Long term asset allocation for the patient investor

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The Basic Problem

Suppose you are saving for retirement (i.e. 20 years away)

- What is your portfolio allocation strategy?
  - i.e. how much should you allocate to bonds, and how much to equities (i.e. an index ETF)
- How should this allocation change through time?
  - Typical rule of thumb: fraction of portfolio in stocks $= 100$ *minus your age*.
- Target Date (Lifecyle) funds
  - Automatically adjust the fraction in stocks (risky assets) as time goes on
  - Use a specified *"glide path"* to determine the risky asset proportion as a function of time to go
  - At the end of 2013, over $600 billion invested in US[1]

[1] Morningstar

Risk-reward tradeoff

This problem (and many others) involve a tradeoff between risk and reward.

Intuitive approach: multi-period mean-variance optimization

- When risk is specified by variance, and reward by expected value
  - → Even non-technical managers can understand the tradeoffs and make informed decisions[2]

In this talk, I will determine the optimal asset allocation strategy

- Objective: minimize risk for specified expected gain
- Use tools of *optimal stochastic control*

[2] I am now a member of the University of Waterloo Pension Committee. I have seen this problem first-hand

Multi-period Mean Variance

Criticism: variance as risk measure penalizes upside as well as downside

I hope to convince you that multi-period mean variance optimization

- Can be modified slightly to be (effectively) a downside risk measure
- Has other good properties: small probability of shortfall

Outcome: optimal strategy for a Target Date Fund

- I will show you that most Target Date Funds being sold in the marketplace use a sub-optimal strategy

"All models are wrong: some are useful" [4]

Let S be the price of an underlying asset (i.e. TSX index).

- A standard model for the evolution of S through time is Geometric Brownian Motion (GBM)
- Basic assumption: price process is stochastic, i.e. unpredictable[3]

$$\frac{dS}{S} = \mu\, dt + \sigma\phi\sqrt{dt}$$

$\mu =$ drift rate,

$\sigma =$ volatility,

$\phi =$ random draw from a standard normal distribution

[3] If this were not true, then I (and many others) would be rich

[4] G. Box, of Box-Jenkins and Box-Muller fame.

Monte Carlo Paths: GBM

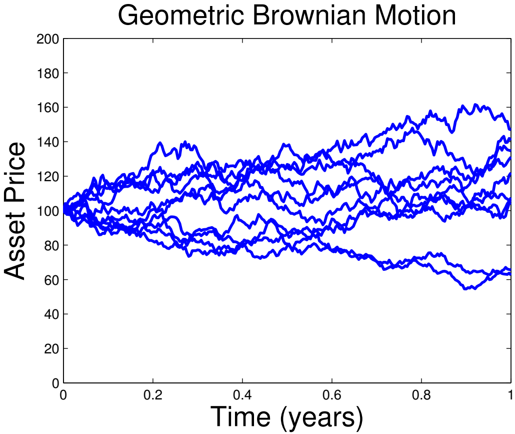


Figure: Ten realizations of possible random paths. Assumption: price processes are stochastic, i.e. unpredictable. $\mu = .10, \sigma = .25$.

What's Wrong with GBM?

- Equity return data suggests market has *jumps* in addition to GBM
  - Sudden discontinuous changes in price
- Most asset allocation strategies ignore the jumps, i.e. market crashes
- But, it seems that we get a financial crisis occurring about once every ten years
- Does it make sense to ignore these events?
- Jumps are also known as:
  - **Black Swans** (see the book with the same title by Nassim Taleb)
  - Fat tail events

TSX Composite monthly log returns 1979-2014

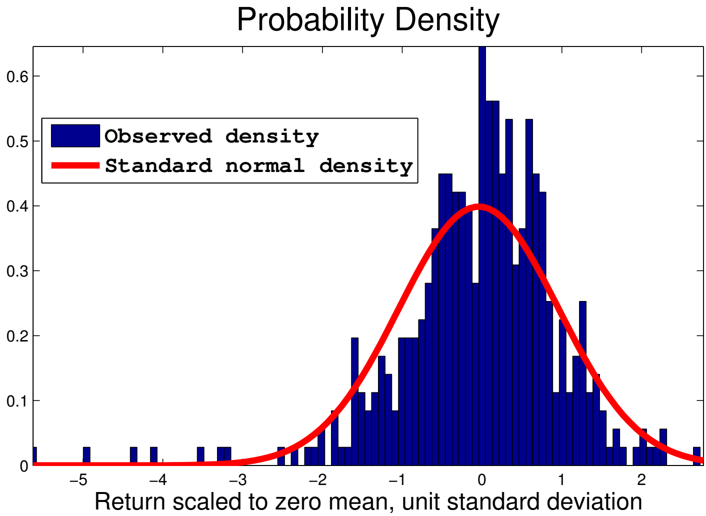


Figure: Higher peaks, heavier tails than normal distribution

A Better Model: Jump Diffusion

$$\frac{dS}{S} = \overbrace{(\mu - \lambda\kappa)\ dt + \sigma\phi\sqrt{dt}}^{GBM} + \overbrace{(J-1)dq}^{Jumps}$$

$$dq = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt, \end{cases}$$

$$\lambda = \text{ mean arrival rate of Poisson jumps; } S \to JS$$

$$J = \text{ Random jump size } \; ; \quad \kappa = E[J-1].$$

- GBM plus jumps (jump diffusion)
- When a jump occurs, $S \to JS$, where J is also random
- This simulates a sudden market crash

Monte Carlo Paths

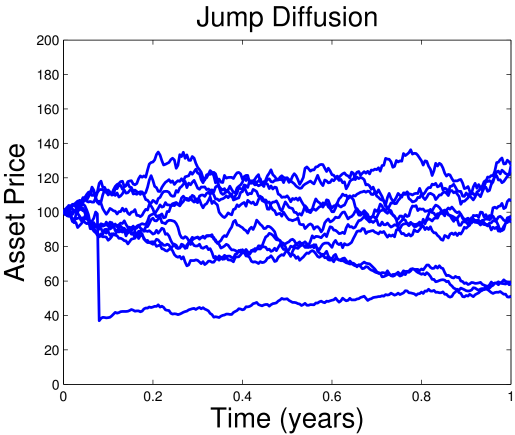


Figure: The arrival rate of the Poisson jump process is .1 per year. Most of the time, the asset follows GBM. In only one of ten stochastic paths, in any given year, can we expect a crash. $\mu = .10, \sigma = .25$.

Example: Target Date (Lifecyle) Fund with two assets

Risk free bond B

$$dB = rB\ dt$$
$$r = \text{ risk-free rate}$$

Amount in risky stock index S

$$dS = \textit{jump diffusion process}$$

Total wealth W

$$W = S + B \tag{1}$$

Objective:

- Optimal allocation of amounts $(S(t), B(t))$, which is multi-period mean-variance optimal
- Optimal strategy is in general a function of (W, t)

Optimal Control

Let:

$$
\begin{aligned}
X &= (S(t), B(t)) = \text{ Process} \\
x &= (S(t) = s, B(t) = b) = (s, b) = \text{ State} \\
(s+b) &= \text{ total wealth}
\end{aligned}
$$

Let $(s, b) = (S(t^-), B(t^-))$ be the state of the portfolio the instant before applying a control

The control $c(s, b) = (d, B^+)$ generates a new state

$$
\begin{aligned}
b &\rightarrow B^+ \\
s &\rightarrow S^+ \\
& \quad S^+ = \underbrace{(s+b)}_{wealth\ \ at\ \ t^-} - B^+ - \underbrace{d}_{withdrawal}
\end{aligned}
$$

Note: we allow cash withdrawals of an amount $d \geq 0$ at a rebalancing time

Semi-self financing policy

Since we allow cash withdrawals

$\rightarrow$ The portfolio may not be self-financing

$\rightarrow$ The portfolio may generate a free cash flow

Let $W_a = S(t) + B(t)$ be the **allocated wealth**

- W_a is the wealth available for allocation into $(S(t), B(t))$.

The non-allocated wealth $W_n(t)$ consists of cash withdrawals and accumulated interest

Constraints on the strategy

The investor can continue trading only if solvent

$$\underbrace{W_a(s, b) = s + b > 0}_{Solvency\ \ condition}. \tag{2}$$

In the event of bankruptcy, the investor must liquidate

$$S^+ = 0 \quad ; \quad B^+ = W_a(s, b) \quad ; \quad \text{if } \underbrace{W_a(s, b) \leq 0}_{bankruptcy} .$$

Leverage is also constrained

$$\frac{S^+}{W^+} \quad \leq \quad q_{\text{max}}$$

$$W^+ = S^+ + B^+ = \text{ Total Wealth}$$

Mean and Variance under control $c(X(t), t)$

Let:

$$\underbrace{E_{t,x}^{c(\cdot)}[W_a(T)]}_{Reward} = \text{Expectation conditional on } (x, t) \text{ under control } c(\cdot)$$

$$\underbrace{\text{Var}_{t,x}^{c(\cdot)}[W_a(T)]}_{Risk} = \text{Variance conditional on } (x, t) \text{ under control } c(\cdot)$$

Important:

- mean and variance of $W_a(T)$ are as observed at time t, initial state x.

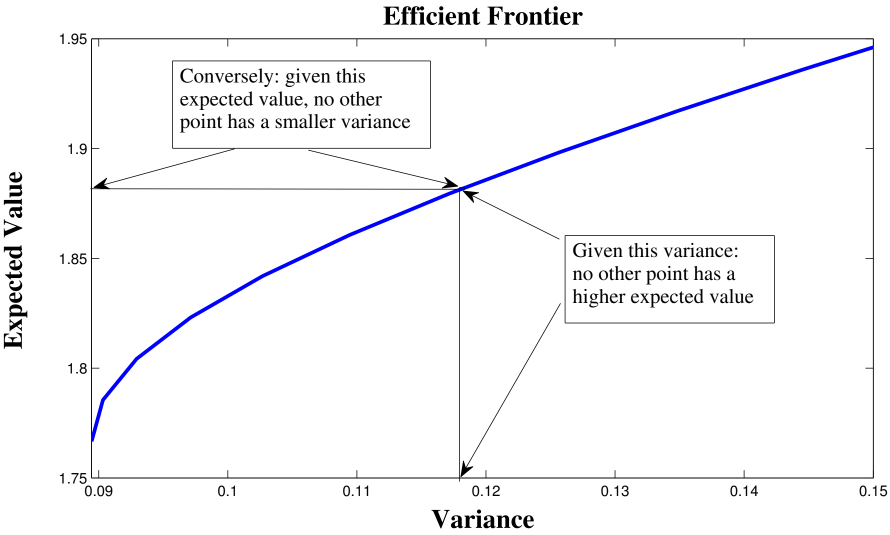
Basic problem: find Efficient frontier

We construct the *efficient frontier* by finding the optimal control $c(\cdot)$ which solves (for fixed λ) [5]

$$\max_{c} \Big\{ \underbrace{E_{t,x}^{c(\cdot)}[W_a(T)]}_{Reward} - \lambda \underbrace{Var_{t,x}^{c(\cdot)}[W_a(T)]}_{Risk} \Big\} \tag{3}$$

- Varying $\lambda \in [0, \infty)$ traces out the efficient frontier
- $\lambda = 0; \rightarrow$ we seek only maximize cash received, we don't care about risk.
- $\lambda = \infty \rightarrow$ we seek only to minimize risk, we don't care about the expected reward.

[5] All investors should pick one of the strategies on the efficient frontier.



Each point on the efficient frontier represents a different strategy $c(\cdot)$.

Mean Variance: Standard Formulation

$$\max_{c(X(u),u\geq t)} \Bigg\{ \underbrace{E^{c(\cdot)}_{t,x}[W_a(T)]}_{Reward\ \ as\ \ seen\ \ at\ \ t} - \lambda \underbrace{Var^{c(\cdot)}_{t,x}[W_a(T)]}_{Risk\ \ as\ \ seen\ \ at\ \ t} \Bigg\} ,$$
$$\lambda \in [0, \infty) \tag{4}$$

- Let $c^*_t(x, u), u \geq t$ be the optimal policy for (4).

Then $c^*_{t+\Delta t}(x, u), u \geq t + \Delta t$ is the optimal policy which maximizes

$$\max_{c(X(u),u\geq t+\Delta t))} \Bigg\{ \underbrace{E^{c(\cdot)}_{t+\Delta t,X(t+\Delta t)}[W_a(T)]}_{Reward\ \ as\ \ seen\ \ at\ \ t+\Delta t} - \lambda \underbrace{Var^{c(\cdot)}_{t+\Delta t,X(t+\Delta t)}[W_a(T)]}_{Risk\ \ as\ \ seen\ \ at\ \ t+\Delta t} \Bigg\} .$$

Pre-commitment Policy

However, in general

$$\underbrace{c_t^*(X(u),u)}_{optimal\ \ policy\ \ as\ \ seen\ \ at\ \ t} \quad \neq \quad \underbrace{c_{t+\Delta t}^*(X(u),u)}_{optimal\ \ policy\ \ as\ \ seen\ \ at\ \ t+\Delta t} \quad ; \quad \underbrace{u \geq t+\Delta t}_{any\ \ time > t+\Delta t} \quad , \tag{5}$$

$\hookrightarrow$ Optimal policy is not *time-consistent*.

The strategy which solves problem (4) has been called the *pre-commitment* policy

Your future self may not agree with your current self!

Ulysses and the Sirens: A pre-commitment strategy

Ulysses had himself tied to the mast of his ship (and put wax in his sailor's ears) so that he could hear the sirens song, but not jump to his death.

Re-formulate MV Problem → Dynamic Programming[6]

For fixed λ, if $c^*(\cdot)$ maximizes

$$\max_{c(X(u),u\geq t)} \left\{ \underbrace{E^c_{t,x}[W_a(T)]}_{Reward} - \lambda \underbrace{Var^c_{t,x}[W_a(T)]}_{Risk} \right\} , \tag{6}$$

→ There exists γ such that $c^*(\cdot)$ minimizes

$$\min_{c(\cdot)} E^{c(\cdot)}_{t,x} \left[\left(W_a(T) - \frac{\gamma}{2} \right)^2 \right] . \tag{7}$$

Once $c^*(\cdot)$ is known

- Easy to determine $E^{c^*(\cdot)}_{t,x}[W_a(T)]$, $Var^{c^*(\cdot)}_{t,x}[W_a(T)]$
- Repeat for different γ, traces out efficient frontier

[6] Li and Ng (2000), Zhou and Li (2000)

Equivalence of MV optimization and target problem

MV optimization is equivalent[7] to investing strategy which

- Attempts to hit a target final wealth of $\gamma/2$
- There is a quadratic penalty for not hitting this wealth target
- From (Li and Ng(2000))

$$\underbrace{\frac{\gamma}{2}}_{wealth\ \ target} = \underbrace{\frac{1}{2\lambda}}_{risk\ \ aversion} + \underbrace{E^{c(\cdot)}_{t=0,x_0}[W_a(T)]}_{expected\ \ wealth}$$

Intuition: if you want to achieve $E[W_a(T)]$, you must aim higher

[7] Vigna, Quantitative Finance, 2014

HJB PIDE

Determination of the optimal control $c(\cdot) \Rightarrow$ find the value function

$$V(x,t) = \min_{c(\cdot)} \left\{ E_{x,t}^{c(\cdot)}[(W_a(T) - \gamma/2)^2] \right\} ,$$

Value function

- Given from numerical solution of a Hamilton-Jacobi-Bellman (HJB) partial integro-differential equation (PIDE)
- This also generates the optimal control $c(\cdot)$.

Optimal semi-self-financing strategy

Let

$$\begin{aligned} F(t) &= \frac{\gamma}{2} e^{-r(T-t)} \\ &= \text{discounted target wealth} \end{aligned}$$

Theorem (Dang and Forsyth (2014))

If $W_a(t) > F(t)$, $t \in [0, T]$, an optimal MV strategy is

- *Withdraw cash $W_a(t) - F(t)$ from the portfolio*
- *Invest the remaining amount $F(t)$ in the risk-free asset.*

What should you do with the cash you withdraw (the free cash)?

- Anything you like (e.g. buy an expensive car).
- You are better off withdrawing the cash!

Intuition: Multi-period mean-variance

Optimal target strategy: try to hit $W_a(T) = \gamma/2 = F(T)$.

If $W_a(t) > F(t) = F(T)e^{-r(T-t)}$, then the target can be hit exactly by

- Withdrawing[8] $W_a(t) - F(t)$ from the portfolio
- Investing $F(t)$ in the risk free account

This strategy dominates any other MV strategy

- We never exceed the target
- No *"upside penalization"*

$\rightarrow$ And the investor receives a bonus in terms of a free cash flow

[8] Idea that withdrawing cash may be mean variance optimal was also suggested in (Ehrbar, J. Econ. Theory (1990))

Numerical Examples

initial allocated wealth ($W_a(0)$)	100
r (risk-free interest rate)	0.04450
T (investment horizon)	20 (years)
$q_{\max}$ (leverage constraint)	1.5
$t_{i+1} - t_i$ (discrete re-balancing time period)	1.0 (years)

	mean downward jumps	mean upward jumps
μ (drift)	0.07955	0.12168
λ (jump intensity)	0.05851	0.05851
σ (volatility)	0.17650	0.17650
mean log jump size	-0.78832	0.10000

Objective: verify that removing cash when wealth exceeds target is optimal.

Efficient Frontier: sometimes its optimal to spend money

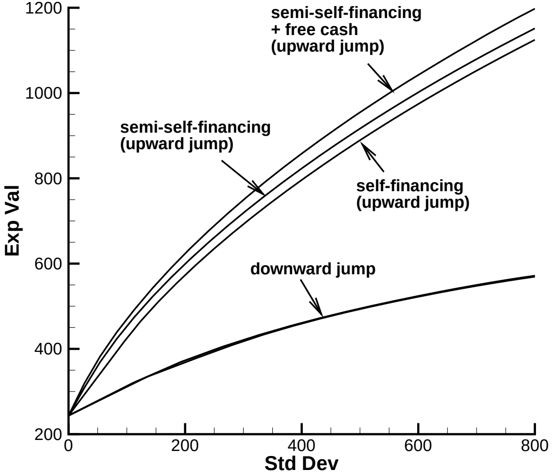


Figure: $T = 20$ years, $W_a(0) = 100$.

Example II

Two assets: risk-free bond, index

- Risky asset follows GBM (no jumps)

According to Benjamin Graham[9], most investors should

- Pick a fraction p of wealth to invest in an index fund (e.g. $p = 1/2$).
- Invest $(1 - p)$ in bonds
- Rebalance to maintain this asset mix

How much better is the optimal asset allocation vs. simple rebalancing rules?

[9] Benjamin Graham, *The Intelligent Investor*

Long term investment asset allocation

Investment horizon (years)	30
Drift rate risky asset μ	.10
Volatility σ	.15
Risk free rate r	.04
Initial investment W_0	100

Benjamin Graham strategy

Constant proportion	Expected Value	Standard Deviation	Quantile
$p = 0.0$	332.01	NA	NA
$p = 0.5$	816.62	350.12	$Prob(W(T) < 800) = 0.56$
$p = 1.0$	2008.55	1972.10	$Prob(W(T) < 2000) = 0.66$

Table: Constant fixed proportion strategy. $p =$ fraction of wealth in risky asset. Continuous rebalancing.

Optimal semi-self-financing asset allocation

Fix expected value to be the same as for constant proportion $p = 0.5$.

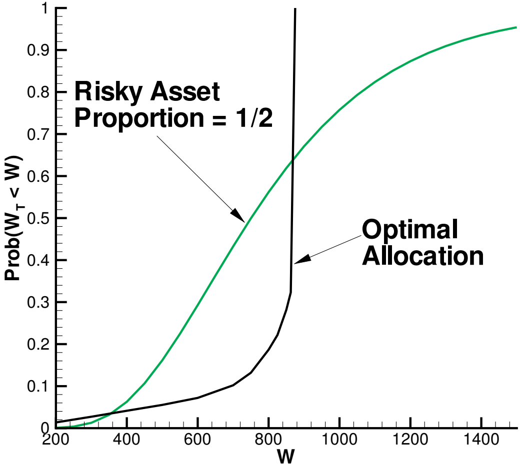
Determine optimal strategy which minimizes the variance for this expected value.

Strategy	Expected Value	Standard Deviation	Quantile
Graham $p = 0.5$	816.62	**350.12**	$Prob(W(T) < 800) = \mathbf{0.56}$
Optimal	816.62	**142.85**	$Prob(W(T) < 800) = \mathbf{0.19}$

Table: $T = 30$ years. $W(0) = 100$. Semi-self-financing: no trading if insolvent; maximum leverage $= 1.5$, rebalancing once/year.

Standard deviation reduced by 250 %, shortfall probability reduced by $3\times$

Cumulative Distribution Functions



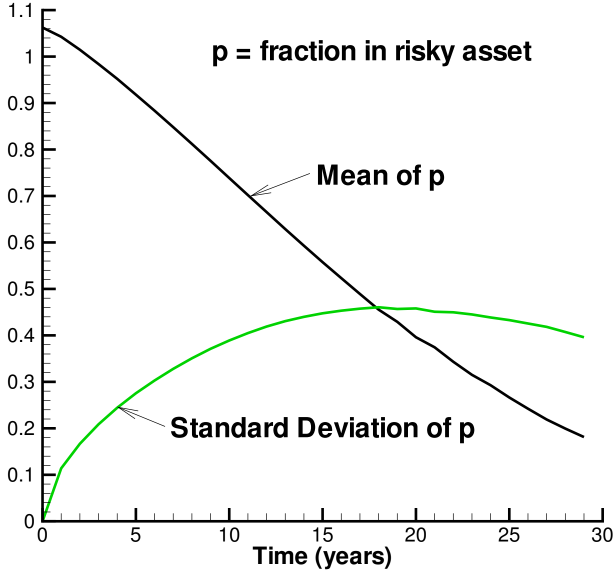
$E[W_T] = 816.62$ for both strategies

Optimal policy: Contrarian: when market goes down $\rightarrow$ increase stock allocation; when market goes up $\rightarrow$ decrease stock allocation

Optimal allocation gives up gains $\gg$ target in order to reduce variance and probability of shortfall.

Investor must pre-commit to target wealth

Mean and standard deviation of the control



Typical Strategy for Target Date Fund: Linear Glide Path

Let p be fraction in risky asset

$$p(t) \quad = \quad p_{start} + \frac{t}{T}(p_{end} - p_{start})$$

Choose parameters so that we get the same expected value as the optimal strategy

$$p_{start} = 1.0 \quad ; \quad p_{end} = 0.0$$

Strategy	Expected Value	Stndrd Dev	$Pr(W(T) < 800)$	Expected Free Cash
$p = 0.5$	817	350	0.56	0.0
Linear[11] Glide Path	817	410	0.58	0.0
Optimal	817	143	0.19	6.3

[11] We can prove that for any deterministic glide path, there exists a superior constant mix strategy

Sensitivity to Market Parameter Estimates

Test: We only know the mean values for the market parameters

- Compute control using mean values
- But: in real market $\rightarrow$ parameters are uniformly distributed in a range centered on mean
- Compute investment result using Monte Carlo simulations

Interest rate range	Drift rate range	Volatility range
$[.02, .06]$	$[.06, .14]$	$[.10, .20]$

	Strategy: computed using fixed parameters			
Market Parameters	Expected Value	Stndrd Dev	$Pr(W(T) < 800)$	Expected Free Cash
Fixed at Mean	817	143	0.19	6.3
Random	807	145	0.19	30.5

Conclusions

- Optimal allocation strategy dominates simple constant proportion strategy by a large margin
  - $\rightarrow$ Probability of shortfall $\simeq$ 3 times smaller!
- But
  - $\rightarrow$ Investors must pre-commit to a wealth target
  - $\rightarrow$ Investors must commit to a long term strategy ($>$ 20 years)
  - $\rightarrow$ Investors buy-in when market crashes, de-risk when near target
- Standard *"glide path"* strategies of Target Date funds
  - $\rightarrow$ Inferior to constant mix strategy[12]
  - $\rightarrow$ Constant mix strategy inferior to optimal control strategy
- Optimal stochastic control: teaches us an important life lesson
  - Decide on a life target ahead of time and stick with it
  - If you achieve your target, do not be greedy and want more

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[12] See also *"The false promise of Target Date funds"*, Esch and Michaud (2014); *"Life-cycle funds: much ado about nothing?"*, Graf (2013)