CS 458 / 658: Computer Security and Privacy

Module 6 - Data Security and Privacy

Part 3 - Differential privacy

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Outline

We are being too honest...

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Q: How about we add noise to the query response?

Formalize our setup

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- The database curator has access to the full database.
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- The data analyst consumes the data by asking a series of queries to the curator. Each query is denoted as S and the curator provides a response to query S with R_S.
  The analyst may be honest or malicious.
- The way in which the curator responds to queries is called the mechanism. Formally, $M : S \to R_S$. We'd like a mechnism that
  - gives statistically useful responses but
  - avoids leaking sensitive information about individuals.

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Attack setup

We consider the database to be a collection of n records

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Each record d_i may consist of k attributes. For simplicity, we assume that the adversary already knows $k-1$ attribute for all records and the only attribute unknown to the adversary is a single bit.

$$D = \left[\begin{array}{cccc|c} a_{\{1,1\}} & a_{\{1,2\}} & \cdots & a_{\{1,k-1\}} & b_1 \\ a_{\{2,1\}} & a_{\{2,2\}} & \cdots & a_{\{2,k-1\}} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{\{n,1\}} & a_{\{n,2\}} & \cdots & a_{\{n,k-1\}} & b_n \end{array}\right]$$

Attack setup example

Name	ZIP	DOB	COVID
Alice	K8V 7R6	5/2/1984	1
Bob	V5K 5J9	2/8/2001	0
Charlie	V1C 7J2	10/10/1954	1
David	R4K 5T1	4/4/1944	0
Eve	G7N 8Y3	1/1/1980	1
. . . 995 more entries . . .			

Threat model

The attacker is allowed to ask aggregated queries, and perhaps the most basic type of aggregate query in this case is a counting query, i.e., how many records in D that satisfies a condition $C(a_{\{*,1\}}, a_{\{*,2\}}, \ldots, a_{\{*,k-1\}})$ have their secret bit set to 1?

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For example: How many rows satisfying condition `(Name = "David" OR DOB > 1980)` have `COVID = 1`.

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For example: How many rows satisfying condition `(Name = "David" OR DOB > 1980)` have `COVID = 1`.

The key point is, the adversary is allowed to pick arbitrary rows in the database using their background knowledge to formulate queries. Formally, $S \in \{0,1\}^n$. An example is $S = [0, 1, 1, 1, \dots, 0]$

Curator mechanism

Upon receiving a query S, the curator will first calculate the true answer $A(S) = S \times [b_1, b_2, \ldots, b_n]$.

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Upon receiving a query S, the curator will first calculate the true answer $A(S) = S \times [b_1, b_2, \ldots, b_n]$.

$$R_S = A(S) + E$$

And subsequently add a random noise E to the true answer.

The inefficient attack

Theorem: If the analyst is allowed to ask 2^n queries to a dataset of n users, and the curator adds noise with some bound E, then based on the results, the adversary can reconstruct the database in all but at most $4E$ positions.

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Algorithm:

- For an attacker, there are only 2^n database candidates.
- For each candidate database $C \in \{0,1\}^n$, if there exists a query S such that $|\Sigma_{i \in S} C[i] - R_S| > E$, rule out C.
- Any database candidate not ruled out (C) differs with the actual database (D) by $4E$ at max.

The inefficient attack proof

Proof: Any database candidate not ruled out (C) differs with the actual database (D) by $4E$ at max

Consider query $I_0 \leftarrow \{i|D[i] = 0\}$, we know that

$$|\Sigma_{i \in I_0} C[i] - R_{I_0}| \leq E, |\Sigma_{i \in I_0} D[i] - R_{I_0}| \leq E, \implies \Sigma_{i \in I_0} |C[i] - D[i]| \leq 2E$$

Consider query $I_1 \leftarrow \{i|D[i] = 1\}$, we know that

$$|\Sigma_{i \in I_1} C[i] - R_{I_1}| \leq E, |\Sigma_{i \in I_1} D[i] - R_{I_1}| \leq E, \implies \Sigma_{i \in I_1} |C[i] - D[i]| \leq 2E$$

The efficient attack

Theorem: If the analyst is allowed to ask $O(n)$ queries to a dataset of n users, and the curator adds noise with some bound $E = O(\alpha\sqrt{n})$, then based on the results, a computationally efficient adversary can reconstruct the database in all but at most $\Theta(\alpha^2 n)$ positions.

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Differential privacy, on the other hand, is a definition on whether a mechanism is private.

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So..., more noise maybe?

We add more noise such that the adversary cannot reconstruct the database. But how much more is more?

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Well, that depends on what your privacy goal is.

An informal privacy goal

Consider a setting where

- I hand in my data to a database D (which is trusted),
- an algorithm A runs over D and releases a set of data T,
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This privacy notion makes no assumption about what background knowledge the adversary might possess:

- If the adversary does not know whether I am in the database, it won't know that either after seeing the result.
- If the adversary already knows whether I am in the database, it won't know more about the secret values I supplied.

An example from the attacker's perspective

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Q: How can you infer whether Alice is enrolled in CS458 or not?

The attack

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A: For a single response, we either get

- $63 \hookleftarrow \frac{C_{29}^{2}}{C_{30}^{3}} = 10\%$
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A: For a single response, we either get

- $63 \hookleftarrow \frac{C_{29}^2}{C_{30}^3} = 10\%$
- $50 \hookleftarrow$ otherwise

For all 5 responses, the chance of getting at least one 63 is
$1 - (1 - \frac{C_{29}^2}{C_{30}^3})^5 = 40.95\%!$

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Informally, the DP notion requires any single element in a dataset to have only a limited impact on the output.

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Background knowledge 1: You know that Alice is a top-performer and always gets ≥ 90 in course scores.

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Algorithm: You are given an algorithm that

- allows you to make 5 queries,
- each query returns the average score of 3 randomly selected students (out of 30 scores in total) plus a random value

Demo time (dp-demo.py)

The data collectors' argument

... on trying to persuade you to join a differentially private survey:

You will not be affected, adversely or otherwise, by allowing your data to be used in any study or analysis, no matter what other studies, data sets, or information sources, are available.

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You will not be affected, adversely or otherwise, by allowing your data to be used in any study or analysis, no matter what other studies, data sets, or information sources, are available.

But this is only true if they tell you what algorithm they use to release your data and you have verified that their algorithm is indeed differentially private.

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- *Unbounded DP*: D_1 and D_2 are neighboring if D_2 can be obtained from D_1 by adding or removing one element
- *Bounded DP*: D_1 and D_2 are neighboring if D_2 can be obtained from D_1 by replacing one element

ϵ-differential privacy

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Definition:
A mechanism $M : X \to Y$ is ϵ-differentially private (ϵ-DP) if for any two neighboring databases $D_1 : X$ and $D_2 : X$:

$$\forall T \subseteq Y, \quad \Pr[M(D_1) \in T] \leq e^{\epsilon} \Pr[M(D_2) \in T]$$

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In the CS458 grades example, for a single query,

- $M : \{\text{Name} \times [0 - 100]\} \to [0 - 100]$
- $T : [60 - 100]$
- $\Pr[M(D_1) \in T] = 10\%$
- $\Pr[M(D_2) \in T] = 0\%$

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Definition (Wrong):
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Suppose we have:

- $\epsilon = 0.01$
- $\Pr[M(D_1) \in T] = 0.005$
- $\Pr[M(D_2) \in T] = 0.001$

- $\epsilon = 0.01$
- $\Pr[M(D_1) \in T] = 0.96$
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Definition (Better):
A mechanism $M : X \to Y$ is ϵ-differentially private (ϵ-DP) if for any two neighboring databases $D_1 : X$ and $D_2 : X$:

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It does not make sense for ϵ to be < 1 or too large.

ϵ-differential privacy

Definition (Almost):
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NOTE: for small ϵ, $e^\epsilon \approx 1 + \epsilon$ by Talor series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Safety against post-processing

Theorem: Suppose mechanism $M : X \to Y$ is ϵ-differentially private. Then, for any mechanism $A : Y \to Z$, we have that $A \circ M : X \to Z$ is also ϵ-differentially private.

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Once the data is privatized, it can't be "un-privatized"

Compositional privacy

Theorem: Given

- $M_1 : X \to Y_1$ being ϵ_1-DP, and
- $M_2 : X \to Y_2$ being ϵ_2-DP.

We define a new mechanism $M : X \to Y_1 \times Y_2$ as $M(X) = (M_1(X), M_2(X))$. Then M is $(\epsilon_1 + \epsilon_2)$-DP.

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This has a gossip analogy:

- If A tells you something (potentially with noise),
- and then B tells you some other things (again, with noise).

At the end of the day you might have learned more information by combining them together.

Group privacy

Theorem: Suppose mechanism $M : X \to Y$ is ϵ-differentially private. Suppose D_1 and D_2 are two datasets which differ in exactly k positions. Then:

$$\forall T \subseteq Y, \quad \Pr[M(D_1) \in T] \leq e^{k\epsilon} \Pr[M(D_2) \in T]$$

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If you need to hide the "effect" if a whole group, you need to prepare a larger privacy budget.

Outline

Sensitivity

Q: How much noise to add?

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Definition: given a query processing function $f : X \to \mathbb{R}^k$, the ℓ_1-sensitivity of f is defined as:

$$\Delta_1^f = \max_{D_1 \sim D_2} \|f(D_1) - f(D_2)\|_1 \quad \text{where } D_1, D_2 \in X$$

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NOTE 1: The range of f is k-dimensional

NOTE 2: ℓ_1-sensitivity is $\|\vec{x_1} - \vec{x_2}\|_1 = \sum_i |\vec{x_1}[i] - \vec{x_2}[i]|$

Sensitivity w/ one pair of neighboring databases

D1 with Alice enrolled:

- Alice: 90
- Everyone else (29 of them): 50

D2 with Alice not enrolled:

- Everyone (30 of them): 50

Algorithm: You are allowed to make a query that returns the average score of this course.

Q: What is the ℓ_1-sensitivity here?

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Algorithm: You are allowed to make a query that returns the average score of this course.

Q: What is the ℓ_1-sensitivity here?
A: $|\text{Avg}(D_1) - \text{Avg}(D_2)| = 1.33$

Sensitivity w/ more database candidates

Q: What if we don't know the scores?

Suppose we only know that each student's score $\in [0 - 100]$, and

- (in bounded DP): there are 30 students enrolled
- (in unbounded DP): there are 29 or 30 students enrolled

Algorithm: You are allowed to make a query that returns the average score of this course.

Q: What is the ℓ_1-sensitivity here?

Sensitivity w/ more database candidates - bounded

Suppose we only know that each student's score $\in [0 - 100]$, and there are 30 students enrolled in the course.

Algorithm: You are allowed to make a query that returns the average score of this course.

$$
\begin{aligned}
\ell_1 &= \max\left(\left|\frac{\sum_{29 \text{ students}} + k_1}{30} - \frac{\sum_{29 \text{ students}} + k_2}{30}\right|\right) \\
&= \frac{1}{30}\max(|k_1 - k_2|) \\
&= \frac{1}{30} \times 100 \qquad \hookleftarrow (k_1 = 0 \wedge k_2 = 100) \vee (k_1 = 100 \wedge k_2 = 0) \\
&= \frac{10}{3}
\end{aligned}
$$

Sensitivity w/ more database candidates - unbounded

Suppose we only know that each student's score $\in [0 - 100]$, and there are either 29 or 30 students enrolled in the course.

Algorithm: You are allowed to make a query that returns the average score of this course.

$$\begin{aligned}
\ell_1 &= \max(|\frac{\sum_{29 \text{ students}}}{29} - \frac{\sum_{29 \text{ students}} + k}{30}|) \\
&= \max(|\frac{\sum_{29 \text{ students}}}{29 \times 30} - \frac{k}{30}|) \\
&\xrightarrow{\text{case1}} \max(\frac{\sum_{29 \text{ students}}}{29 \times 30}) - \min(\frac{k}{30}) \\
&\xrightarrow{\text{case2}} \max(\frac{k}{30}) - \min(\frac{\sum_{29 \text{ students}}}{29 \times 30}) \\
&= \frac{10}{3} \text{ for both cases}
\end{aligned}$$

Laplace distribution

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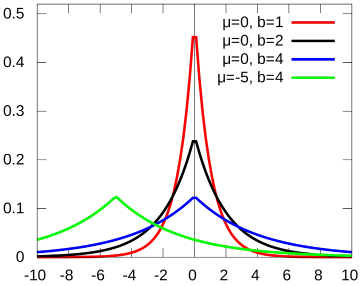
- Usually, for DP, we set $\mu = 0$, so you may see Lap(b) which is essentially Lap(0, b)
- Lap(μ, b) has variance $\sigma^2 = 2b^2$
- As b increases, the distribution becomes more flat



Laplace mechanism

Definition: Let $f : X \to \mathbb{R}^k$ is the function that calculates the "true" value of a query. The Laplace mechanism is defined as:

$$M(D) = f(D) + (Y_1, Y_2, \cdots, Y_k)$$

where Y_i are independent and identically distributed (i.i.d) random variables sampled from Lap($\frac{\Delta_1^f}{\epsilon}$)

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Demo time (average-demo.py)

Does the Laplace mechanism work in our example?

Let's first update the PDF by replacing $b = \frac{\Delta}{\epsilon}$:

$$\Pr[x = v] = \frac{\epsilon}{2\Delta} \exp\left(\frac{-\epsilon|v - \mu|}{\Delta}\right)$$

For D_1, $\mu = 50$,

$$\Pr_1[x = 51.33] = \frac{\epsilon}{2\Delta} \exp\left(\frac{-\epsilon|51.33 - 50|}{\Delta}\right) = C \times e^{-0.1}$$

For D_2, $\mu = 51.33$,

$$\Pr_2[x = 51.33] = \frac{\epsilon}{2\Delta} \exp\left(\frac{-\epsilon|51.33 - 51.33|}{\Delta}\right) = C \times e^{-0.075}$$

$$\frac{\Pr_2[x = 51.33]}{\Pr_1[x = 51.33]} = \frac{C \times e^{-0.075}}{C \times e^{-0.1}} = e^{0.025} \approx 1.025$$

The Laplace mechanism is ϵ-DP

Proof:

- Let D_1 and D_2 be any neighboring databases
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$$\frac{\Pr[M(D_1) = z]}{\Pr[M(D_2) = z]} = \frac{\prod_{i=1}^{k} \frac{\epsilon}{2\Delta} \exp\left(\frac{-\epsilon}{\Delta} |f(D_1)[i] - z[i]|\right)}{\prod_{i=1}^{k} \frac{\epsilon}{2\Delta} \exp\left(\frac{-\epsilon}{\Delta} |f(D_2)[i] - z[i]|\right)}$$

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$$\frac{\Pr[M(D_1) = z]}{\Pr[M(D_2) = z]} \leq \exp(\epsilon)$$

Outline

Approximate differential privacy

Definition:
A mechanism $M : X \to Y$ is (ϵ, δ)-differentially private ((ϵ, δ)-DP) if for any two neighboring databases $D_1 : X$ and $D_2 : X$:

$$\forall T \subseteq Y, \quad \Pr[M(D_1) \in T] \leq e^{\epsilon} \Pr[M(D_2) \in T] + \delta$$

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This definition allows us to add a much smaller noise.

Local differential privacy

Local differential privacy (LDP) is a model of differential privacy with the added restriction that even if an adversary has access to the personal responses of an individual in the database, that adversary will still be unable to learn too much about the user's personal data.

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Example: Randomized response to a survey