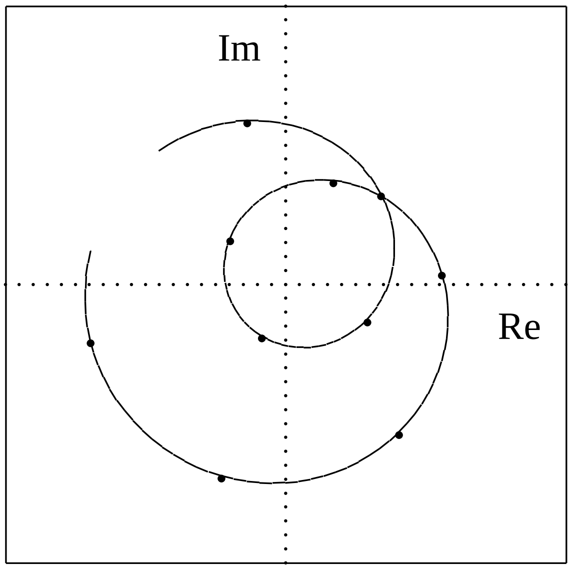
Quadrature Pairs

Filtering: Let $g(x) = G * I(x)$ and $h(x) = H * I(x)$, with $G(x)$ and $H(x)$ a quadrature pair (i.e. $\mathcal{G}(k) = \text{sign}(k)\mathcal{H}(k)$).



Complex Polar Representation:

$$
\begin{aligned}
(G(x) + iH(x)) * I =& g(x) + ih(x) \\
=& e^{i\phi(x)} A(x).
\end{aligned}
$$

For a bandpass $G(x)$ and $H(x)$:

- **Local Energy:** $A^2(x)$ is a low-pass (slowly varying) signal.
- **Local Phase:** $\phi(x) \approx \phi_0 + k_0(x - x_0)$ provides a roughly linear progression.

Example: G2 and H2

An approximate quadrature pair is given by [Freeman et al, '91]:

$$G2(x, y; \sigma) = [0.9213((x/\sigma)^2 - 1)]e^{-(x^2+y^2)/(2\sigma^2)},$$

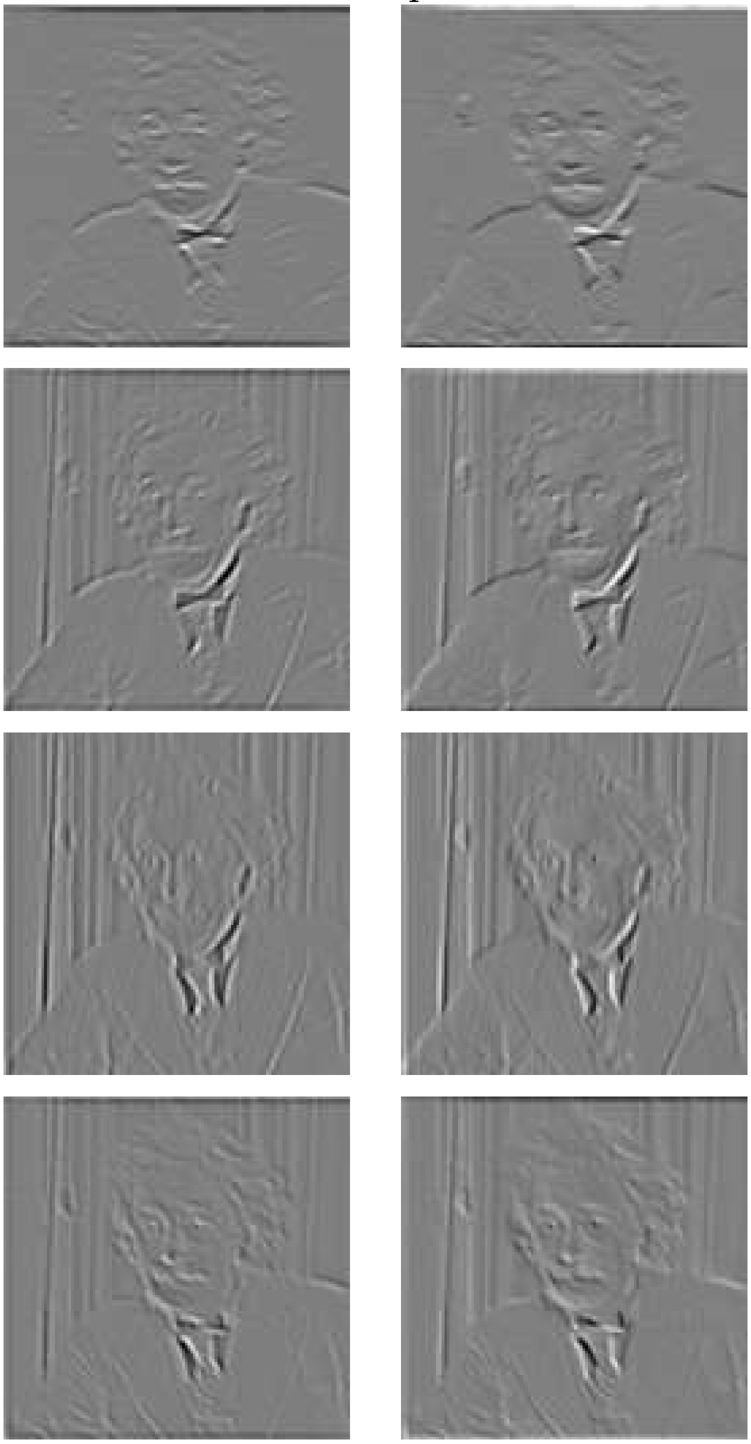
$$H2(x, y; \sigma) = [(x/\sigma)(0.3458(x/\sigma)^2 - 1.559)]e^{-(x^2+y^2)/(2\sigma^2)},$$

Here $G2(x, y; \sigma) \propto \frac{\partial^2}{\partial x^2} e^{-(x^2+y^2)/(2\sigma^2)}$.

An image pyramid using G2 and H2.

- Define the scale using the wavelength, λ, for the most sensitive frequency in the fourier transform. Use $\sigma = \lambda/3.918$.
- Use 4 scales, with wavelengths (say) $\lambda = 4, 8, 16,$ and 32 pixels.
- Spatially rotate $G2(x, y)$ and $H2(x, y)$ by 0, 45, 90, 135 degrees.
- Sample each pair at the rate $\lambda/4$ in each direction.
- Efficient implementation uses separable filters.
- The orientation of the filters is continuously steerable from 7 real-valued basis filters (or 4 sampled orientations).

G2 $\lambda = 8.0$ pixels H2



128×128 images

G2 $\lambda = 16.0$ pixels H2

64×64 images


Amp
$\lambda = 4.0$ pixels
Phase
$\lambda = 8.0$ pixels
$\lambda = 16.0$ pixels
$\lambda = 32.0$ pixels

References:

1. C.S. Burrus, R.A. Gopinath, and H. Guo, Introduction to Wavelets and Wavelet Transforms, Prentice Hall.
2. W T Freeman and E H Adelson. The design and use of steerable filters. IEEE Pat. Anal. Mach. Intell., 13(9), 1991, pp.891-906.
3. Olshausen BA, Field DJ (1996). Emergence of simple-cell receptive field properties by learning a sparse code for natural images. Nature, 381: 607-609.
   `URL: /redwood.ucdavis.edu/bruno/papers.html`
4. E P Simoncelli, W T Freeman, E H Adelson, and D J Heeger. Shiftable multi-scale transforms, IEEE Trans. Information Theory, 38(2), 1992, pp.587-607.