Non-Commutative Gröbner Bases in Poincaré-Birkhoff-Witt Extensions

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Abstract. Commutative Gröbner Bases are a well established technique with many applications, including polynomial solving and constructive approaches to commutative algebra and algebraic geometry. Noncommutative Gröbner Bases are a focus of much recent research activity. For example, combining invariant theory and elimination theory, or elimination in moving frames of partial differential operators invariant under an equivalence group, requires the use of noncommutative Gröbner bases. This paper presents theory and algorithms for noncommutative Gröbner bases in Poincaré-Birkhoff-Witt extensions. These extension rings generalize the previous domains over which noncommutative Gröbner bases have been applied. Our approach to noncommutative Gröbner bases differs from previous work which assumes that the coefficients are from a field or commutative ring. In applications such as Cartan's method of moving frames, this is not the case, and the theory that we present can be applied.

1 Introduction

Since the mid-1980's, noncommutative Gröbner bases have developed as an active research area in Computer Algebra, with many applications. See, for example, Chyzak and Salvy [6] for Ore algebras, Green [10] for path algebras, Kandri-Rody and Weispfenning [12] for algebras of solvable type, Mora [17] for free algebras over fields. Generally speaking, there are two streams in these studies. One is free algebras, which preserve properties of semigroups. The other is algebras of solvable type (including rings of differential operators) which preserve Dickson's lemma. In most of the above papers, the authors assume that the coefficients are from a field or commutative ring, and that these commute with the indeterminates (although the indeterminates may not commute with each other).

There are many interesting and useful rings which the above papers do not address. Examples include some kinds of homogenous partial differential equations with non-constant coefficients (see Adams et al. [2]). The method of choosing coordinates which are invariant under a given symmetry group (e.g. polar coordinates), in its most general form requires the introduction of a moving frame of non-commuting partial differential operators (Cartan's famous equivalence method). Elimination theories for such systems, by necessity, require a non-commutative Gröbner Basis method of the type presented in this paper (see Lisle & Reid [14], and Mansfield [15]). Another example is the skew enveloping algebra $R\#U(L)$ (see McConnell and Robson [16]), which is important in associative ring theory. This motivates us to define Gröbner bases in Poincaré-Birkhoff-Witt (PBW) extensions. We prove the left division rule and many fundamental properties of such Gröbner bases, and give an algorithm to construct them. As a special case, we consider the graded lexicographic ordering, and reduce computing Gröbner bases of PBW extensions to the commutative case. Finally we apply this theory to the moving frame approach in Section 4.

Differential elimination algorithms have been effective in pre-processing and simplifying systems for the subsequent application of the methods of scientific computation. Such methods include numerical integration techniques and symmetry techniques.

A popular new research area is the area of Geometric Integration [18]. The general philosophy of that area is to include as many qualitative features of the physical system being studied as possible in the tools used to study the system. For example numerical integrators, which are invariant under the symplectic group (*geometric integrators*) are used to numerically solve Hamilton's equations, which are also symplectically invariant. This paper represents progress in this direction for differential elimination algorithms by, for example, enabling the differentiations and eliminations of such algorithms to be executed in a moving frame, invariant under a group admitted by the given problem. Our extension of such algorithms

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to coefficients which do not come from a field (and are not necessarily invertible) are potentially relevant for matrix formulations arising in non-commutative field theories. In such instances it is helpful to be able to perform such calculations, as physicists would, in the non-commutative matrix formalism, instead of breaking it down to components and using commutative differential algebra, as is the current practice. This is an area which we are investigating.

2 Poincaré-Birkhoff-Witt Extensions

We first introduce the definition of PBW extensions used in this paper, which is given by Bell and Goodearl [5]. This definition leads to a unified treatment of many polynomial rings currently studied in associative ring theory and Computer Algebra.

Definition 1. *Let R and T be two associative rings with $R \subseteq T$. T is called a (finite) PBW extension of R if there exist $x_1, x_2, \cdots, x_n \in T$ such that*
(1) the monomials $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ form a basis for T as a free left R-module, where $i_1, i_2, \cdots, i_n \in \mathbb{N}$;
(2) $x_i r - r x_i := [x_i, r] \in R$ for each $i = 1, \cdots, n$ and any $r \in R$;
(3) $x_i x_j - x_j x_i := [x_i, x_j] \in R + Rx_1 + \cdots + Rx_n$ for all $i, j = 1, \cdots, n$.
We write $T = R\langle x_1, \cdots, x_n \rangle$.

It is clear that every element of $R\langle x_1, \cdots, x_n\rangle$ can be uniquely represented as a finite sum $\sum r_i x_1^{i_1} \cdots x_n^{i_n}$, where $r_i \in R, i_1, \cdots, i_n \in \mathbb{N}$. We call the monomials of the form $x_1^{i_1} \cdots x_n^{i_n}$ *standard monomials.* Note also that any non-standard monomial $x_{j_1}^{i_1} \cdots x_{j_n}^{i_n}$ can be expressed as a finite sum of standard monomials.

Remark 1. We remind the reader of the differences between PBW extensions and other similar algebraic structures. For example, comparing PBW extensions with algebras of solvable type defined in [12], we note a number of important differences. Algebras of solvable type require coefficients and indeterminates commute, whereas PBW extensions make no such requirement. For example, $R\#U(L)$ is not an algebra of solvable type in general, but is a PBW extension. On the other hand, algebras of solvable type define a "quantum" or "ordering" version of condition (3) above, which states roughly that the commutator is smaller than the product under a term ordering. In an upcoming paper we define skew-PBW extensions which include both algebras of solvable type and PBW extensions. We also suggest that reader should compare PBW extensions with free algebras and note the differences.

Example 1. There are several prototypical examples of PBW extensions:

(1) The usual multivariate polynomial rings over R, Ore algebras and PBW algebras discussed in Computer algebra. The skew enveloping algebra (or smash product) $R\#U(L)$ is an example of PBW extensions. In particular, the universal enveloping algebra $U(L)$, the n-Weyl algebra $A_n(K)$ and skew polynomial ring (derivation type) $R[x, \delta]$ also are examples of PBW extensions (see McConnell and Robson [16]).
(2) Some kinds of PDEs with non-constant coefficients. For example, let $R = K(x_1, \cdots, x_n)$ be a function field over a field K. It is well-known that all partial derivations over R form a (possibly infinite dimensional) Lie algebra under the usual brace product. In the language of PDEs, one partial differential equation is written as:

$$a_n(x_1, \cdots, x_n) \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}} + \cdots + a_0(x_1, \cdots, x_n) = 0.$$

Corresponding to the skew enveloping algebra, the above equation is in the ring $K(x_1, \cdots, x_n)\#U(L)$, and can be written as:

$$a_n(x_1, \cdots, x_n) \bar{x}_1^{i_1} \cdots \bar{x}_n^{i_n} + \cdots + a_0(x_1, \cdots, x_n) = 0$$

where the new (non-commuting) indeterminates $\bar{x}_1, \ldots, \bar{x}_n$ represent the operators $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ respectively.
(3) Moving frames (see Section 4) and some systems of linear homogeneous partial differential equations with non-constant coefficients (see [2], [14] and [15]).

We denote by $\mathcal{M}(X)$ or $\mathcal{M}(x_1, \ldots, x_n)$ the set $\{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$ of all standard monomials of $\{x_1, \cdots, x_n\}$. For simplicity, we write $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$. Given a total order $\prec$ on the set of standard monomials, we define the *leading monomial* $\mathrm{lm}(f)$ of $f \in R\langle x_1, \cdots, x_n\rangle$ to be the largest standard monomial occurring in f with non-zero coefficient, the *leading coefficient* $\mathrm{lc}(f)$ to be the coefficient of $\mathrm{lm}(f)$ and *leading term* $\mathrm{lt}(f) = \mathrm{lc}(f) \cdot \mathrm{lm}(f)$. For a subset $S \subseteq R\langle x_1, \cdots, x_n\rangle$, $\mathrm{lm}(S) = \{\mathrm{lm}(s) | s \in S\}$ while $\mathrm{lc}(S)$ and $\mathrm{lt}(S)$ are similarly defined and ${}_\ell(S)$ will be the left ideal generated by S in $R\langle x_1, \cdots, x_n\rangle$. The *degree* of $X^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is $deg(X^\alpha) = |\alpha| := \alpha_1 + \cdots \alpha_n$.

Definition 2. *An admissible order $\prec$ on $\mathcal{M}(X)$ is a total order of $\mathcal{M}(X)$ which satisfies:*

(1) multiplicative, i.e., $r \prec X^\alpha$ and $X^\alpha \prec X^\beta$ imply that $\mathrm{lm}(X^\eta X^\alpha X^\gamma) \prec \mathrm{lm}(X^\eta X^\beta X^\gamma)$, where $X^\eta, X^\alpha, X^\beta, X^\gamma \in \mathcal{M}(X)$ and $r \in R$.

(2) degree compatible, i.e., $deg(X^\alpha) \prec deg(X^\beta)$ implies $X^\alpha \prec X^\beta$, where $X^\alpha, X^\beta \in \mathcal{M}(X)$.

Remark 2. From condition (2), we get the *descending chain condition*, i.e., there are no infinite strictly descending chains of standard monomials, which is used to prove that the reductions will stop eventually.

Example 2. The typical example of this order is the graded lexicographic ordering. That is, $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \prec x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$ if and only if the first nonzero component of $(\sum_{k=1}^n (\alpha_k - \beta_k), \alpha_1 - \beta_1, \cdots, \alpha_n - \beta_n)$ is negative.

Before computing Gröbner bases in PBW extensions, we first need a notion of the coefficient ring being computable.

Definition 3. *An associative (though not necessarily commutative) ring R is (left) computable, if in addition to the usual arithmetic operations being computable, the following two conditions hold:*

(1) (left ideal membership) Given $a, a_1, \cdots, a_m \in R$, there is an algorithm which decides whether a is in the left ideal $R(a_1, \cdots, a_m)$ and if so, finds $b_1, b_2, \cdots, b_m \in R$ such that $a = \sum_{i=1}^m b_i a_i$.

(2) (left syzygies) Given $a_1, \cdots, a_m \in R$ there is an algorithm which finds a finite set of generators for the R-module

$$Syz(a_1, \cdots, a_m) := \{(b_1, \cdots, b_m) \in R^m | \sum_{i=1}^m b_i a_i = 0\}.$$

If R is a field, the condition "left ideal membership" is trivial. But if R is a ring, it is a useful and necessary condition. The condition "left syzygies" is needed to guarantee that the algorithm `GröbnerPBW` which follows is implementable, since from the noetherian condition we only know that there exist a finite number of generators. In fact these conditions have been used in many papers, for example, Gianni, Trager and Zacharias [9]. It is a common condition when one considers Gröbner theory on rings instead of fields. There are many rings satisfying this condition, for example, the usual polynomial ring over a field and the universal enveloping Lie algebra over a field (see Apel and Lassner [4]).

In the remainder of the paper, we assume that R is a left computable and noetherian (not necessarily commutative) ring with a finite PBW extension $R\langle x_1, \cdots, x_n\rangle$, and $\prec$ is an admissible order on $\mathcal{M}(X)$.

3 Gröbner Bases in PBW Extensions

In [8], Galligo first considered Gröbner bases in the ring of linear differential operators. Later, many authors extended Galligo's idea to various rings. For example, Chyzak and Salvy [6] consider Gröbner bases in Ore algebras, and Insa and Pauer [11] for Gröbner bases in the ring of differential operators, assuming the coefficient form a subring of a function field. In this paper we consider the more general case, PBW extensions.

Definition 4. *For $f, g \in \mathcal{M}(X)$, f is a factor of g if there exist $p, q \in \mathcal{M}(X)$ such that $g = \mathrm{lm}(pfq)$. Also g is said to be divisible by f, denoted by $f|g$.*

It is easy to see that if $f = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $g = x_1^{\beta_1} \cdots x_n^{\beta_n}$, then f is a factor of g if and only if $\alpha_k \leq \beta_k$, for all $1 \leq k \leq n$.

In the following definition, we only require that the leading terms are reduced. The conditions involving reduced polynomials (over rings) are no longer required, and so reducing non-leading terms is unnecessary (see [1], p.211).

Definition 5. *Let G be a finite subset of $R\langle x_1, \cdots, x_n\rangle$ and $f, h \in R\langle x_1, \cdots, x_n\rangle$. We say h is a one step reduction of f modulo G, denoted $f \to^G h$, if for the leading term aX^α in f, there exist $g_1, \cdots, g_t \in G$ and $r_1, \cdots, r_t \in R$ such that*
(1) $\mathrm{lm}(g_i)$*'s are factors of* X^α*, say* $\mathrm{lm}(X^{\beta_i} g_i X^{\gamma_i}) = X^\alpha$*, for all* $i = 1, \cdots, t$*;*
(2) $a = \sum_{i=1}^t r_i \,\mathrm{lc}(g_i)$ *for some* $r_i \in R, i = 1, \cdots, t$*;*
(3) $h = f - \sum_{i=1}^t r_i X^{\beta_i} g_i X^{\gamma_i}$.
Furthermore, we say f reduces to h modulo G if and only if there exist $h_1, \cdots, h_s \in R\langle x_1, \cdots, x_n\rangle$ such that $f \to^G h_1 \to^G h_2 \to^G \cdots \to^G h_s \to^G h$.

Remark 3.
(1) Note that the conditions of (1) and (3) are equivalent to:
(1)' $\mathrm{lm}(g_i)$'s are factors of X^α, say $\mathrm{lm}(X^{\beta_i} g_i) = X^\alpha$, for all $i = 1, \cdots, t$;
(3)' $h = f - \sum_{i=1}^t r_i X^{\beta_i} g_i$.
(2) We remind the reader that Gauss reduction does not work in PBW extensions since the elements of R are not necessarily invertible. Therefore we have to use "sum" to cancel some terms. For example, consider the PBW extension $K[y_1, y_2, y_3]\langle x_1, x_2, x_3\rangle$, where $K[y_1, y_2, y_3]$ is the usual polynomial ring over a field K and $\{x_1, x_2, x_3\}$ is the 3-dimensional Lie algebra with $[x_1, x_2] = x_1, [x_1, x_3] = -2x_1$ and $[x_2, x_3] = -2x_3$. Set $G = \{g_1 := y_1 x_1, g_2 := y_2 x_2\}$ and $f := (2y_1 + 3y_2)x_1 x_2 + x_1 + 1$. Since $K[y_1, y_2, y_3]$ is just a ring, not a field, we do elimination as follows:

$$\begin{aligned} h &= f - (2g_1 x_2 + 3g_2 x_1) = (2y_1 + 3y_2)x_1 x_2 + x_1 + 1 - (2(y_1 x_1)x_2 + 3(y_2 x_2)x_1) \\ &= (2y_1 + 3y_2)x_1 x_2 + x_1 + 1 - (2(y_1 x_1 x_2) + 3y_2(x_1 x_2 - x_3)) = x_1 - 3y_2 x_3 + 1. \end{aligned}$$

Definition 6. *An element $f \in R\langle x_1, \cdots, x_n\rangle$ is said to be in reduced form with respect to G if f can not be reduced modulo G. A reduced form of f modulo G is an element $h \in R\langle x_1, \cdots, x_n\rangle$ such that h is in reduced form with respect to G and $f \to^G h$.*

As in the commutative case we have a division rule, but here it is one-sided.

Proposition 1. (Left division rule) *Let $G := \{g_1, \cdots, g_t\} \subseteq R\langle x_1, \cdots, x_n\rangle$ and $f \in R\langle x_1, \cdots, x_n\rangle$. Then there exist $h_1, \cdots, h_t, \psi \in R\langle x_1, \cdots, x_n\rangle$ such that $f = h_1 g_1 + \cdots + h_t g_t + \psi$, where ψ is reduced modulo G and* $\mathrm{lm}(f) = max\{max\{\mathrm{lm}(\mathrm{lm}(h_i)\,\mathrm{lm}(g_i))\}_{i=1}^t, \mathrm{lm}(\psi)\}$.

Proof. If f is reduced modulo G, then there is nothing to do. Assume then that there is a reduction chain $f \to \psi_1 \to \psi_2 \to \cdots$. By the definition of reduction, the leading term is decreased, that is, $\mathrm{lm}(f) \succ \mathrm{lm}(\psi_1) \succ \mathrm{lm}(\psi_2) \succ \cdots$. Since $\succ$ is a well-ordering, the reduction chain has to stop, say, $f \to \psi_1 \to \cdots \to \psi_s \to \psi$, where ψ is reduced modulo G. By the remark 3, we have $f - \psi_1 = r_{11} X^{\alpha_{11}} g_1 + \cdots + r_{1t} X^{\alpha_{1t}} g_t$, where $\{r_{1i}\} \in R$, $\mathrm{lt}(f) = r_{11}\,\mathrm{lt}(X^{\alpha_{11}} g_1) + \cdots + r_{1t}\,\mathrm{lt}(X^{\alpha_{1t}} g_t)$ (some r_{ji} and α_{ji} maybe zero) and $\mathrm{lm}(f) = \mathrm{lm}(X^{\alpha_{1i}} g_i)$ for all i such that $r_{1i} \neq 0$ (since R maybe has zero-divisors). Similarly we have the representation for $\psi_1 - \psi_2 = r_{21} X^{\alpha_{21}} g_1 + \cdots + r_{2t} X^{\alpha_{2t}} g_t$. Therefore

$$f - \psi_2 = (f - \psi_1) + (\psi_1 - \psi_2) = (r_{11} X^{\alpha_{11}} + r_{21} X^{\alpha_{21}}) g_1 + \cdots + (r_{1t} X^{\alpha_{1t}} + r_{2t} X^{\alpha_{2t}}) g_t.$$

Note that the coefficients of $\{g_i\}$ are not all zero if $s > 2$. Continuing in this way, we get the representation for $f - \psi$ as required. □

The above proposition gives a method to calculate the reduced form:

Algorithm: `ReducedForm`
Input: ▸ $G = \{g_1, \cdots, g_t\} \subseteq R\langle x_1, \cdots, x_n\rangle$;
▸ $f \in R\langle x_1, \cdots, x_n\rangle$;
Output: ▸ a reduced form of f modulo G: $h_1, \cdots, h_t, \psi \in R\langle x_1, \cdots, x_n\rangle$ such that $f = h_1 g_1 + \cdots + h_t g_t + \psi$, where ψ is reduced modulo G and $\mathrm{lm}(f) = max\{max\{\mathrm{lm}(\mathrm{lm}(h_i)\,\mathrm{lm}(g_i))\}_{i=1}^t, \mathrm{lm}(\psi)\}$;
Set $\psi := f$ and $h_1, \ldots, h_t := 0$;
While $\psi \neq 0$ and $\mathrm{lc}(\psi) \in {}_\ell(\mathrm{lc}(g) : g \in G, \mathrm{lm}(g) | \mathrm{lm}(\psi))$ Do
Find $\{r_i\}_1^t \in R$, $\{X^{\alpha_i}\}_1^t$ so $\mathrm{lc}(\psi) = \sum_{i=1}^t r_i\,\mathrm{lc}(g_i)$, $\mathrm{lm}(X^{\alpha_i} g_i) = \mathrm{lm}(\psi)$;
$\psi := \psi - \sum_{i=1}^t r_i X^{\alpha_i} g_i$.
For $i = 1$ to t Do $h_i := h_i + r_i X^{\alpha_i}$;
End.

Definition 7. *Let I be a left ideal of $R\langle x_1, \cdots, x_n\rangle$ and G a subset of I. Then G is called a (left) Gröbner basis of I if for all $f \in I$, $f \to^G 0$.*

While some of the definitions and theorems in the PBW extensions case are equivalent to those in the commutative case, others do not hold.

Theorem 1. *Let I be a left ideal of $R\langle x_1, \cdots, x_n\rangle$ and let G be a finite subset of I. The following assertions are equivalent:*

(1) G is a left Gröbner basis of I;

(2) For all $0 \neq f \in I$, f is reducible modulo G;

(3) For all $0 \neq f \in I$, there exist $g_1, \cdots, g_t \in G$ such that $\operatorname{lm}(g_i), i = 1, \cdots, t$ are factors of $\operatorname{lm}(f)$ and $\operatorname{lc}(f) \in {}_\ell(\operatorname{lc}(g_1), \cdots, \operatorname{lc}(g_t))$;

(4) For $\alpha \in \mathbb{N}^n$, let ${}_\ell(\alpha, I) := {}_\ell(\operatorname{lc}(f) : f \in I, \operatorname{lm}(f) = X^\alpha)$. Then for all $\alpha \in \mathbb{N}^n$ the left ideal ${}_\ell(\alpha, I)$ is generated by $\{\operatorname{lc}(g) : g \in G, \operatorname{lm}(g) | X^\alpha\}$.

Proof. (1) $\Rightarrow$ (2) follows from the definition of Gröbner bases.

(2) $\Rightarrow$ (1): Let $G := \{g_1, \cdots, g_t\}$ and $0 \neq f \in I$. Since f is reducible, by induction and Proposition 1, there exist $h_1, \cdots h_t, \psi \in R\langle x_1, \cdots, x_n\rangle$ such that $f = h_1 g_1 + \cdots h_t g_t + \psi$, where ψ is reduced modulo G. This implies that $\psi = f - (h_1 g_1 + \cdots + h_t g_t) \in I$, and so ψ is reducible by (2), a contradiction. Therefore $\psi = 0$, that is, f can be reduced to 0 modulo G.

(2) $\Leftrightarrow$ (3) follows from the definition of reduction.

(3) $\Rightarrow$ (4): For fixed $\alpha \in \mathbb{N}^n$, for any $r \in {}_\ell(\alpha, I)$, since ${}_\ell(\alpha, I)$ is usually not a principal ideal, there exist $f_1, \cdots, f_t \in I$ with $\operatorname{lm}(f_i) = X^\alpha, i = 1, \cdots, t$ and $a_1, \cdots, a_t \in R$ such that $r = a_1 \operatorname{lc}(f_1) + \cdots + a_t \operatorname{lc}(f_t)$. From (3), for each f_i, we have $\operatorname{lc}(f_i) = b_1 \operatorname{lc}(g_{i1}) + \cdots + b_s \operatorname{lc}(g_{is})$, where $g_{ij} \in G, \operatorname{lm}(g_{ij}) | \operatorname{lm}(f_i), b_j \in R, j = 1, \cdots, s$. Thus $r \in {}_\ell(\operatorname{lc}(g) : g \in G, \operatorname{lm}(g) | X^\alpha)$.

Conversely, let $a \in {}_\ell(\operatorname{lc}(g) : g \in G, \operatorname{lm}(g) | X^\alpha)$. Then $a = r_1 \operatorname{lc}(g_1) + \cdots + r_t \operatorname{lc}(g_t)$ for some $r_i \in R, g_i \in G, \operatorname{lm}(g_i) | X^\alpha, i = 1, \cdots, t$. Choose $\{X^{\alpha_i}\}_1^t \in \mathcal{M}(X)$ such that $\operatorname{lm}(X^{\alpha_i} g_i) = X^\alpha$ for $i = 1, \cdots, t$. Note that $X^{\alpha_i} g_i \in I$, we get

$$\begin{aligned} a &= r_1 \operatorname{lc}(g_1) + \cdots + r_t \operatorname{lc}(g_t) = r_1 \operatorname{lc}(X^{\alpha_1} g_1) + \cdots + r_t \operatorname{lc}(X^{\alpha_t} g_t) \\ &\in {}_\ell(\operatorname{lc}(f) : f \in I, \operatorname{lm}(f) = X^\alpha). \end{aligned}$$

(4) $\Rightarrow$ (3): For any $0 \neq f \in I$, let $\operatorname{lm}(f) = X^\alpha$. Then $\operatorname{lc}(f) \in {}_\ell(\alpha, I)$. By (4), $\operatorname{lc}(f) \in {}_\ell(\operatorname{lc}(g) : g \in G, \operatorname{lm}(g) | X^\alpha)$. □

Corollary 1.

(1) If G is a Gröbner basis for the left ideal I in $R\langle x_1, \cdots, x_n\rangle$, then $I = {}_\ell(G)$.

(2) If G is a Gröbner basis and $f \in {}_\ell(G)$ and $f \to^G h$, where h is reduced, then $h = 0$.

(3) Let G be the Gröbner basis of left ideal I and $f \in R\langle x_1, \cdots, x_n\rangle$. Then $f \in I$ if and only if $f = 0$ modulo G.

Next we give a method to construct Gröbner bases in PBW extensions. We remind the reader that in general rings the S-pair criteria do not work. A simple example can be found in ([4], p.248).

Theorem 2. *Let I be the left ideal of $R\langle x_1, \cdots, x_n\rangle$ generated by a finite set G. For $F := \{g_1, \cdots, g_s\} \subseteq G$, let $\operatorname{lcm}(F)$ be the least common multiple of $\{\operatorname{lm}(g_i)\}_{i=1}^s$. Let $\mathcal{B}_F$ be a finite set of generators of $Syz(\operatorname{lc}(g_1), \cdots, \operatorname{lc}(g_s))$. Then the following assertions are equivalent:*

(1) G is a Gröbner basis of I;

(2) For all $F := \{g_i\}_1^s \subseteq G$ and for all $\{b_1, \ldots, b_s\} \in \mathcal{B}_F$, we have that $\sum_{i=1}^s b_i (X^{\operatorname{lcm}(F)} / \operatorname{lm}(g_i)) g_i$ reduces to zero with respect to G.

Proof. (1) $\Rightarrow$ (2) follows from $\sum_{i=1}^s b_i (X^{\operatorname{lcm}(F)} / \operatorname{lm}(g_i)) g_i \in {}_\ell(G)$.

(2) $\Rightarrow$ (1): Let $f \in I$. By Theorem 1 we need to show: $\operatorname{lc}(f) \in {}_\ell(\operatorname{lc}(g) : g \in G, \operatorname{lm}(g) | \operatorname{lm}(f))$. Let $G := \{g_1, \cdots, g_t\}$. Then $f \in I$ implies that there exists a representation $f = \sum_{i=1}^t f_i g_i$. Furthermore we may choose $\{f_i\}_1^t$ such that $max_\prec \{\operatorname{lm}(\operatorname{lm}(f_i) \operatorname{lm}(g_i))\}_{i=1}^t$ is minimal, say X^{α_0}. Let $F := \{g_i \in G | \operatorname{lm}(\operatorname{lm}(f_i) \operatorname{lm}(g_i)) = X^{\alpha_0}\}$. Without loss of generality, we may assume that $F = \{g_1, \cdots, g_s\}, 1 \leq s \leq t$. The following argument considers cases based on the leading monomial of f.

If $\mathrm{lm}(f) = X^{\alpha_0}$, then $\mathrm{lt}(f) = \sum_{i=1}^{s} \mathrm{lt}(f_i g_i)$ and $\mathrm{lc}(f) = \sum_{i=1}^{s} \mathrm{lc}(f_i)\,\mathrm{lc}(g_i) \in {}_{\ell}(\mathrm{lc}(g) : g \in F)$. Note that $g_i \in F$ implies $\mathrm{lm}(f) = \mathrm{lm}(\mathrm{lm}(f_i)\,\mathrm{lm}(g_i))$. Thus $\mathrm{lm}(g_i) | \mathrm{lm}(f)$. Therefore $\mathrm{lc}(f) \in {}_{\ell}(\mathrm{lc}(g) : g \in G, \mathrm{lm}(g) | \mathrm{lm}(f))$.

If $\mathrm{lm}(f) \prec X^{\alpha_0}$, then $\sum_{i=1}^{s} \mathrm{lc}(f_i)\,\mathrm{lc}(g_i) = 0$ and $(\mathrm{lc}(f_1), \cdots, \mathrm{lc}(f_s)) \in Syz(\mathrm{lc}(g_1), \cdots, \mathrm{lc}(g_s))$. Let $\mathcal{B}_F$ be a basis of $Syz(\mathrm{lc}(g_1), \cdots, \mathrm{lc}(g_s))$, say, $\mathcal{B}_F := \{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_k\} := \{(b_{11}, \cdots, b_{1s}), \cdots, (b_{k1}, \cdots, b_{ks})\}$. There exist $r_1, \cdots, r_k \in R$ such that $(\mathrm{lc}(f_1), \cdots, \mathrm{lc}(f_s)) = r_1\boldsymbol{b}_1 + \cdots + r_k\boldsymbol{b}_k = (r_1 b_{11} + \cdots + r_k b_{k1}, \cdots, r_1 b_{1s} \cdots + r_k b_{ks})$. That is, for $1 \le i \le s$, we have $\mathrm{lc}(f_i) = r_1 b_{1i} + \cdots + r_k b_{ki}$. Now

$$
\begin{aligned}
f = \sum_{i=1}^{t} f_i g_i &= \sum_{i=1}^{s} f_i g_i + \sum_{i=s+1}^{t} f_i g_i \\
&= \sum_{i=1}^{s} (f_i - \sum_{j=1}^{k} r_j b_{ji}\,\mathrm{lm}(f_i)) g_i + \sum_{i=1}^{s}\sum_{j=1}^{k} r_j b_{ji}\,\mathrm{lm}(f_i) g_i + \sum_{i=s+1}^{t} f_i g_i.
\end{aligned}
$$

For the first sum of equation (3.2), from $\mathrm{lm}(f_i - \sum_{j=1}^{k} r_j b_{ji}\,\mathrm{lm}(f_i)) \prec lm(f_i)$, we have that $\mathrm{lm}(\mathrm{lm}(f_i - \sum_{j=1}^{k} r_j b_{ji}\,\mathrm{lm}(f_i))\,\mathrm{lm}(g_i)) \prec X^{\alpha_0}, i = 1, \cdots, s$.

For the third sum of equation (3.2), by the definition of F, we have $max\{\mathrm{lm}(\mathrm{lm}(f_i)\,\mathrm{lm}(g_i))\}_{i=s+1}^{t} \prec X^{\alpha_0}$.

In order to get the contradiction to the minimality of α_0, we have to rewrite the second sum of equation (3.2). Note that for all $g_i \in F$, $\mathrm{lm}(\mathrm{lm}(f_i)\,\mathrm{lm}(g_i)) = X^{\alpha_0}$. Thus there is an element $0 \neq \gamma \in \mathbb{N}^n$ such that $\alpha_0 = \mathrm{lcm}(F) + \gamma$. Then

$$
\begin{aligned}
\sum_{i=1}^{s}\sum_{j=1}^{k} r_j b_{ji}\,\mathrm{lm}(f_i) g_i = &\sum_{j=1}^{k} r_j X^{\gamma} (\sum_{i=1}^{s} b_{ji} \frac{X^{\mathrm{lcm}(F)}}{\mathrm{lm}(g_i)} g_i) \\
&+ \sum_{j=1}^{k} r_j (\sum_{i=1}^{s} (b_{ji}\,\mathrm{lm}(f_i) - X^{\gamma} b_{ji} \frac{X^{\mathrm{lcm}(F)}}{\mathrm{lm}(g_i)}) g_i).
\end{aligned}
$$

Clearly, the leading monomials of every product in the second sum are smaller than X^{α_0}. Thus we only need to consider the first sum. By the assumption, $\sum_{i=1}^{s} b_{ji} \frac{X^{\mathrm{lcm}(F)}}{\mathrm{lm}(g_i)} g_i$ reduces to zero modulo G. Then by the division rule there exist $h_1, \cdots, h_t \in R\langle x_1, \cdots, x_n\rangle$ such that $\sum_{i=1}^{s} b_{ji} \frac{X^{\mathrm{lcm}(F)}}{\mathrm{lm}(g_i)} g_i - \sum_{i=1}^{t} h_i g_i = 0$ and $\mathrm{lm}(\mathrm{lm}(h_i)\,\mathrm{lm}(g_i)) \prec X^{\alpha_0 - \gamma}$. Therefore

$$
\sum_{j=1}^{k} r_j X^{\gamma} (\sum_{i=1}^{s} b_{ji} \frac{X^{m(F)}}{\mathrm{lm}(g_i)} g_i) = \sum_{j=1}^{k} r_j X^{\gamma} (\sum_{i=1}^{t} h_i g_i) = \sum_{j=1}^{k}\sum_{i=1}^{t} X^{\gamma} h_i g_i.
$$

It is easy to see that for all i, $\mathrm{lm}(\mathrm{lm}(X^{\gamma} h_i)\,\mathrm{lm}(g_i)) \prec X^{\alpha_0}$, a contradiction as required. □

The above theorem suggests an algorithm to construct Gröbner bases:

Algorithm: `GröbnerPBW`

Input: ▸ $F = \{f_1, \cdots, f_s\} \subseteq R\langle x_1, \cdots, x_n\rangle$.

Output: ▸ $G = \{g_1, \cdots, g_t\}$, a Gröbner basis for ${}_{\ell}(f_1, \cdots, f_s)$.

- Initialization: $G := F, G' :=$ all subsets of F;
- While $G' \neq \emptyset$ Do
  - Choose $\emptyset \neq S \in G'$, say, $S := \{f_{i_1}, \cdots, f_{i_k}\}$;
  - $G' := G' \setminus S$;
  - Compute $\mathcal{B}_S$, a generating set for $Syz(\mathrm{lc}(f_{i_1}), \cdots, \mathrm{lc}(f_{i_k}))$ and $\mathrm{lcm}(S)$;
  - For each $b := (b_{i_1}, \cdots, b_{i_k}) \in \mathcal{B}_S$ Do
    - $b_{i_1}(X^{\mathrm{lcm}(S)}/\,\mathrm{lm}(f_{i_1})) f_{i_1} + \cdots + b_{i_k}(X^{\mathrm{lcm}(S)}/\,\mathrm{lm}(f_{i_k})) f_{i_k} \to^{G} \psi$, ψ is reduced module G.
    - If $\psi \neq 0$, then
      - $G := G \cup \{\psi\}$;
      - $G' := \{S' \cup \{\psi\}\}$; add ψ to every nonempty subset S'.
  - End Do
- End Do

By the noetherian properties of the ring R, we know that $R\langle x_1, \cdots, x_n\rangle$ is also noetherian and we obtain the following proposition.

Corollary 2. *Let I be a non-zero left ideal of $R\langle x_1, \cdots, x_n\rangle$. Then I has a finite Gröbner basis.*

If R has some special properties, we *can* define a notion of S-pairs. For example, when R is a field, we can define the reductions and S-pairs similar to the commutative case, and easily prove that:

Corollary 3. *Assume that R is a field. Let I be a left ideal of $R\langle x_1, \cdots, x_n\rangle$ generated by a finite set G. Then G is a Gröbner basis if and only if all S-pairs reduce to zero.*

For noncommutative PIDs, we can define S-pairs in a similar fashion to commutative PIDs, though the situation becomes much more complex. For example, in noncommutative PIDs and UFDs, factors are only unique with respect to some invariant factors and left great common divisors are may not same as right great common divisors. We will discuss these issues in forthcoming paper.

Discussing Gröbner bases from the graded point of view has a long history; for example, Robbiano [19], and more recently Apel [3], consider Gröbner bases on general graded rings. In PBW extensions the graded lexicographic ordering is the most popular ordering. Let $T = R\langle x_1, \cdots, x_n\rangle$ be a PBW extension and $\prec$ a term ordering compatible with the graded lexicographic ordering. The associated graded ring $\mathrm{gr}\, T$ is defined as $R[\bar{x}_1, \cdots, \bar{x}_n]$ with $r\bar{x}_i = \bar{x}_i r$ and $\bar{x}_i\bar{x}_j = \bar{x}_j\bar{x}_i$ for all $r \in R$, that is, the usual polynomial ring over R.

We set up a relation between the Gröbner bases of $R\langle x_1, \cdots, x_n\rangle$ and its associated graded ring $\mathrm{gr}\, R\langle x_1, \cdots, x_n\rangle$. For an element $f \in R\langle x_1, \cdots, x_n\rangle$, $\bar{f}$ will be the image of f in $\mathrm{gr}\, R\langle x_1, \cdots, x_n\rangle$. Recall that the standard filtration of $R\langle x_1, \cdots, x_n\rangle$ is $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_n$ where $\mathcal{F}_i$ is the R-subspace generated by all X^α with $|\alpha| \le i$ and $\mathrm{gr}\, R\langle x_1, \cdots, x_n\rangle = \oplus \mathcal{F}_n/\mathcal{F}_{n-1} = R[\bar{x}_1, \cdots, \bar{x}_n]$. For any left ideal I of $R\langle x_1, \cdots, x_n\rangle$, there is a graded ideal $\mathrm{gr}\, I$ of $\mathrm{gr}\, R\langle x_1, \cdots, x_n\rangle$ which is defined by setting $(\mathrm{gr}\, I)_i = (I + \mathcal{F}_{i-1}) \cap \mathcal{F}_i/\mathcal{F}_{i-1} \simeq I \cap \mathcal{F}_i / I \cap \mathcal{F}_{i-1} \subset \mathcal{F}_i/\mathcal{F}_{i-1}$, and $\mathrm{gr}\, I = \oplus_i (\mathrm{gr}\, I)_i$.

Lemma 1. *([16]) Let $T = R\langle x_1, \cdots, x_n\rangle$ be a PBW extension. If R is a noetherian ring, then T and $\mathrm{gr}\, T$ are noetherian rings.*

Theorem 3. *Let I be a left ideal of $T = R\langle x_1, \cdots, x_n\rangle$. If $G = \{f_1, \cdots, f_m\}$ is a Gröbner bases of I, then $\{\bar{f}_1, \cdots, \bar{f}_m\}$ is a Gröbner basis of $\mathrm{gr}\, I$. Conversely if $\{g_1, \cdots, g_t\}$ is a Gröbner basis of $\mathrm{gr}\, I$, then choose $\{f_1, \cdots, f_t\} \subseteq I$ such that $\bar{f}_1 = g_1, \cdots, \bar{f}_t = g_t$ and $\{f_1, \cdots, f_t\}$ is a Gröbner basis of I.*

Proof. Let $G = \{f_1, \cdots, f_m\}$ be a Gröbner bases of I. For a homogeneous element $g \in (\mathrm{gr}\, I)_i$, there exists $f \in (I + \mathcal{F}_{i-1}) \cap \mathcal{F}_i/\mathcal{F}_{i-1}$ such that $\bar{f} = g$. Since $f \in I$, we assume that $f = h_1 f_1 + \cdots + h_t f_t$ with $\mathrm{lm}(\mathrm{lm}(h_j)\,\mathrm{lm}(f_j)) \prec \mathrm{lm}(f), 1 \le j \le t$. Thus $g = \bar{f} = f + \mathcal{F}_{i-1} = h_1 f_1 + \cdots + h_t f_t + \mathcal{F}_{i-1} = (h_1 f_1 + \mathcal{F}_{i-1}) + \cdots + (h_t f_t + \mathcal{F}_{i-1}) = \bar{h}_1 \bar{f}_1 + \cdots + \bar{h}_t \bar{f}_t$. Therefore $\{\bar{f}_1, \cdots, \bar{f}_m\}$ is a Gröbner basis of $\mathrm{gr}\, I$. We leave the remainder of the proof to the reader and remind that S-pairs also do not work on grT. □

4 Application to Moving Frames

Standard differential elimination algorithms are based on commuting derivations. In many applications it is natural to choose instead a basis for these derivations which is adapted to the geometry of the application (e.g., a basis which is invariant under some given symmetry group of the application).

In these cases, the given commuting frame of commuting partial differential operators $\frac{\partial}{\partial x_i}$ may not be well suited to the application.

One class of moving frames is moving frames of differential operators of the form

$$\Delta_i = \sum_i A_i^j(z) \frac{\partial}{\partial z^j}, \tag{1}$$

and these satisfy frame commutation relations of the form:

$$[\,\Delta_i, \Delta_j\,] = \sum_k \gamma_{ij}^k(z) \Delta_k \ . \tag{2}$$

A significant abstraction and generalization of these ideas was initiated by Cartan, and in recent times developed and applied by Fels and Olver [7].

A simple example of such an approach is using polar coordinates for cylindrically invariant problems (where the operators $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial r}$ commute). Given an arbitrary Lie group $\mathcal{G}$ the power of the general method of moving frames is that, on some sufficiently prolonged space, a $\mathcal{G}$-invariant frame always exists.

A study of differential-elimination methods in moving frames of differential operators was given by Lisle in his PhD thesis [13]. However Lisle did not give a rigorous Gröbner basis theory for his approach. He was able, however, to do very complex *classification problems* which were beyond the power of differential elimination packages based on commuting derivations.

In this section we show that the Gröbner theory developed in this paper can be applied to moving frames of differential operators for systems of linear homogeneous partial differential equations.

We treat an illustrative example which arises from the group classification problem for the class of nonlinear diffusion equations of the form

$$u_t = \big(D(u)u_x\big)_x \,. \tag{3}$$

This example was used by Lisle and Reid [14] and Lisle [13] to illustrate Lisle's moving frame method. The method of group classification attempts, for every form of $D(u)$, to describe the symmetry properties of the above partial differential equation. This is easy for the illustrative example, but in general leads to intractable overdetermined systems of partial differential equations for the symmetries when commuting derivations are used. The idea of Lisle's method, was to exploit equivalence transformations which mapped members of the class of partial differential differential equations to another member of the class. In the above case $u \mapsto au + b, x \mapsto cx + d, t \mapsto ex + f$, are simple examples of such transformations. Then the method constructs a moving frame of differential operators invariant under such an equivalence group.

One branch of the calculation, for the nonlinear diffusion equation, leads to the following system of partial differential equations in the *frame standard form* of Lisle and Reid [14]:

$$\begin{array}{lll} \Delta_1\Delta_1\theta^1 = 0 & \Delta_1\theta^2 = 0 & \Delta_1\theta^3 = 0 \\ \Delta_2\theta^1 = 0 & \Delta_2\theta^2 = 2\Delta_1\theta^1 - \theta^3 & \Delta_2\theta^3 = 0 \\ \Delta_3\theta^1 = -\frac{1}{2}\theta^1 & \Delta_3\theta^2 = 0 & \Delta_3\theta^3 = 0. \end{array}$$

This is the system of equations just after equation (18) of Lisle and Reid [14]. In this case the frame derivations Δ_j, $j = 1, 2, 3$, have vanishing commutators except for

$$[\Delta_1, \Delta_3] = -\frac{1}{2}\Delta_1. \tag{4}$$

In terms of the original physical variables x, t, u, and the commuting coordinate frame $\frac{\partial}{\partial x}, \frac{\partial}{\partial t}, \frac{\partial}{\partial u}$, the frame derivations are given by

$$\Delta_1 := D^{1/2}\frac{\partial}{\partial x}, \quad \Delta_2 := \frac{\partial}{\partial t}, \quad \Delta_3 := D/\dot{D}\,\frac{\partial}{\partial u}, \tag{5}$$

and the dependent variables $\theta^1, \theta^2, \theta^3$, yield the infinitesimal symmetries via the relation (14) given in Lisle and Reid [14].

Notice that the theory of this paper, which is directed to linear homogeneous systems is not directly applicable, since the above system has 3 dependent variables. To transform it to an equivalent system, with one dependent variable, we use the *Drach Transformation* which, although written for the commutative case, easily generalizes to the non-commutative case. Consider systems with n independent variables $x_1, ..., x_n$ (here $n = 3$) and m dependent variables (here $m = 3$). The Drach transformation proceeds by introducing m new independent variables $x_{n+j}, j = 1, ..., m$ and is defined by:

$$\theta^j := \Delta_{n+j}w, j = 1, ..., m, \quad \Delta_{n+j} := \frac{\partial}{\partial x_{n+j}} \tag{6}$$

together with the additional relations

$$\Delta_{n+j}\Delta_{n+k}w = 0, \quad 1 \le j, k \le m. \tag{7}$$

The only non-vanishing commutators remain as

$$[\,\Delta_i, \Delta_j\,] = \sum_k \gamma_{ij}^k(z)\Delta_k, \;\; 1 \le i, j \le n\,. \tag{8}$$

Under this transformation our system becomes

$$\begin{array}{lll} \Delta_1\Delta_1\Delta_4 w = 0 & \Delta_1\Delta_5 w = 0 & \Delta_1\Delta_6 w = 0 \\ \Delta_2\Delta_4 w = 0 & \Delta_2\Delta_5 w = 2\Delta_1\Delta_4 w - \Delta_6 w & \Delta_2\Delta_6 w = 0 \\ \Delta_3\Delta_4 w = -\frac{1}{2}\Delta_4 w & \Delta_3\Delta_5 w = 0 & \Delta_3\Delta_6 w = 0, \end{array}$$

together with the extra relations

$$\Delta_{3+j}\Delta_{3+k} w = 0, 1 \le j, k \le 3, \tag{9}$$

and the single non-vanishing commutator amongst the $\Delta_1, ..., \Delta_6$ remains as $[\Delta_1, \Delta_3] = -\frac{1}{2}\Delta_1$. Now the system for w is a system to which our PBW Gröbner methods can be applied.

Next we outline the main idea. First, we define the ranking on a moving frame. Set the independent variables $x = \{x_1, \cdots, x_m\}$, dependent variables $u = \{u_1, \cdots, u_n\}$, derivatives $\{\Delta_1, \cdots, \Delta_m\}$ and $\Delta = \{\Delta^\alpha u_i | \alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m, i \in \{1, \cdots, n\}\}$. For $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m$, let $|\alpha| = \alpha_1 + \cdots + \alpha_m$. Now we define the ranking on Δ as following:

Without loss of generality, we may assume that $x_1 \prec x_2 \prec \cdots \prec x_m$ and $u_1 \prec u_2 \prec \cdots \prec u_n$. The *total degree ordering* on Δ is given by:

$$\begin{aligned} \Delta^\alpha u_i \prec \Delta^\beta u_j \iff & |\alpha| < |\beta|, \text{ or } |\alpha| = |\beta|, \text{ and } i < j, \text{ or} \\ & |\alpha| = |\beta|, \ i = j \text{ and } \alpha_1 < \beta_1, \text{ or} \\ & |\alpha| = |\beta|, \ i = j, \ \alpha_1 = \beta_1, \cdots, \alpha_{k-1} = \beta_{k-1}, \\ & \qquad \text{and } \alpha_k < \beta_k \text{ for some } 2 \le k \le m-1. \end{aligned}$$

By the commutative rule, we also can write $\Delta_j \Delta^\alpha u_i$ as a polynomial of standard monomials and u_i. Thus we can define $\text{HD}\Delta_j\Delta^\alpha u_i$ as the highest derivative (highest with respect to $\prec$).

Definition 8. *A positive ranking $\prec$ of Δ is a total degree ordering of Δ which is compatible with differentiation and well-ordering:*

$$\Delta^\alpha u_i \prec \Delta^\beta u_j \Rightarrow \text{HD}\Delta^\gamma\Delta^\alpha u_i \prec \text{HD}\Delta^\gamma\Delta^\beta u_j \tag{10}$$

$$\text{HD}\Delta^\alpha u_i \prec \text{HD}\Delta^\gamma\Delta^\alpha u_i \ \textit{for}\ |\gamma| \neq 0. \tag{11}$$

It is easy to see that positive ranking is compatible with Definition 2 and the Drach transformation keeps the positive ranking invariant. Let I be the left ideal generated by a w-system. Then $f(u) = 0$ for all $f \in I$. This point of view enables us to study w-systems through Gröbner bases for left ideals in PBW extensions. Given fw, gw in w-system, the S-pair is defined to be $S(fw, gw) = \text{lcm}(\text{lm}(f), \text{lm}(g))/\text{lm}(f) \cdot fw - lc(f)/lc(g) \cdot \text{lcm}(\text{lm}(f), \text{lm}(g))/\text{lm}(g) \cdot gw$. In particular if we assume that all S-pairs are reduced to zero in the original untransformed system, then it is easy to show that all the S-pairs in the w-system are reduced to zero and we can use Corollary 3 to construct Gröbner bases. Thus we have a Buchberger-like algorithm for moving frames.

The above example from Lisle and Reid [14] is fairly simple to do, even in the original non-variant commuting coordinate frame. However Lisle and Reid [14] apply their moving frame method to some highly non-trivial systems (earlier given in Lisle's thesis [13]). These include group classification of a class of potential convection diffusion equation, and also a large class of linear wave equations admitting an infinite equivalence group.

Lisle and Reid conjectured [14], but did not prove, that if their linear homogeneous frame systems had their S-pairs reduce to zero, then they would obtain a Gröbner basis. This conjecture is rigorously proved in the current paper.

Some of the linear homogeneous systems in [14] involve functions of the dependent variables (the *class variables*) in their coefficients. The sometimes nonlinear auxiliary relations satisfied by the class functions, need a separate treatment, and had to be checked on a case by case basis. A fully algorithmic approach involving the class functions in addition to the linear homogeneous frame systems, is an important open problem. Finally, we note the we are currently working on generalizing the treatment of this paper to PBW extensions over modules, and this would allow the example to be treated directly without using the indirect Drach Transformation.

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