Lecture 2 Solving Recurrences

Examples from previous courses - Sorting

MergeSort(A):

- Divide array into left and right halves.
- 2 Subproblems: Left and right half subarrays, recurse on both.
- Merge the two sorted arrays $\Rightarrow O(n)$ work to conquer.

Analysis:

Best-case/Worst-case: $T(n) = 2T(n/2) + O(n) \in O(n \log n)$

Rigorously, $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + cn$

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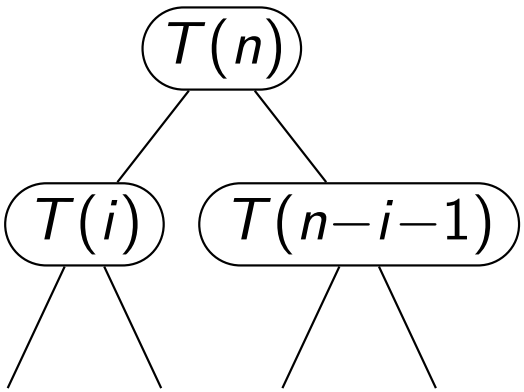

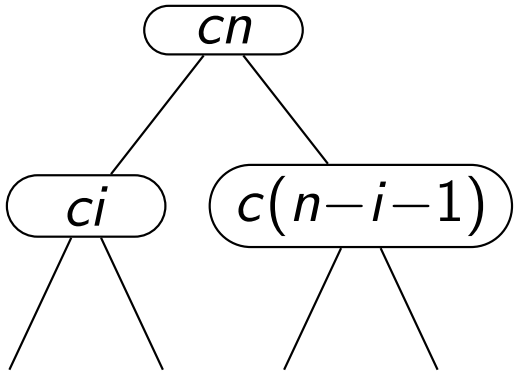

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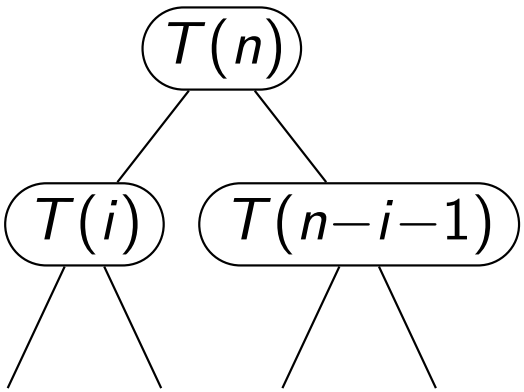

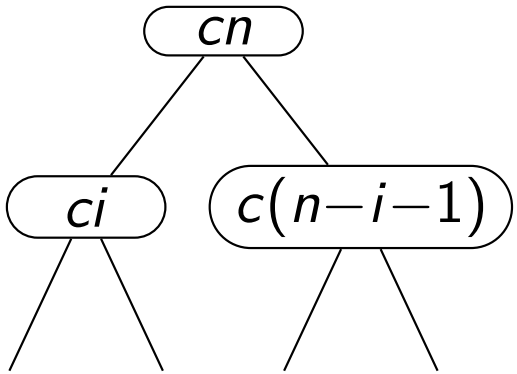

Worst-Case (worst luck): $T(n) = T(n-1) + cn \in \Theta(n^2)$
Best-Case (best luck): $T(n) = 2T(n/2) + cn \in \Theta(n \log n)$
Expected Runtime: $\Theta(n \log n)$

Recursion Tree for MergeSort

$T(n) = 2T(n/2) + cn$, if n is even
$T(1) = 0$, if counting number of comparisons

Recursion Tree where n is a power of 2:

Total work in the recursion tree sums to: $c \cdot n \log n$

Solving Mergesort

If we want to be precise, the math is not trivial:

$$T(n) = \begin{cases} T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor) + (n-1) & \text{if } n > 1 \\ 0 & \text{if } n = 1 \end{cases}$$

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When given n that is not a power of 2, we can also argue that performing the analysis using $n' = s$mallest power of 2 larger than n will also give a precise result since $n' < 2 \cdot n$, runtimes are typically increasing and we only want the growth rate.

Solving Recurrences by Substitution

Also known as "Lucky Guess", "Guess and Check", "Guess and Prove", ...

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Induction Hypothesis: Assume that $T(k) \leq c \cdot k \log k$ for all $k < n$ where $k \geq 2$.

Induction Step: One method to give a rigorous proof is to separate into even and odd cases. If n is even, we don't need floors and ceilings. If n is odd, ...

Careful! Watchout!

$$T(n) = 2T(n/2) + n$$

Claim: $T(n) \in O(n)$

Prove $T(n) \leq cn, \forall n \geq n_0$.

Induction Hypothesis: Assume $T(k) \leq c \cdot k, \forall k < n, k \geq n_0$.

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &\leq 2c(n/2) + n \text{ by the Induction Hypothesis} \\ &= (c+1)n \end{aligned}$$

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The conclusion is clearly incorrect!

The problem is the constant is continuously growing.

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Guess $T(n) \in O(n)$.
Prove by induction that $T(n) \leq cn$ for some constant c.

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$T(n) \leq c \cdot \lceil \frac{n}{2} \rceil - 1 + c \cdot \lfloor \frac{n}{2} \rfloor - 1 + 1 = cn - 1$

Substituion - Changing Variables

$$T(n) = 2T(\lfloor \sqrt{n} \rfloor) + \log n$$

Let $m = \log n$ so $n = 2^m$.
Our new recurrence relation is:

$$T(2^m) = 2T(2^{m/2}) + m$$

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$S(m) \in O(m \log m)$ so $T(2^m) \in O(m \log m)$ and

$T(n) \in O((\log n)(\log \log n))$

Common Recurrences

We often see recurrences of the form:

$$T(n) = aT(\frac{n}{b}) + cn^k$$

Example: Mergesort $k = 1, a = 2, b = 2$

$$T(n) = 2T(\frac{n}{2}) + cn \in O(n \log n)$$

Example: $k = 1, a = 1, b = 2$

$$T(n) = T(\frac{n}{2}) + cn \in O(n)$$

Example: $k = 1, a = 4, b = 2$

$$T(n) = 4T(\frac{n}{2}) + cn \in O(n^2)$$

Master Theorem

Theorem: Given

$$T(n) = aT(\frac{n}{b}) + cn^k$$

where $a \geq 1, b > 1, c > 0, k \geq 0$, then

$$T(n) \in \begin{cases} \Theta(n^k) & \text{if } a < b^k \text{ i.e. } \log_b a < k \\ \Theta(n^k \log n) & \text{if } a = b^k \\ \Theta(n^{\log_b a}) & \text{if } a > b^k \end{cases}$$

Proof: For a rigorous proof, use induction.
Less rigorous, think about the recursion tree.