Optimal Asset Allocation for DC Pension Decumulation with a Variable Spending Rule

Peter A. Forsyth\textsuperscript{a}  Kenneth R. Vetzal\textsuperscript{b}  Graham Westmacott\textsuperscript{c}

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Abstract

We determine the optimal asset allocation to bonds and stocks using an Annually Recalculated Virtual Annuity (ARVA) spending rule for DC pension plan decumulation. Our objective function minimizes downside withdrawal variability for a given fixed value of total expected withdrawals. The optimal asset allocation is found using optimal stochastic control methods. We formulate the strategy as a solution to a Hamilton Jacobi Bellman (HJB) Partial Integro Differential Equation (PIDE). We impose realistic constraints on the controls (no shorting, no leverage, discrete rebalancing), and solve the HJB PIDEs numerically. Compared to a fixed weight strategy which has the same expected total withdrawals, the optimal strategy has a much smaller average allocation to stocks, and tends to de-risk rapidly over time. This conclusion holds in the case of a parametric model based on historical data, and also in a bootstrapped market based on the historical data.

Keywords: DC pension plans, decumulation, optimal control, HJB equation, annually recalculated virtual annuity

1 Introduction

Throughout the developed economies, defined benefit (DB) pension plans are disappearing. In an effort to reduce corporate risk exposures, firms are moving employees to defined contribution (DC) plans. In a typical DC plan, the employee and employer contribute a fraction of the employee’s annual salary to a tax-advantaged fund. The employee then can choose to invest the funds in a variety of investment vehicles. These usually include bond and equity index funds. A common default choice in the US is a Target Date Fund (TDF). In a TDF, the fraction invested in stocks declines over time in a prescribed manner, according to the anticipated retirement date.

However, once employees reach retirement age, they are faced with constructing a decumulation strategy. In other words, pensioners then face the difficult problems of (i) calculating how much to withdraw from their investment portfolios each year; and (ii) determining their asset allocation strategies.

\textsuperscript{a}David R. Cheriton School of Computer Science, University of Waterloo, Waterloo ON, Canada N2L 3G1, paforsyt@uwaterloo.ca, +1 519 888 4567 ext. 34415.

\textsuperscript{b}School of Accounting and Finance, University of Waterloo, Waterloo ON, Canada N2L 3G1, kvetzal@uwaterloo.ca, +1 519 888 4567 ext. 36518.

\textsuperscript{c}PWL Capital, 20 Erb Street W., Suite 506, Waterloo, ON, Canada N2L 1T2, gwestmacott@pwlcapital.com, +1 519 880 0888.
The shift to a DC plan transfers income and longevity risk from the firm to the plan member. The plan member is now exposed to investment risk during both the accumulation and decumulation phases, as well as longevity risk during decumulation. It might seem that use of an annuity would be a good strategy for decumulation, since this eliminates both of these risks during that phase. However, it is well known that when given a choice, few investors annuitize (Peijnenburg et al., 2016). While this has been traced to the investor’s desire to retain control over their portfolio, there are many other potential factors in play as well. For example, MacDonald et al. (2013) list more than three dozen reasons to avoid annuitization. Among these are the general lack of true inflation protection and the fact that in many cases, annuities are poorly priced in practice.

A well-known decumulation strategy relies on the 4% rule. This can be traced to the work of Bengen (1994), which was based on historical backtests of a portfolio that had 50% invested in stocks and 50% invested in bonds, rebalanced annually. The final recommendation in Bengen (1994) was that withdrawing initially at a rate of 4% of the initial portfolio value and with successive withdrawals adjusted for inflation was a safe strategy, in that the investor would have never run out of funds over any rolling 30-year historical period considered. More generally, there is a large academic literature on decumulation strategies. A small but representative sample includes Blake et al. (2003), Gerrard et al. (2004, 2006), Smith and Gould (2007), Milevsky and Young (2007), Freedman (2008), and Liang and Young (2018). For overviews of various strategies for decumulation, see MacDonald et al. (2013) and Bernhardt and Donnelly (2018).

Rather than specifying a strategy which withdraws a fixed real amount each year, we consider a strategy that responds to the actual investment experience. A recent suggestion is based on an Annually Recalculated Virtual Annuity (ARVA) (Waring and Siegel, 2015; Westmacott and Daley, 2015). An ARVA rule can summarized as follows:

“Each year, one should spend (at most) the amount that a freshly purchased annuity— with purchase price equal to the then-current portfolio and priced at the current interest rates and number of years of required cash flows remaining—would pay out in that year.”

(Waring and Siegel, 2015, p. 91)

The trade-off here is that real withdrawals will fluctuate in order to prevent the possibility of running out of cash. Of course, the downside is that in the event of very poor portfolio investment performance, the withdrawals may become minuscule (though the value of the ARVA portfolio will not drop to zero). Waring and Siegel (2015) justify this strategy by contending that DC plan decumulation is basically an annuitization problem. This does not require an actual annuity (hence the use of virtual in the ARVA designation), but does require annuity thinking. Waring and Siegel (2015) focus exclusively on the withdrawal rule, as opposed to the asset allocation strategy. Our objective here is to optimize the asset allocation strategy for an ARVA type decumulation rule. We consider two measures of performance:

- The expected total real (i.e. inflation-adjusted) withdrawals over the lifetime of the strategy.
- The expected downside variability of the withdrawals from one year to the next.

This results in a multi-objective optimization problem, which we solve using a scalarization approach. In other words, for a fixed value of the expected total withdrawal, we find the strategy which gives us the smallest possible withdrawal variability. We formulate this problem as a Hamilton-Jacobi-Bellman (HJB) Partial Integro Differential Equation (PIDE), which we solve numerically. This allows us to impose realistic constraints on our strategy, i.e. no short-selling or leverage and infrequent (yearly) rebalancing.
Using parameters of a stochastic model estimated from 90 years of market data, we compute and store the optimal strategy from the numerical solution of the HJB equation. We then use this strategy in Monte Carlo simulations to generate statistics of interest. One set of simulations is based on the same parametric stochastic model used to determine the investment strategy. We label this type of simulation the \textit{synthetic market}. As a robustness check, we also use the stored strategy in historical bootstrap backtests. We refer to the market based on bootstrap resampling as the \textit{historical market}. For comparison purposes, in addition to our optimal strategies we also test fixed weight strategies with ARVA withdrawals in both the synthetic and historical markets.

\section{Formulation}

For simplicity we assume that there are only two assets available in the financial market, namely a risky asset and a risk-free asset. In practice, the risky asset would be a broad equity market index fund. We assume that the investor’s allocation to these two assets is rebalanced periodically, not continuously.

The investment horizon (i.e. the total time of the decumulation phase) is \(T\). \(S_t\) and \(B_t\) respectively denote the \textit{amounts} invested in the risky and risk-free assets at time \(t\), \(t \in [0, T]\). The investor’s total wealth at time \(t\) is defined as

\[
\text{Total wealth} \equiv W_t = S_t + B_t. \tag{2.1}
\]

To reduce subscript clutter, we will occasionally use the notation \(S_t \equiv S(t)\), \(B_t \equiv B(t)\) and \(W_t \equiv W(t)\). Since we focus on real cash flows, \(S_t\) and \(B_t\) should be seen as being in real terms.

In general, \(S_t\) and \(B_t\) will depend on the investor’s strategy over time, as well as changes in the real unit prices of these assets. Absent a control determined by the investor (i.e. withdrawing funds or rebalancing the portfolio), \(S_t\) and \(B_t\) will only change as a result of movements in real asset prices. In this case (absence of control), we assume that \(S_t\) follows a jump diffusion process of the form

\[
\frac{dS_t}{S_t} = (\mu - \lambda \xi E[\xi - 1]) \, dt + \sigma \, d\mathcal{Z} + d\left(\sum_{i=1}^{\pi_t} (\xi_i - 1)\right), \tag{2.2}
\]

where \(S_{t-} = \lim_{\epsilon \to 0^+} S_{t-\epsilon}\), \(\mu\) is the uncompensated drift rate, \(\sigma\) is the diffusive volatility, \(\mathcal{Z}\) is a standard Brownian motion, \(\pi_t\) is a Poisson process with positive intensity parameter \(\lambda\), and \(\xi_i\) are i.i.d. positive random variables. We assume that \(\xi_i\), \(\pi_t\), and \(\mathcal{Z}\) are all mutually independent. Equation (2.2) augments standard geometric Brownian motion with occasional discontinuous jumps. Adding in jumps allows us to incorporate the effects of large market movements (e.g. market crashes) into our analysis. As we only consider cases where the portfolio is discretely rebalanced, the jump process models the cumulative effects of substantial changes in the real price of the risky asset between rebalancing times.

When a jump occurs, \(S_t = \xi S_{t-}\). We assume that \(\log(\xi)\) follows a double exponential distribution \footnote{Kou, 2002}. Conditional on a jump occurring, \(p_u\) is the probability of an upward jump, while \(p_d = 1 - p_u\) is the chance of a downward jump. The density function for \(y = \log(\xi)\) is

\[
f(y) = p_u \eta_1 e^{-\eta_1 y} 1_{y \geq 0} + p_d \eta_2 e^{\eta_2 y} 1_{y < 0}. \tag{2.3}
\]

Given (2.3), the expected value of the jump multiplier \(\xi\) is

\[
E[\xi] = \frac{p_u \eta_1}{\eta_1 - 1} + \frac{p_d \eta_2}{\eta_2 + 1}. \tag{2.4}
\]
Letting $\kappa = E[\xi - 1]$, we can write equation (2.2) more informally as

$$\frac{dS_t}{S_t} = (\mu - \lambda \xi \kappa) \ dt + \sigma \ dZ + (\xi - 1) \ dQ$$

(2.5)

where $dQ = 1$ with probability $\lambda \xi \ dt$ and $dQ = 0$ with probability $1 - \lambda \xi \ dt$.

In the absence of control, we assume that the dynamics of the amount $B_t$ invested in the risk-free asset are simply

$$dB_t = rB_t \ dt,$$

(2.6)

where $r$ is the constant real risk-free rate. This is obviously a gross simplification of the actual bond market, but it allows us to compute a relatively simple asset allocation strategy.

Summarizing the model, we assume that there is a constant risk-free interest rate so that the amount invested in the risk-free asset is described by equation (2.6), apart from times when the investor either rebalances the portfolio or withdraws cash from it. Similarly excluding rebalances and withdrawals, the amount invested in the risky asset is described by the jump diffusion model (2.2). Equations (2.2) and (2.6) represent what we referred to earlier as the synthetic market. These equations are highly simplified models of actual stock and bond index processes. We use them to determine the asset allocation strategy in the synthetic market. We first evaluate the performance of the strategy using Monte Carlo simulations driven by the same underlying assumptions (i.e. in the synthetic market), but we subsequently assess the performance of the same strategy in the historical market, i.e. using bootstrapped historical bond and stock market returns. This second set of tests subjects the strategy to complications such as stochastic interest rates, correlations in stock and bond indices, and random changes in asset volatilities. As will be seen below, the performance of the strategy in the historical market is very similar to that found in the synthetic market, indicating that the parsimonious representation (2.2)-(2.6) suffices to determine an asset allocation strategy in our context.

3 ARVA Spending Rule

The ARVA spending rule is based on the following idea. Each year, a virtual annuity is constructed based on the current portfolio value, the current number of remaining years of required cash flows, and the prevailing real interest rate. The withdrawal amount in a given year is then based on the fixed virtual annuity payment per year. The virtual annuity is recomputed each year, so the annual payments will fluctuate in response to the investment experience. In contrast to the ubiquitous 4% rule, ARVA is efficient, in the sense that all accumulated assets are withdrawn at the end of the period of required cash flows.

As noted by Waring and Siegel (2015) and Westmacott and Daley (2015), taking into account mortality can front load spending into periods when retirees are more active. We use the methods suggested in Westmacott and Daley (2015) to add a mortality boost to the ARVA spending rule. Suppose that a retiree is $x$ years old at $t = 0$. Assuming that the $x + t$ year old retiree is alive at time $t$, let $T_x^*(t)$ be the time at which $80\%$ of the cohort of $x + t$ year olds have passed away, given that all members of the cohort were alive at time $t$. In other words, conditional on an investor being alive at time $t$, $T_x^*(t)$ is the time at which there is just a 20% chance that this investor will still be alive.

Given the real interest rate $r$, the continuous real annuity payment $a(t)$ for $T_x^*(t) - t$ years that
Figure 3.1: ARVA annuity factor and spending amounts per year. The real interest rate \( r = .0048 \), CPM 2014 mortality tables are used. The investor is assumed to be a 65 year old male at \( t = 0 \). In panel (a), it is assumed that the initial portfolio value at \( t = 0 \) is \( W(0) = \$1,000,000 \) and that the portfolio is invested entirely in the risk-free asset. Mortality Effects: assumes equation (3.3) used. No Mortality: fixed payments which exhaust all wealth after 30 years.

Consider a set of withdrawal times \( \mathcal{T} \)

\[
\mathcal{T} \equiv \{ t_0 = 0 < t_1 < \cdots < t_M = T \}, \tag{3.2}
\]

where \( t = 0 \) denotes the time that the \( x \) year old retiree begins to withdraw money from the DC plan. We specify \( \mathcal{T} \) to be equidistant with \( t_i - t_{i-1} = \Delta t = T/M \), \( i = 1, \ldots, M \). In the following we will let \( \Delta t = 1 \) year. If we restrict ourselves to annual payments at times \( t_i \), we can convert the continuous payment above into a lump sum \( A(t_i) \) received in advance over the interval \([t_i, t_{i+1}]\).

This lump sum is

\[
lump \text{ sum payment} = W(t_i)A(t_i), \quad A(t_i) = \int_{t_i}^{t_{i+1}} \frac{r}{1 - \exp[-r(T_x^*(t') - t)]]} \, dt'. \tag{3.3}
\]

We use the CPM 2014 mortality tables (male) from the Canadian Institute of Actuaries\textsuperscript{1} to compute \( T_x^*(t) \) with \( x = 65 \).

Figure 3.1(a) shows the annuity factor \( A(t) \) (assuming lump sum annual payments) for our Canadian male who begins withdrawing at age 65 at \( t = 0 \). Assuming an initial wealth of \( W(0) = \$1,000,000 \), Figure 3.1(b) shows the withdrawal amounts per year based on the ARVA spending rule. The portfolio is entirely invested in the risk-free asset with an assumed real return of \( r = .004835 \), which was the average real return of one month US T-bills over the period 1926:1-2016:12 (see

\textsuperscript{1}www.cia-ica.ca/docs/default-source/2014/214013e.pdf
Section 6). If mortality effects are ignored, then the real fixed lump sum yearly payment that
precisely exhausts the initial wealth after 30 years would be about $34,650 per year.\footnote{In contrast, assuming an initial capital of $1,000,000 a fairly priced real life annuity would generate about $49,960 per year. However, in the Canadian context real annuities are essentially unavailable. As of February 2019, online posted rates for a life annuity (no guarantee) for a 65-year old Canadian male were in the range of $58,000 to $65,160 per year (nominal). A 2% annual inflation rate would reduce the real value of a payment of $60,000 to about $33,000 after 30 years.}

We can see from Figure 3.1(b) that the ARVA rule with a mortality boost shifts spending to
earlier years, but this comes at the cost of reduced spending compared to a fixed term annuity after
about year 20, i.e. age 85 (assuming, of course, that the investor has not passed away). As points
of reference, the CPM 2014 tables indicate that the probabilities that a 65 year old Canadian male
attains the ages of 85, 95, or 100 are .58, .13, and .02 respectively.

However, this pattern of reducing spending in the latter stages of retirement is perhaps not
unreasonable. Studies show a decline in spending by about 1% per year after age 70, followed by
a 2% decline per year after age 80 (Vettese 2018). In addition, in the Canadian context, deferring
government benefits (Canada Pension Plan) from age 65 to age 70 results in a 48% increase in
annuity income. This deferred government annuity strategy can be used to offset the declining
ARVA payments.

Our objective is to improve these spending patterns with a high probability by investing the
portfolio in a combination of risky and risk-free assets. The comparison with the fixed term annuity
is not quite fair, since the ARVA with mortality effects will not exhaust the portfolio at \( t = 30 \).
In fact, at \( t = 30 \), \( A(t = 30) = .24 \), so that the final portfolio value (after the last payment) is
equal to approximately three times the last payment. In other words, there is enough wealth left
in the portfolio to provide equal payments at ages 95, 96, 97, and 98, which would fund the retiree
through to his 99\textsuperscript{th} birthday.

4 Optimization Problem

Let \( T \) be the set of withdrawal/rebalancing times. At each \( t_i \in T \), the investor (i) withhdraws
an amount of cash \( Q_i \) from the portfolio and then (ii) rebalances the portfolio. As noted in the
introduction, we consider a multi-objective problem which involves attempting to maximize reward
while minimizing risk. As a measure of reward, we consider

\[
E \left[ \sum_{i=0}^{M} Q_i \right].
\]

Equation (4.1) is the expected sum of all real withdrawals. This captures the simple intuition that
the investor seeks to maximize total real withdrawals. We do not explicitly consider the present
value of the withdrawals, in part because it is not clear what discount rate should be used given
the risk of the withdrawals. One way to interpret equation (4.1) is to think of the discount rate as
being zero. Alternatively, if we discount the withdrawals using the long run average real risk-free
rate of 0.4835\%, we do not obtain a very different present value.\footnote{For example, suppose the investor takes 30 annual constant real withdrawals. Under our criterion, the reward measure would be 30 times the annual withdrawal. Discounting at the long run average real risk-free rate (assuming withdrawals are at the start of each year) would give a reward measure of about 28 times the annual withdrawal.} In addition, note that if we apply
the mortality boost as described in Section 3 above, cash flows in earlier periods will tend to be
higher. This will reduce duration, so the present value calculated using the long run average real
risk free rate would be closer to the sum of the withdrawals.
Our measure of risk is
\[ E \left[ \sum_{i=M}^{i=M} (Q_i - Q_{i-1})^- \right]^2 \]
(4.2)
where \((Q_i - Q_{i-1})^- \equiv \min(Q_i - Q_{i-1}, 0)\). This is a measure of the downside variability in withdrawals, and reflects the idea that the retiree generally wants to avoid year-to-year declines in withdrawals. However, keep in mind that applying the mortality boost as described in Section 3 will tend to reduce withdrawals over time. These reductions will be reflected in the risk measure given in equation 4.2, even though they are in a sense a deliberate choice (from the mortality boost), not a consequence of poor investment allocation or performance. Overall, the investor wants to maximize reward (4.1) while minimizing risk (4.2). These are clearly conflicting goals, and we search for Pareto optimal strategies using a scalarization approach.

Given a time dependent function \(f(t)\), we use the shorthand notation
\[
\begin{align*}
&F^+_i = f(t_i^+) \equiv \lim_{\epsilon \to 0^+} f(t_i + \epsilon) \\
&F^-_i = f(t_i^-) \equiv \lim_{\epsilon \to 0^+} f(t_i - \epsilon).
\end{align*}
\]
Let \(S_i^- = S_{i-1}^- \), \(S_i^+ = S_{i-1}^+ \), \(B_i^- = B_{i-1}^- \), \(B_i^+ = B_{i-1}^+ \) Similarly, define total wealth as \(W_i^- = S_i^- + B_i^- \) and \(W_i^+ = S_i^+ + B_i^+ \).

Define the state variable \(Q(t)\) for \(t \in (t_{i-1}, t_i)\) as \(Q(t) = Q_{i-1}\) for \(t \in (t_{i-1}, t_i)\). In other words, for any time between withdrawal dates, \(Q(t)\) represents the withdrawal amount at the previous withdrawal time. Finally, denote by \(X(t) = (S(t), B(t), Q(t))\), \(t \in [0, T]\), the multi-dimensional controlled underlying process, and let \(x = (s, b, q)\) be the realized state of the system.

The control for our problem is the fraction allocated to equities at \(t_i^+\), \(p_i = p_i(X_i^-, t_i^-) = S_i^+/W_i^+\), where \(X_i^- = (S_i^-, B_i^-, Q_i^-)\). Our optimization problem is then
\[
\begin{align*}
\max_{\{p_0, \ldots, p_{M-1}\}} & \left\{ E \left[ \sum_{i=0}^{i=M} Q_i \right] - \lambda E \left[ \sum_{i=1}^{i=M} (Q_i - Q_{i-1})^- \right]^2 \right\} \\
\text{subject to} & \begin{cases} & (S_i, B_i) \text{ follow processes (2.2)-(2.6); } t \not\in \mathcal{T} \\ & Q_i = \min(A_i W_i^-, Q_{\max}) \\ & W_i^+ = W_i^- - Q_i; \ S_i^+ = p_i W_i^+; \ B_i^+ = (1 - p_i) W_i^+; \ t \in \mathcal{T} \\ & p_i = p_i(X_i^-, t_i^-); \ p_i \in Z; \ Z = [0,1] \end{cases}
\end{align*}
\]
(4.3)
where \(\lambda > 0\) is the scalarization parameter, \(A_i \equiv A(t_i)\) is defined in equation (3.3) and \(Z\) is the admissible set. In problem (4.3), we impose the constraints that no-shorting and no-leverage are permitted (i.e. \(p_i \in Z = [0,1]\)). We also restrict the withdrawal amount to be at most \(Q_{\max}\) in order to minimize the effects of large low probability withdrawals. We solve problem (4.3) using dynamic programming, working backwards from the investment horizon \(t = T\) to \(t = 0\).

In the interval \((t_i, t_{i+1})\), we define the value function \(V(s, b, q, t)\) as
\[
\begin{align*}
V(s, b, q, t) = \max_{\hat{p}_{i+1}} & \left[ \sum_{k=i+1}^{M} Q_k - \lambda \sum_{k=i+1}^{M} (Q_k - Q_{k-1})^- \right]^2 \left| S(t) = s, B(t) = b, Q(t) = q \right]
\end{align*}
\]
(4.4)
where \(\hat{p}_{i+1} = \{p_{i+1}, \ldots, p_{M-1}\}\). For \(t \in (t_i, t_{i+1})\), there are no external cash flows or controls applied, as well as no discounting (all quantities are real). Thus the tower property gives for
\[ h < (t_{i+1} - t_i) \]

\[
V(s, b, q, t) = E \left[ V(S(t + h), B(t + h), Q(t + h), t + h) \middle| S(t) = s, B(t) = b, Q(t) = q \right] \\
\quad \quad \quad \quad \quad t \in (t_i, t_{i+1} - h). \tag{4.5}
\]

Assuming \((S_t, B_t)\) follow the processes \([2.2]-[2.6]\) and noting that \(Q(t)\) is constant in \((t_i, t_{i+1})\), Itô’s Lemma (for a jump diffusion) with \(h \to 0\) gives the PIDE for \(V(s, b, q, t)\) in the interval \((t_i, t_{i+1})\):

\[
V_t + \frac{\sigma^2 s^2}{2} V_{ss} + (\mu - \lambda_\xi \kappa) V_s - \lambda_\xi V + rbV_b + \int_{-\infty}^{\infty} V(e^y s, b, q, t)f(y) \, dy = 0. \tag{4.6}
\]

Across the rebalancing/withdrawal time \((t^-_i, t^+_i)\), the value function satisfies

\[
V(s, b, q, t^-_i) = \max_{p^i \in \mathbb{Z}} \left\{ V \left( p^i W^+, (1 - p^i) W^+, q^+_i, t^+_i \right) \right\} + q^+_i - \lambda \left( (q^+_i - q^-)^2 \right)
\]

\[
W^+ = s + b - q^+_i \\
q^+_i = \min (A_i(s + b), Q_{\text{max}}) \tag{4.7}
\]

Equations \([4.7]\) can be simplified for implementation purposes. Define

\[
q^+_i (W^-) = \min (A_i W^-, Q_{\text{max}}) \\
W^+ (W^-) = W^- - q^+_i (W^-) \\
p^*_i (W^-) = \arg \max_{p^i \in \mathbb{Z}} V \left( p^i W^+ (W^-), (1 - p^i) W^+ (W^-), q^+_i (W^-), t^+_i \right) \\
\hat{V}_i (W^-) = V \left( p^*_i (W^-) W^+ (W^-), (1 - p^*_i (W^-)) W^+ (W^-), q^+_i (W^-), t^+_i \right) \tag{4.8}
\]

so that across the rebalancing time \((t^-_i, t^+_i)\) we have

\[
V(s, b, q, t^-_i) = \hat{V}_i (W^-) + q^+_i (W^-) - \lambda \left( (q^+_i (W^-) - q)^2 \right). \tag{4.9}
\]

From equation \([4.9]\) it can be seen that the optimal rebalancing fraction \(p_i \left( s, b, q, t^-_i \right) = p^*_i (W^-, t^+_i)\)

is a function of only \(W^-\) and time. Note that this contrasts with typical glide path strategies in to and through target date funds, where the fraction invested in equities is a function of time only

\cite{Forsyth et al., 2019}.

\section{5 Numerical Method}

We use dynamic programming to solve the optimization problem \([4.3]\) on the computational domain

\[
\Omega = (s, b, q, t) \in [s_{\text{min}}, s_{\text{max}}] \times [0, b_{\text{max}}] \times [0, q_{\text{max}}] \times [0, T]. \quad \text{At} \quad t = T \quad \text{we have}
\]

\[
V(s, b, q, T^+) = 0; \quad (s, b, q, T) \in \Omega. \tag{5.1}
\]

We use equation \([4.7]\) to advance the solution (backwards in time) from \(t^+_i \to t^-_i\). Then we use equation \([4.6]\) to advance the solution (backwards in time) from \(t^-_i \to t^+_{i-1}\).

We discretize the intervals \([0, b_{\text{max}}]\) and \([0, q_{\text{max}}]\) using an unequally spaced grid having \(n_b \times n_q\) nodes. Setting \(q_{\text{max}} = Q_{\text{max}}\), then no boundary conditions are required at \(q = 0, q_{\text{max}}\). No boundary conditions are required at \(b = 0\), and we artificially set interest payments to zero at \(b = b_{\text{max}}\).
use the Fourier-based method described in Forsyth and Labahn (2019) with an equally spaced $x = \log s$ grid in the $s$ direction with $n_x$ nodes. To avoid wrap-around pollution, we use a buffer zone where we extend the solution by constant values for $s < s_{\min}$, $s > s_{\max}$, as described in Forsyth and Labahn (2019). The local maximization problem in equation (4.7) is solved using exhaustive search by discretizing the admissible range of $p$ using an equally spaced grid with $n_u$ nodes. Linear interpolation is used to evaluate $V(\cdot)$ at off-grid points. For further details, see Forsyth and Labahn (2019) and Dang and Forsyth (2014).

In order to determine

$$E \left[ \sum_{i=0}^{i=M} Q_i \right] ; \quad E \left[ \sum_{i=1}^{i=M} ((Q_i - Q_{i-1})^-)^2 \right]$$

(5.2)

separately, we solve an additional PIDE for $U(s,b,q,t)$ defined by

$$U(s,b,q,t) = E[p_i^r, \ldots, p_{M-1}^r] \left[ \sum_{k=i+1}^{M} Q_k \left| S(t) = s, B(t) = b, Q(t) = q \right\} \right] ; \quad t \in (t_i^-, t_{i+1}^-).$$

(5.3)

where $p_i^r$ are the optimal controls determined from equation (4.9). Across the rebalancing times $(t_i^-, t_i^+)$ we have

$$U(s, b, q, t^-) = \left\{ U(p_i^r W^+, (1 - p_i^r) W^+, q_i^+ (W^-), t_i^+) \right\} + q_i^+ (W^-)$$

$$W^- = (s + b); \quad W^+ = W^- - q_i^+ (W^-).$$

(5.4)

At $t = T$, we have the initial condition

$$U(s, b, q, T^+) = 0; \quad (s, b, q, T) \in \Omega.$$ 

(5.5)

From $t_i^- \to t_{i-1}^+$, we have

$$U_t + \frac{\sigma^2}{2} U_{s s} + (\mu - \lambda \xi \kappa) U_s - \lambda \xi U + rbU_b + \int_{-\infty}^{+\infty} U(e^y s, b, q, t)f(y) dy = 0.$$ 

(5.6)

Given an initial portfolio value $W_0$ along with $V(0, W_0, 0, 0^-)$ and $U(0, W_0, 0, 0^-)$, it is straightforward to determine the quantities of interest in equation (5.2).

### 6 Data and Parameters

The data we use was obtained from Dimensional Returns 2.0 under licence from Dimensional Fund Advisors Canada. In particular, we use the Center for Research in Security Prices Deciles (1-10) index. This is a total return value-weighted index of US stocks. We also use one month Treasury bill returns for the risk-free asset. Both the equity returns and the Treasury bill returns are in nominal terms, so we adjust them for inflation by using the US CPI index. All of the data used was at the monthly frequency, with a sample period of 1926:1 to 2016:12.

To avoid known problems with other approaches, we use the method described in Dang and Forsyth (2016) and Forsyth and Vetzal (2017) based on the thresholding technique of Mancini (2009) and Cont and Mancini (2011). A tuning parameter $\alpha$ is required which, in intuitive terms, identifies a jump if the absolute value of the detrended log return is more than $\alpha \sigma \sqrt{\Delta t}$, where $\sigma$ is the annualized diffusive volatility and $\Delta t$ is the time interval (measured in years) between
Table 6.1: Annualized parameter estimates for jump diffusion model (see equation (2.2)) of the real CRSP value-weighted equity index and mean annualized real rate of return for 1-month US Treasury bills (log\[B(T)/B(0)]/T). Also reported are the annualized volatility of the real rate of return for Treasury bills and the correlation between real returns for the Treasury bill and value-weighted equity indexes. Sample period 1926:1 to 2016:12. Data obtained from Dimensional Returns 2.0 under licence from Dimensional Fund Advisors Canada.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu)</td>
<td>0.08753</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>0.14801</td>
</tr>
<tr>
<td>(\lambda)</td>
<td>0.34065</td>
</tr>
<tr>
<td>(p_u)</td>
<td>0.25806</td>
</tr>
<tr>
<td>(\eta_1)</td>
<td>4.67877</td>
</tr>
<tr>
<td>(\eta_2)</td>
<td>5.60389</td>
</tr>
</tbody>
</table>

Table 7.1: Convergence Test

We begin by conducting a convergence test of our numerical method. We consider the scenario documented in Table 7.1. Monetary units in the table are in thousands of dollars, so that the initial portfolio value \(W_0 = 1,000\) implies an initial wealth of $1 million. The investor withdraws cash immediately and at the end of each of the next 30 years, and is not permitted to withdraw more than $100,000 per year. The portfolio is rebalanced annually, at the cash withdrawal times. As indicated in Table 7.1, the market parameters used are from Table 6.1. The summary statistics provided here are based on the average expected withdrawal \(\bar{Q}\) and the average withdrawal variability \(\bar{V}_q\).

These two quantities are defined as follows:

Average expected withdrawal \(\bar{Q} = \frac{1}{M+1} E \left[ \sum_{i=0}^{i=M} Q_i \right] \)

Average withdrawal variability \(\bar{V}_q = \sqrt{\frac{1}{M} E \left[ \sum_{i=1}^{i=M} \left( (Q_i - Q_{i-1})^{-} \right)^2 \right]}

\((Q_i - Q_{i-1})^{-} = \min(Q_i - Q_{i-1}, 0)\).

We take the square root in equation (7.1) so that \(\bar{V}_q\) and \(\bar{Q}\) have the same units.

We solve the optimization problem (4.3) by solving the PIDEs (4.4)-(4.7) and equations (5.3)-(5.6). We discretize the problem in the \((s, b, q)\) directions using \(s_{\text{min}} = 0.04, s_{\text{max}} = b_{\text{max}} = 10^4\),
Table 7.1: Input data for examples. Monetary units: thousands of dollars.

<table>
<thead>
<tr>
<th>Investment horizon $T$ (years)</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity market index CRSP value-weighted index (real)</td>
<td></td>
</tr>
<tr>
<td>Risk-free asset index 1-month Treasury bill index (real)</td>
<td></td>
</tr>
<tr>
<td>Initial portfolio value $W_0$</td>
<td>1000</td>
</tr>
<tr>
<td>Cash withdrawal times $t = 0, 1, \ldots, 30$</td>
<td></td>
</tr>
<tr>
<td>$Q_{\max}$</td>
<td>100</td>
</tr>
<tr>
<td>Rebalancing interval (years)</td>
<td>1</td>
</tr>
<tr>
<td>Market parameters</td>
<td>See Table 6.1</td>
</tr>
</tbody>
</table>

Table 7.2: Convergence test of the solution of equations (4.4) - (4.7) and equations (5.3) - (5.6) used to compute the optimal asset allocation. Monte Carlo results based on 640,000 simulated paths in the synthetic market, with controls computed from the PIDE at the indicated grid size. Input data provided in Table 7.1. Monetary units: thousands of dollars.

| Grid $(n_x, n_b, n_q)$ | PIDE | | Monte Carlo |
|------------------------|------|---|-----------------|---|---|
|                        | Value Function | $\bar{Q}$ | $\bar{V}_q$ | $\bar{Q}$ | $\bar{V}_q$ |
|                        | $\lambda = 2.0$ |      |     |      |     |
| $256 \times 153 \times 157$ | 1257.23 | 61.4008 | 3.2284 | 61.3628 | 3.1231 |
| $512 \times 305 \times 313$ | 1290.10 | 61.2590 | 3.1338 | 61.2232 | 3.1050 |
| $1024 \times 609 \times 625$ | 1297.95 | 61.2060 | 3.1094 | 61.1830 | 3.1006 |
| $\lambda = 1.0$ |      |     |      |      |     |
| $256 \times 153 \times 157$ | 1647.22 | 67.8729 | 3.8388 | 67.8621 | 3.7580 |
| $512 \times 305 \times 313$ | 1661.53 | 67.8883 | 3.7803 | 67.8786 | 3.7559 |
| $1024 \times 609 \times 625$ | 1665.24 | 67.8931 | 3.7651 | 67.8808 | 3.7552 |

and $Q_{\max} = Q_{\max}$. Increasing $s_{\max}, b_{\max}$ (by a factor of ten) and decreasing $s_{\min}$ (dividing by ten) resulted in no change to the solution to six figures. We use the Fourier method described in Forsyth and Labahn [2019], which requires that $s_{\min} > 0$. There is no timestepping error for the Fourier method between rebalancing times. Table 7.2 provides a convergence study in which we compute various quantities of interest for a sequence of grid sizes. The value function is $V(0, W_0, 0^-, \cdot)$ where $V(\cdot)$ is defined in equation (4.4). We compute and store the optimal asset allocations from the PIDE solver, and then carry out Monte Carlo simulations to verify the solution. The rate of convergence appears to be between first and second order for the PIDE solution. Results reported in the remainder of the paper use the control from the finest PIDE grid.

8 Synthetic Market Examples

We now explore some examples based on the input data given in Table 7.1. This section presents results in the synthetic market. Recall that this means that we compute the control using the parameters from Table 6.1 and then assess performance by Monte Carlo simulation assuming exactly the same parameters. More specifically, we use the following steps:
Table 8.1: Results for ARVA spending rule when the portfolio is rebalanced at each rebalancing date to a fixed risky asset weight \( p \). Monetary units: thousands of dollars. Results computed using Monte Carlo simulations in the synthetic market with 640,000 paths. Input data provided in Table 7.1.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \bar{Q} )</th>
<th>( \bar{V}_q )</th>
<th>( p )</th>
<th>( \bar{Q} )</th>
<th>( \bar{V}_q )</th>
<th>( p )</th>
<th>( \bar{Q} )</th>
<th>( \bar{V}_q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>33.0</td>
<td>1.11</td>
<td>0.45</td>
<td>55.7</td>
<td>3.45</td>
<td>0.75</td>
<td>66.1</td>
<td>4.56</td>
</tr>
<tr>
<td>0.20</td>
<td>41.9</td>
<td>1.88</td>
<td>0.50</td>
<td>58.0</td>
<td>3.68</td>
<td>0.80</td>
<td>67.0</td>
<td>4.73</td>
</tr>
<tr>
<td>0.25</td>
<td>44.6</td>
<td>2.21</td>
<td>0.55</td>
<td>60.1</td>
<td>3.88</td>
<td>0.85</td>
<td>67.9</td>
<td>4.90</td>
</tr>
<tr>
<td>0.30</td>
<td>47.4</td>
<td>2.56</td>
<td>0.60</td>
<td>62.0</td>
<td>4.06</td>
<td>0.90</td>
<td>68.5</td>
<td>5.07</td>
</tr>
<tr>
<td>0.35</td>
<td>50.3</td>
<td>2.90</td>
<td>0.65</td>
<td>63.6</td>
<td>4.24</td>
<td>0.95</td>
<td>69.1</td>
<td>5.24</td>
</tr>
<tr>
<td>0.40</td>
<td>53.1</td>
<td>3.88</td>
<td>0.70</td>
<td>64.9</td>
<td>4.40</td>
<td>1.00</td>
<td>69.6</td>
<td>5.42</td>
</tr>
</tbody>
</table>

1. We solve problem (4.3) to determine the optimal asset allocation strategy. This assumes that the value of the risky equity market and risk-free bond indexes evolve according to equations (2.2) and (2.6) respectively, with the parameters provided in Table 6.1. We store the generated optimal controls.

2. We generate Monte Carlo simulated paths of the two indexes over the investment horizon, calculating values at each rebalancing date according to processes (2.2) and (2.6) with the parameters in Table 6.1.

3. We then apply the stored controls to each path, calculating statistics such as the average withdrawal and withdrawal variability. We then compute averages and percentiles of the relevant path statistics across the simulated paths.

For purposes of comparison, we also evaluate the performance of fixed weight strategies. In these cases Step 1 above is skipped, and in Step 3 we just rebalance to constant specified portfolio weights.

As a first case, we consider the ARVA spending rule with a fixed (constant) equity allocation at each rebalancing time. Table 8.1 shows the results for this constant weight asset allocation strategy. As the fixed equity weight is increased, both the average expected withdrawal \( \bar{Q} \) and the average withdrawal variability \( \bar{V}_q \) rise monotonically. Note that even for \( p = 0 \) (all money invested in bonds), \( \bar{V}_q > 0 \). This is because the ARVA spending rule results in declining payments over time, due to the front end loading of the mortality boost as indicated in Figure 3.1(b).

We next solve problem (4.3) to determine the optimal strategy according to our criteria. Optimal asset allocation results are shown for various values of the scalarization parameter \( \lambda \) in Table 8.2. Higher values of \( \lambda \) correspond to higher risk-aversion since more weight is placed on the risk term in the objective function. As the table shows, reducing \( \lambda \) leads to monotonically increasing reward \( \bar{Q} \) and risk \( \bar{V}_q \).

Table 8.2 shows that the average expected withdrawal \( \bar{Q} \) is 67.9 when \( \lambda = 1.0 \). From Table 8.1 the same average expected withdrawal is obtained for a fixed weight strategy with \( p = 0.85 \). However, the fixed weight strategy has higher average withdrawal variability \( \bar{V}_q \) of 4.73 compared to 3.76 for the optimal asset allocation. An overall indication of the general pattern of withdrawals over time is provided in Figure 8.1 which shows the 5th, 50th, and 95th percentiles of the distribution of withdrawals for the fixed weight strategy in panel (a) and the optimal strategy in panel (b). At a broad level, the two cases appear to be quite similar. The initial withdrawal is around $40,000. The 95th percentile of withdrawals rises rapidly over about the first 5 years to the maximum specified amount of $100,000 and remains there throughout the horizon for each case. Conversely, the 5th
<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\bar{Q}$</th>
<th>$\bar{V}_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>36.4</td>
<td>1.34</td>
</tr>
<tr>
<td>4.0</td>
<td>39.1</td>
<td>1.54</td>
</tr>
<tr>
<td>3.0</td>
<td>53.1</td>
<td>2.55</td>
</tr>
<tr>
<td>2.0</td>
<td>61.2</td>
<td>3.10</td>
</tr>
<tr>
<td>1.0</td>
<td>67.9</td>
<td>3.76</td>
</tr>
<tr>
<td>0.5</td>
<td>70.2</td>
<td>4.29</td>
</tr>
</tbody>
</table>

Table 8.2: Results for ARVA spending rule when the portfolio is rebalanced according to the optimal asset allocation, defined as the solution to problem (4.3). Monetary units: thousands of dollars. Results computed using Monte Carlo simulations in the synthetic market with 640,000 paths. Input data provided in Table 7.1.

Figure 8.1: Percentiles of withdrawal amounts over time for fixed weight strategy with $p = 0.85$ and optimal strategy with $\lambda = 1.0$. Results computed using Monte Carlo simulations in the synthetic market with 640,000 paths. Input data provided in Table 7.1.

percentile of withdrawals quickly drops below $30,000 and remains there over most of the horizon, before tailing off a bit further in the final few years. The median withdrawal rises more slowly than the 95th percentile, but does attain the allowed maximum in each case. This happens slightly faster under the optimal strategy compared to the fixed weight strategy. The median withdrawal remains constant at the maximum amount throughout the horizon for the fixed weight strategy, but it drops off during the last 3 years of the horizon for the optimal strategy.

Figure 8.2 depicts the 5th, 50th, and 95th percentiles of the fraction of the portfolio allocated to equities over time for the optimal strategy with $\lambda = 1$. Keep in mind that this strategy produces the same average expected withdrawal as a fixed weight strategy that annually rebalances to having 85% invested in the risky equity market index. The optimal strategy starts out with all funds in the risky asset. In the 5th percentile case, the portion of the portfolio in the risky asset drops very quickly, down to about 30% after 5 years and reaching zero after about 15 years. The median fraction has $p = 1$ for the first 7 years. This declines to around 10-15% for years 20-25, and thereafter increases back to about 30% at the end of the horizon. In the 95th percentile case, the portfolio is entirely invested in the risky asset for almost 20 years, and then falls off to being about
30% at risk at the end of the horizon. Overall, an investor who follows the optimal strategy will need
to initially put all of his funds in the risky asset, but he will likely to be able to reduce his equity
market risk exposure substantially over time. Reaching the same average expected withdrawal
with a fixed weight strategy requires keeping a consistently high equity weighting throughout the
horizon. Of course, this leads to higher withdrawal variability, as measured by $\bar{V}_q$.

![Figure 8.2: Percentiles of control $p$ over time for optimal strategy with $\lambda = 1.0$. Results computed
using Monte Carlo simulations in the synthetic market with 640,000 paths. Input data provided in
Table 7.1.](image)

Similar results are shown for $\lambda = 2.0$ in Figures 8.3 and 8.4. From Table 8.2 the average
expected withdrawal in this case is $\bar{Q} = 61.2$. The constant weight strategy which gives the same
value of $\bar{Q}$ is $p^* = .58$, found by interpolating the results reported in Table 8.1. This constant
weight strategy has $\bar{V}_q = 3.99$, compared to the optimal strategy which has $\bar{V}_q = 3.10$. Unlike in
Figure 8.1 above, here (Figure 8.3) the median withdrawal amount rises over the first several years
and subsequently falls, but it never comes close to the maximum allowed withdrawal. Comparing
Figures 8.2 and 8.4 we see that in the more risk-averse case ($\lambda = 2$), the fraction optimally put at
risk declines from the initial value of $p = 1$ much earlier.

Table 8.3 shows some statistics about the distributions of final wealth for the optimal strategies
with $\lambda = \{1.0, 2.0\}$ and the constant weight strategies which generate the same average expected
withdrawals $\bar{Q}$. The final wealth is at $t = 30$ years. Recall that there is enough cash remaining to
fund 3 years of payments (after the payment at $t = 30$). As a result, this takes the retiree through
to his 99th birthday. The final wealth values at the 5th percentiles are comparable with the fixed
weight strategies with the same $\bar{Q}$. However, the fixed weight strategies have much higher median
and 95th percentile terminal portfolio values, which is to be expected due to the higher average
allocation to equities. These large values of final wealth are due to low probability very favourable
investment results, coupled with the cap on withdrawals of $100,000 per year.

As another point of comparison between the fixed weight strategies and the optimal strategy,
Table 8.4 shows the time averaged median fraction invested in the risky asset for the cases of
$\lambda = \{1.0, 2.0\}$ compared with the fixed weight strategies which give the same $\bar{Q}$. When $\lambda = 1.0$,
the time averaged median value of $p$ is .57, versus the fixed weight of .85. Similarly, for the case
with $\lambda = 2.0$, the time averaged median of $p$ is .40, compared to the fixed weight of $p = .58$.

From Figure 8.4 we can see that the optimal strategy ($\lambda = 2$) has a median fraction in stocks
of 1.0 during the early years of retirement, which then drops rapidly. This is contrary to the
usual advice given to retirees. However, from Table 8.4 we can see that time averaged fraction in
stocks for this strategy is 0.40. In order to generate the same average expected withdrawal, a fixed
weight strategy requires $p = 0.58$, with considerably greater withdrawal variability. In other words,
although the optimal strategy has a maximum equity fraction larger than the fixed weight strategy
with the same expected withdrawal, it is the time averaged equity fraction which contributes to
overall risk. We discuss this in greater detail in the next section.

8.1 Analysis of the Objective Function

We now provide a heuristic analysis as to why the optimal strategy for objective function (4.3)
tends to reduce the average amount in the risky asset. Consider the risk term in problem (4.3):

$$E \left[ \sum_{i=1}^{i=M} ((Q_i - Q_{i-1})^-)^2 \right] = \sum_{i=1}^{i=M} E \left[ ((Q_i - Q_{i-1})^-)^2 \right].$$

(8.1)
Figure 8.4: Percentiles of control $p$ over time for optimal strategy with $\lambda = 2.0$. Results computed using Monte Carlo simulations in the synthetic market with 640,000 paths. Input data provided in Table 7.1.

<table>
<thead>
<tr>
<th>Time (years)</th>
<th>Fraction in stocks</th>
<th>Median</th>
<th>5th percentile</th>
<th>95th percentile</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>0.6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>0.7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>0.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>0.9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>1.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8.4: Results for optimal strategy as in problem (4.3) and fixed weight strategies having the same expected average withdrawal $\bar{Q}$. Monetary units: thousands of dollars. Results computed using Monte Carlo simulations in the synthetic market with 640,000 paths. Input data provided in Table 7.1.

<table>
<thead>
<tr>
<th>Optimal</th>
<th>Fixed weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$\bar{Q}$</td>
</tr>
<tr>
<td>1.0</td>
<td>67.9</td>
</tr>
<tr>
<td>2.0</td>
<td>61.2</td>
</tr>
</tbody>
</table>

For ease of exposition, assume that $t_i - t_{i-1} = \Delta t$ is small. If $t_i \ll T, T_x^*(t_i)$, then we can also assume that $A(t_i) = O(\Delta t); A(t_i) = A(t_{i-1}) + O(\Delta t)^2$. Examining one term of the sum in equation (8.1) gives

$$
\begin{align*}
((Q_i - Q_{i-1})^{-})^2 & \leq ((Q_i - Q_{i-1}))^2 \\
& = A_i^2 \left(W_i^- - W_{i-1}^- + O(\Delta t)^2\right)^2 \\
& \simeq A_i^2 \left(W_i^- - \frac{W_{i-1}^+}{1 - A_i} + O(\Delta t)^2\right)^2 \\
& = A_i^2 \left(\Delta W_{i-1}^+ + O(\Delta t)\right)^2, \quad (8.2)
\end{align*}
$$

where $\Delta W_{i-1}^+ = \left(W_i^- - W_{i-1}^+\right)$. If we continuously rebalance to a fixed weight $p_{i-1}$ for $t \in (t_{i-1}, t_i)$, then from equations (2.5) and (2.6), and Itô’s Lemma for jump processes, we obtain

$$
\frac{dW}{W} = \left[p_{i-1}(\mu - r) + r\right] dt - \lambda \kappa dt + p_{i-1} \sigma dZ + p_{i-1}(\xi - 1) d\bar{Q}. \quad (8.3)
$$
If we assume that $\Delta t = t_i - t_{i-1} \simeq dt$ and that $dW \simeq (W_i^+ - W_{i-1}^+) \equiv \Delta W_{i-1}^+$, then equation (8.3) becomes
\[
\frac{\Delta W_{i-1}^+}{W_{i-1}^+} = [p_{i-1}(\mu - r) + r] \, dt - \lambda \xi \, dt + p_{i-1}\sigma \, dZ + p_{i-1}(\xi - 1) \, d\mathbb{Q}.
\] (8.4)

Substituting equation (8.4) into equation (8.2) and taking expectations gives
\[
E \left[ (Q_i - Q_{i-1}) \right]^2 \leq E \left[ (Q_i - Q_{i-1}) \right]^2
= \left( W_{i-1}^+ \right)^2 \sum_{i=1}^{i=M} A_i^2 \left[ p_{i-1}^2 \sigma^2 \, dt + p_{i-1}^2 E \left[ (\xi - 1)^2 \right] \lambda \xi \, dt \right] + o(dt)
\simeq \left( W_{i-1}^+ \right)^2 \sum_{i=1}^{i=M} A_i^2 \left[ p_{i-1}^2 \sigma^2 \Delta t + p_{i-1}^2 E \left[ (\xi - 1)^2 \right] \lambda \xi \, \Delta t \right] + o(\Delta t).
\] (8.5)

Substituting equation (8.5) into equation (8.1) and ignoring terms of $o(\Delta t)$ gives
\[
E \left[ \sum_{i=1}^{i=M} (Q_i - Q_{i-1})^2 \right] \leq \sum_{i=1}^{i=M} p_{i-1}^2 \left( W_{i-1}^+ \right)^2 A_i^2 \left[ \sigma^2 + E \left[ (\xi - 1)^2 \right] \lambda \xi \right] \Delta t.
\] (8.6)

From equation (8.6) we can see that minimizing the risk term (8.1) minimizes the weighted average fraction invested in the risky asset $(p_{i-1})$, which is consistent with the numerical results.

As well, we can see that when $W_{i-1}^+$ becomes small (for large times) the weight multiplying the risky asset fraction in equation (8.6) becomes small, hence maximizing problem (4.3) would focus on maximizing the expected total withdrawals, which would tend to increase the fraction invested in the risky asset at later times. This effect can be seen in Figures 8.2 and 8.4. The non-smooth percentile curves in these plots for larger times arise because with little time remaining the control has a small influence on maximizing the expected total withdrawals.

Finally, note that in the limit as $t_i - t_{i-1} \to 0$, the risk term on the right hand side of (8.6) becomes a weighted portfolio quadratic variation. This has previously been suggested as a standalone risk measure in sources such as Brugiere (1996), Forsyth et al. (2012), and van Staden et al. (2019).

9 Bootstrap Tests

The results reported above have all been in the synthetic market, following the 3 step procedure outlined at the start of Section 8. We now replace the second step involving Monte Carlo simulation by bootstrap resampling of the historical data to generate simulated paths of the values of the risk and risk-free assets, in the absence of control. In this historical market, the other two steps remain as before. Although we still compute the optimal asset allocation strategy by solving problem (4.3), assuming as before that $S_t$ and $B_t$ follow processes (2.2) and (2.6) respectively, the performance tests themselves make no assumptions regarding the stochastic processes followed by the value of the equity and bond market indexes.

To construct a single bootstrap resampled path for asset returns, we use the stationary block bootstrap to account for possible serial dependence (see, e.g. Politis and White (2004) Patton et al. (2009)). We start at a random month in the 1926:1 to 2016:12 sample period. We draw a block of data starting in that month (we simultaneously sample both the bond and the stock indexes). The length of the block is determined by drawing a random value from a geometric distribution having mean (i.e. expected blocksize) $\hat{b}$. We continue to draw blocks of data in this way and paste them together until we have a path that covers the entire horizon of $T = 30$ years. This procedure is repeated many times to generate a large number of resampled paths. Note that we draw the blocks...
of data with replacement, so it is possible for us to use a historical period more than once in a single path. We wrap the data around so that if the size of a particular block extends past the end of the sample period in 2016:12, values for the remaining duration of that block are taken from the start of the sample period, beginning in 1926:1. See Forsyth and Vetzal (2019) for a detailed description of the bootstrap algorithm.

In principle, it is possible to estimate the optimal expected blocksize \( \hat{b} \). However, if we apply the algorithm described in Patton et al. (2009) to our data, we find very different estimates for the two indexes: the value for the equity market index is about 3.5 months, while the value for the bond market index is around 57 months. This poses a problem since we sample simultaneously from both indexes. Consequently, we give results for several expected blocksizes.

One final point should be noted about our procedures. In our bootstrap tests, the bond and stock returns are computed using the actual historical returns. The ARVA annuity factor \( (3.3) \) is determined using the long term average real T-bill rate (recall that this is \( r = .0048 \)). Since this rate is very low, this is a conservative approach, which essentially means that fluctuations in withdrawals are primarily driven by the actual observed asset returns, instead of projections about future real interest rates. Alternatively, it would be possible to use the most recently observed historical short rate in the ARVA annuity factor computation. However, this can cause volatility in the withdrawals solely due to the bootstrapping procedure, even when the portfolio returns are not volatile.

Table 9.1 shows the results. We also provide comparable results for the fixed weight strategy which gives the same value of \( \bar{Q} \) in the synthetic market. For any given expected blocksize, the optimal strategy has a much smaller average allocation to the risky asset, while having a very similar average total withdrawal. We also observe that the results in Table 9.1 are relatively insensitive to expected blocksize, which suggests that the strategies are quite robust. Comparing Table 9.1 with the earlier Table 8.4 from the synthetic market, we observe that the results for the expected average withdrawal \( \bar{Q} \) are quite similar. For example, with \( \lambda = 1 \) we had \( \bar{Q} = 67.9 \) and \( \bar{V}_q = 3.76 \) in the synthetic market (Table 8.4) for the optimal strategy. The corresponding bootstrap resampled values for \( \bar{Q} \) in Table 9.1 range from 67.6 to 69.3 as the expected blocksize increases from 6 months to 5 years, and the corresponding resampled values for \( \bar{V}_q \) range from 3.81 to 3.97. Overall, for the optimal strategy the average expected withdrawal \( \bar{Q} \) in the historical market is quite close to that for the idealized synthetic market. The average withdrawal variability \( \bar{V}_q \) is slightly higher in the historical market, but this is not surprising since the resampled paths will have stochastic interest rates and randomly changing volatility, neither of which are features of our synthetic market. The historical market results for the fixed weight strategies in Table 9.1 are also quite close to their synthetic market counterparts in Table 8.4. For example, with \( p = .85 \) the average expected withdrawal ranges from 68.0 to 69.4 in the historical market, compared to 67.9 in the synthetic market. Using this fixed weight gives average withdrawal variability that ranges from 4.76 to 5.12 in the historical market, versus 4.90 in the synthetic market.

Figure 9.1 shows the percentiles of withdrawals over time for the fixed weight strategy with \( p = .85 \) and the optimal strategy with \( \lambda = 1 \) in the historical market with an expected blocksize of two years. The two panels here are quite similar to the corresponding synthetic market plots from Figure 8.1, another signal that the synthetic market strategy is robust when tested on historical market data. Figure 9.2 shows the percentiles of the fraction of the investment portfolio allocated to equities over time, based on bootstrap resampling with an expected blocksize of two years. These results are also quite similar to the corresponding synthetic market results shown in Figure 8.2.
An ARVA spending rule results in variable withdrawals, which eliminates the possibility of ruin over the specified horizon. The risk of ruin is effectively replaced by the risk of withdrawal variability. The main positive feature of an ARVA rule is the fact that withdrawals reflect the investment experience. In addition, a mortality boost can be used to front end load the withdrawals. On the other hand, compared to an annuity, there is some possibility of very low withdrawals later on in life. Combining an ARVA rule with investing in a portfolio of risky assets and risk-free assets leads to a higher average expected withdrawal compared to a fairly priced annuity. Under an ARVA rule the investor retains full control over their portfolio, unlike for an annuity.

We compared two possible approaches to managing the investment portfolio under an ARVA spending rule: a fixed weight strategy, and a strategy based on optimal control. The optimal control strategy minimized a downside measure of withdrawal variability, for a given expected average withdrawal. For the same expected average withdrawal, the optimal strategy has smaller withdrawal variability, smaller average investment over time in the risky asset, and similar final wealth at the 5th percentile, compared to a fixed weight strategy. However, the fixed weight strategy has a higher median terminal wealth compared to the optimal strategy. This is to be expected due to the higher average weight in risky assets (for the same expected average withdrawal) compared to the optimal strategy and the cap imposed on withdrawals. These results hold for both a parametric model based on historical time series, as well as bootstrap resampled backtests.

The synthetic market results (parametric model) and the bootstrapped historical market results

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Table 9.1: Results for optimal strategy as in problem 4.3 and fixed weight strategy with the same expected average withdrawal $\bar{Q}$ in the synthetic market. Monetary units: thousands of dollars. Results computed in the historical market with 100,000 bootstrap resampled paths. Input data provided in Table 7.1. Controls computed in the synthetic market using parameters from Table 6.1 and stored, then applied to resampled historical data.
Figure 9.1: Percentiles of withdrawal amounts over time for fixed weight strategy with $p = .85$ and optimal strategy with $\lambda = 1.0$. Results computed in the historical market with 100,000 bootstrap resampled paths and expected blocksize of 2 years. Input data provided in Table 7.1. Control computed in the synthetic market using parameters from Table 6.1 and stored, then applied to resampled historical data.

are very similar for either the optimal strategy or the fixed weight strategies. This suggests that an ARVA spending rule which adapts withdrawals to investment experience results in a very robust strategy, i.e. insensitive to market parameter misspecification.

Overall, a combination of an ARVA spending rule and an optimal control approach to reduce withdrawal variability, result in a decumulation strategy which has a high probability of achieving desirable outcomes. This does, however, come at the cost of high median equity fractions for short periods of time. Nevertheless, the time averaged (median) equity fraction is much smaller than the equivalent constant weight strategy, which we argue is the appropriate risk measure in this case.

A possible avenue for future research is to incorporate ARVA spending rules into the decumulation phase of a full life cycle analysis (accumulation and decumulation phases) of DC plans.
Figure 9.2: Percentiles of control $p$ over time for optimal strategy with $\lambda = 1.0$. Results computed in the historical market with 100,000 bootstrap resampled paths and expected blocksize of 2 years. Input data provided in Table 7.1. Control computed in the synthetic market using parameters from Table 6.1 and stored, then applied to resampled historical data.

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Declarations of Interest

The authors have no conflicts of interest to declare.

References


