\ \ seen\ \ at\ \ t+\Delta t} - \lambda \underbrace{Var^{c(\cdot)}_{t+\Delta t, X(t+\Delta t)}[W_a(T)]}_{Risk\ \ as\ \ seen\ \ at\ \ t+\Delta t} \right\} .$$

Pre-commitment Policy

However, in general

$$\underbrace{c_t^*(X(u),u)}_{optimal\ \ policy\ \ as\ \ seen\ \ at\ \ t} \neq \underbrace{c_{t+\Delta t}^*(X(u),u)}_{optimal\ \ policy\ \ as\ \ seen\ \ at\ \ t+\Delta t} \quad ; \quad \underbrace{u \geq t+\Delta t}_{any\ \ time > t+\Delta t} \quad , \tag{4}$$

$\hookrightarrow$ Optimal policy is not *time-consistent*.

The strategy which solves problem (3) has been called the *pre-commitment* policy[5]

Can force time consistency [6]
$\hookrightarrow$ sub-optimal compared to pre-commitment solution.

We will look for the pre-commitment solution

- Pre-commitment is difficult for most investors!

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[5]Basak,Chabakauri: 2010; Bjork et al: 2010
[6]Wang and Forsyth (2011)

Reformulate MV Problem $\Rightarrow$ Dynamic Programming

Embedding technique[7]: for fixed λ, if $c^*(\cdot)$ maximizes

$$\sup_{c(X(u),u\geq t),c(\cdot)\in\mathbb{Z}} \left\{ \underbrace{E^c_{t,x}[W_a(T)]}_{Reward} - \lambda \underbrace{Var^c_{t,x}[W_a(T)]}_{Risk} \right\} ,$$

$$\mathbb{Z} \text{ is the set of admissible controls} \tag{5}$$

$\rightarrow \exists\ \gamma$ such that $c^*(\cdot)$ minimizes

$$\inf_{c(\cdot)\in\mathbb{Z}} E^{c(\cdot)}_{t,x}\left[\left(W_a(T) - \frac{\gamma}{2}\right)^2\right] . \tag{6}$$

[7]Does not require that we have convex constraints. Can be applied to problems with nonlinear transaction costs. Contrast with Lagrange multiplier approach. (Zhou and Li (2000), Li and Ng (2000))

Construction of Efficient Frontier

Regard γ as a parameter $\Rightarrow$ determine the optimal strategy $c^*(\cdot)$ which solves

$$\inf_{c(\cdot)\in\mathbb{Z}} E_{t,x}^{c(\cdot)}\left[(W_a(T) - \frac{\gamma}{2})^2\right]$$

Once $c^*(\cdot)$ is known

- Easy to determine $E_{t,x}^{c^*(\cdot)}[W_a(T)]$, $Var_{t,x}^{c^*(\cdot)}[W_a(T)]$
- Repeat for different γ, traces out efficient frontier[8]

[8] Strictly speaking, since some values of γ may not represent points on the original frontier, we need to construct the upper left convex hull of these points (Tse, Forsyth, Li (2014), SIAM J. Control Optimization) .

Equivalence of MV optimization and target problem

MV optimization is equivalent[9] to investing strategy which

- Attempts to hit a target final wealth of $\gamma/2$
- There is a quadratic penalty for not hitting this wealth target
- From (Li and Ng(2000))

$$\underbrace{\frac{\gamma}{2}}_{wealth\ \ target} = \underbrace{\frac{1}{2\lambda}}_{risk\ \ aversion} + \underbrace{E^{c(\cdot)}_{t=0,x_0}[W_a(T)]}_{expected\ \ wealth}$$

Intuition: if you want to achieve $E[W_a(T)]$, you must aim higher

[9] Vigna, Quantitative Finance, to appear, 2014

HJB PIDE

Determination of the optimal control $c(\cdot)$ is equivalent to determining the value function

$$V(x,t) = \inf_{c \in \mathcal{Z}} \left\{ E_c^{x,t}[(W_a(T) - \gamma/2)^2] \right\} ,$$

Define:

$$\begin{aligned}
\mathcal{L}V &\equiv \frac{\sigma^2 s^2}{2} V_{ss} + (\mu - \lambda\kappa) s V_s + rbV_b - \lambda V \ , \\
\mathcal{J}V &\equiv \int_0^\infty p(\xi) V(\xi s, b, \tau) \ d\xi \\
& \quad\ p(\xi) = \text{ jump size density}
\end{aligned}$$

and the intervention operator $\mathcal{M}(c) \ V(s,b,t)$

$$\mathcal{M}(c) \ V(s,b,t) \quad = \quad V(S^+(s,b,c), B^+(s,b,c), t)$$

HJB PIDE II

The value function (and the control $c(\cdot)$) is given by solving the impulse control HJB equation

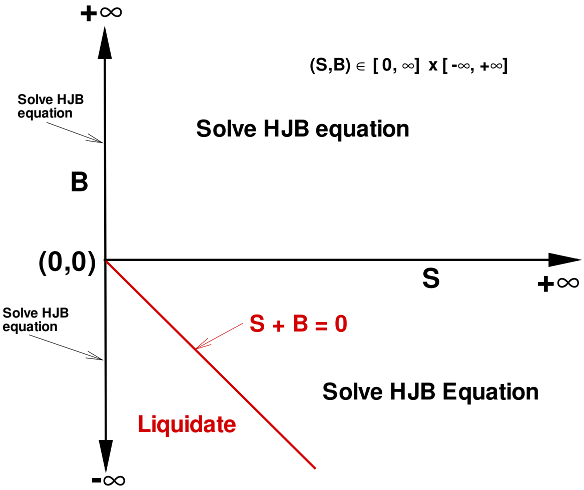
$$\max\left[V_t + \mathcal{L}V + \mathcal{J}V, V - \inf_{c \in \mathcal{Z}}(\mathcal{M}(c)\ V)\right] = 0$$
$$\text{if } (s + b > 0) \qquad (7)$$

Along with liquidation constraint if insolvent

$$V(s, b, t) = V(0, W_a(s, b), t)$$
$$\text{if } (s + b) \leq 0 \text{ and } s \neq 0 \qquad (8)$$

We can easily generalize the above equation to handle the discrete rebalancing case.

Computational Domain[10]



[10] If $\mu > r$ it is never optimal to short S

Global Optimal Point

Examination of the HJB equation allows us to prove the following result (Dang and Forsyth (2014))

Proposition

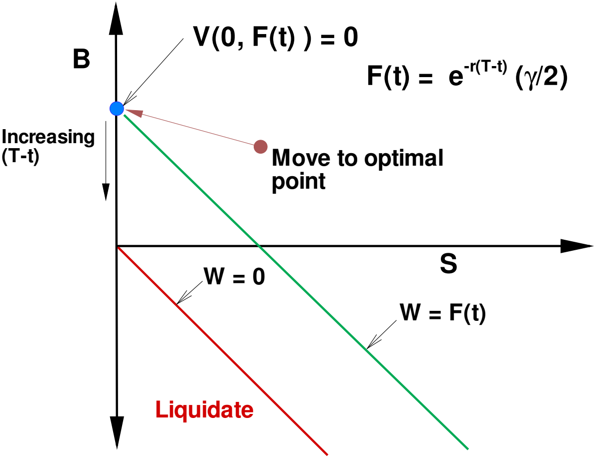
$\forall \gamma > 0$, *the value function* $V(s, b, t)$ *is identically zero at*

$$V(0, F(t), t) \;\; \equiv \;\; 0 \; ; \; F(t) = \frac{\gamma}{2} e^{-r(T-t)} \; , \; \forall t$$

Since $V(s, b, t) \geq 0$

- $V(0, F(t), t) = 0$ is a global minimum
- Any admissible policy which allows moving to this point is an optimal policy
- Once this point is attained, it is optimal to remain at this point

Globally Optimal Point [11]



[11] This is admissible only if $\gamma > 0$

Optimal semi-self-financing strategy

Theorem (Dang and Forsyth (2014))

If $W_a(t) > F(t) \geq 0$,[12] $t \in [0, T]$, *an optimal MV strategy is*[13]

- *Withdraw cash* $W_a(t) - F(t)$ *from the portfolio*
- *Invest the remaining amount* $F(t)$ *in the risk-free asset.*

Criticism of quadratic utilities

- Correspond to a decreasing utility function if $W_a(T) \geq F(T)$
- But for optimal semi-self-financing MV strategy, $W_a(T) \leq F(T)$

$\Rightarrow$ The optimal semi-self-financing MV strategy is

- Equivalent to maximizing a well behaved quadratic utility function

[12] $F(t)$ is the discounted wealth target

[13] A similar semi-self-financing strategy for the discrete rebalancing case was first suggested in (Cui, Li, Wang, Zhu (2012) *Mathematical Finance*).

Intuition: Multi-period mean-variance

Optimal target strategy: try to hit $W_a(T) = \gamma/2 = F(T)$.

If $W_a(t) > F(t) = F(T)e^{-r(T-t)}$, then the target can be hit exactly by

- Withdrawing[14] $W_a(t) - F(t)$ from the portfolio
- Investing $F(t)$ in the risk free account

Optimal control for the target problem $\equiv$ optimal control for the Mean Variance problem

This strategy dominates any other MV strategy

$\rightarrow$ And the investor receives a bonus in terms of a free cash flow

[14] Idea that withdrawing cash may be mean variance optimal was also suggested in (Ehrbar, J. Econ. Theory (1990))

Numerical Method

We solve the HJB impulse control problem numerically using a finite difference method

- We use a semi-Lagrangian timestepping method
- Can impose realistic constraints on the strategy
    - Maximum leverage, no trading if insolvent
    - Arbitrarily shaped solvency boundaries
- Continuous or discrete rebalancing
- Nonlinearities
    - Different interest rates for borrowing/lending
    - Transaction costs
- Regime switching (i.e. stochastic volatility and interest rates)

We can prove[15] that the method is monotone, consistent, ℓ_∞ stable

$\rightarrow$ Guarantees convergence to the viscosity solution

[15] Dang and Forsyth (2014) Numerical Methods for PDEs

Numerical Examples

initial allocated wealth ($W_a(0)$)	100
r (risk-free interest rate)	0.04450
T (investment horizon)	20 (years)
$q_{\max}$ (leverage constraint)	1.5
$t_{i+1} - t_i$ (discrete re-balancing time period)	1.0 (years)

	mean downward jumps	mean upward jumps
μ (drift)	0.07955	0.12168
λ (jump intensity)	0.05851	0.05851
σ (volatility)	0.17650	0.17650
mean log jump size	-0.78832	0.10000
compensated drift	0.10862	0.10862

Objective: verify that removing cash when wealth exceeds target is optimal (i.e. if we win the lottery $\rightarrow$ withdraw cash).

Efficient Frontier: discrete rebalancing

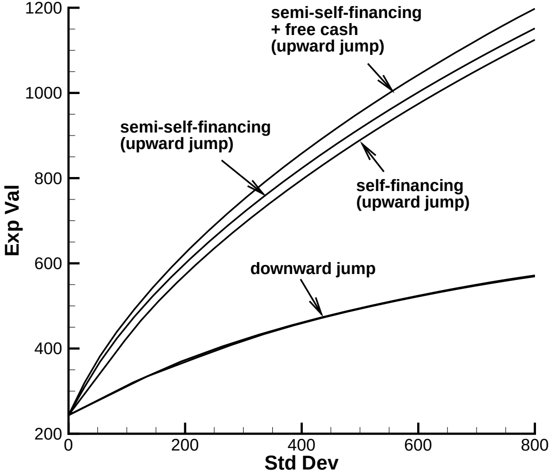


Figure: $T = 20$ years, $W_a(0) = 100$.

Example II

Two assets: risk-free bond, index

- Risky asset follows GBM (no jumps)

According to Benjamin Graham[16], most investors should

- Pick a fraction p of wealth to invest in an index fund (i.e. $p = 1/2$).
- Invest $(1 - p)$ in bonds
- Rebalance to maintain this asset mix

How much better is the optimal asset allocation vs. simple rebalancing rules?

[16]Benjamin Graham, *The Intelligent Investor*

Long term investment asset allocation

Investment horizon (years)	30
Drift rate risky asset μ	.10
Volatility σ	.15
Risk free rate r	.04
Initial investment W_0	100

Benjamin Graham strategy

Constant proportion	Expected Value	Standard Deviation	Quantile
$p = 0.0$	332.01	NA	NA
$p = 0.5$	816.62	350.12	$Prob(W(T) < 800) = 0.56$
$p = 1.0$	2008.55	1972.10	$Prob(W(T) < 2000) = 0.66$

Table: Constant fixed proportion strategy. $p =$ fraction of wealth in risky asset. Continuous rebalancing.

Optimal semi-self-financing asset allocation

Fix expected value to be the same as for constant proportion $p = 0.5$.

Determine optimal strategy which minimizes the variance for this expected value.

- We do this by determining the value of $\gamma/2$ (the wealth target) by Newton iteration

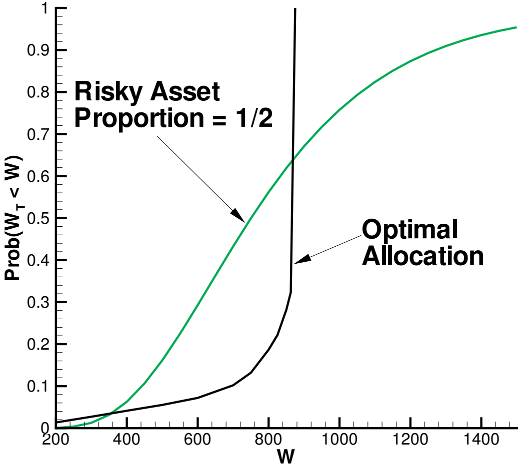
Strategy	Expected Value	Standard Deviation	Quantile
Graham $p = 0.5$[18]	816.62	**350.12**	$Prob(W(T) < 800) = \mathbf{0.56}$
Optimal	816.62	**142.85**	$Prob(W(T) < 800) = \mathbf{0.19}$

Table: $T = 30$ years. $W(0) = 100$. Semi-self-financing: no trading if insolvent; maximum leverage $= 1.5$, rebalancing once/year.

Standard deviation reduced by 250 %, shortfall probability reduced by $3\times$

[18] Continous rebalancing

Cumulative Distribution Functions



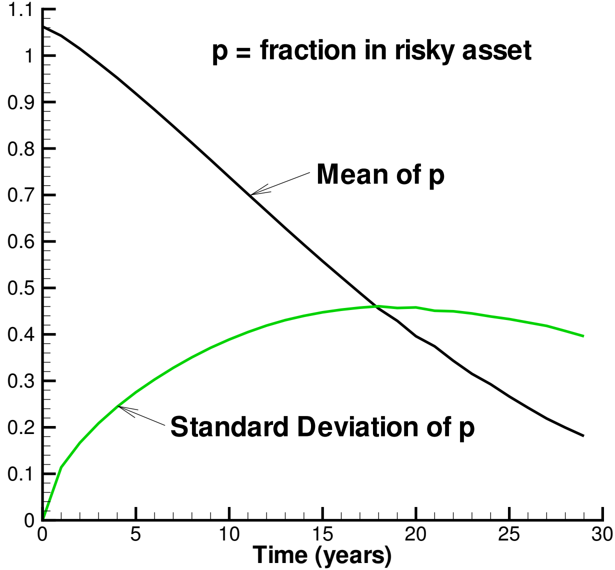
$E[W_T] = 816.62$ for both strategies

Optimal policy: Contrarian: when market goes down $\rightarrow$ increase stock allocation; when market goes up $\rightarrow$ decrease stock allocation

Optimal allocation gives up gains $\gg$ target in order to reduce variance and probability of shortfall.

Investor must pre-commit to target wealth

Mean and standard deviation of the control



Sensitivity to Market Parameter Estimates

Test: We only know the mean values for the market parameters

- Compute control using mean values
- But: in real market → parameters are uniformly distributed in a range centered on mean
- Compute investment result using Monte Carlo simulations

Interest rate range	Drift rate range	Volatility range
$[.02, .06]$	$[.06, .14]$	$[.10, .20]$

	Strategy: computed using fixed parameters			
Market Parameters	Expected Value	Stndrd Dev	$Pr(W(T) < 800)$	Expected Free Cash
Fixed at Mean	817	143	0.19	6.3
Random	807	145	0.19	30.5

Typical Strategy for Target Date Fund: Linear Glide Path

Let p be fraction in risky asset

$$p(t) \quad = \quad p_{start} + \frac{t}{T}(p_{end} - p_{start})$$

Choose parameters so that we get the same expected value as the optimal strategy

$$p_{start} = 1.0 \quad ; \quad p_{end} = 0.0$$

Strategy	Expected Value	Stndrd Dev	$Pr(W(T) < 800)$	Expected Free Cash
$p = 0.5$	817	350	0.56	0.0
Linear[19] Glide Path	817	410	0.58	0.0
Optimal	817	143	0.19	6.3

[19] We can prove that for any deterministic glide path, there exists a superior constant mix strategy

Conclusions

- Optimal allocation strategy dominates simple constant proportion strategy by a large margin
  - $\rightarrow$ Probability of shortfall $\simeq$ 3 times smaller!
- But
  - $\rightarrow$ Investors must pre-commit to a wealth target
  - $\rightarrow$ Investors must commit to a long term strategy ($>$ 20 years)
  - $\rightarrow$ Investors buy-in when market crashes, de-risk when near target
- Standard *"glide path"* strategies of Target Date funds
  - $\rightarrow$ Inferior to constant mix strategy[20]
  - $\rightarrow$ Constant mix strategy inferior to optimal control strategy
- Optimal stochastic control: teaches us an important life lesson
  - Decide on a life target ahead of time and stick with it
  - If you achieve your target, do not be greedy and want more

[20] See also *"The false promise of Target Date funds"*, Esch and Michaud (2014); *"Life-cycle funds: much ado about nothing?"*, Graf (2013)