Algorithmic reduction of polynomially nonlinear PDE systems to parametric ODE systems

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Abstract. Differential-elimination algorithms apply a finite number of differentiations and eliminations to systems of partial differential equations. For systems that are polynomially nonlinear with rational number coefficients, they guarantee the inclusion of missing integrability conditions and the statement of of existence and uniqueness theorems for local analytic solutions of such systems. Further, they are useful in obtaining systems in a form more amenable to exact and approximate solution methods.
Maple's `dsolve` and `pdsolve` algorithms for solving PDE and ODE often automatically call such routines during applications. Indeed even casual users of `dsolve` and `pdsolve` have probably unknowingly used Maple's differential-elimination algorithms.
Suppose that a system of PDE has been reduced by differential-elimination method to a system whose automatic existence and uniqueness algorithm has been determined to be finite-dimensional. We present an algorithm for rewriting the output as a system of parameterized ODE. Exact methods and numerical methods for solving ODE and DAE can be applied to this form.

Keywords: numerical analysis, partial differential equations, algebraic geometry, computer algebra, ordinary differential equations, DAE

1 Introduction

Maple has three powerful differential-elimination packages, including the RIF package [8], the DifferentialAlgebra [7] package and the Differential Thomas package [20].

As an illustrative example, used throughout this article, consider the system of PDE R given by:

$$\frac{\partial^2}{\partial x^2}u(x,y) - \frac{\partial^2}{\partial x \partial y}u(x,y) = 0, \left(\frac{\partial u(x,y)}{\partial y}\right)^2 + \frac{\partial}{\partial y}u(x,y) - u(x,y) = 0 \qquad (1.1)$$

We note that this is polynomially nonlinear with rational coefficients, so the above algorithms can be applied. We also note that the system has two equations for one unknown function $u(x, y)$ and so is over-determined. In particular, differentiating the first equation with respect to y and the second equation with respect to x will yield an integrability condition for the derivative $\frac{\partial^3}{\partial x^2 \partial y} u$. This results in a non-trivial integrability condition, and this system is one for which the above differential-elimination packages are useful.

In the abstract, we claimed that most casual users of Maple's `dsolve` and `pdsolve` had probably unknowingly also used such differential-elimination packages. Indeed such users usually apply `dsolve` and `pdsolve` to a single differential equation, which are not over-determined and have no nontrivial integrability conditions seemingly contradicting our claim.

However, consider the following ODE:

$$6\left(\frac{d}{dx} y(x)\right)\left(\frac{d^2}{dx^2} y(x)\right) + 2y(x)\left(\frac{d^3}{dx^3} y(x)\right) + y(x)^2 = 0 \tag{1.2}$$

When we use the Maple commands `dsolve(DE)` and `infolevel[rifsimp] := 4`, we see that multiple calls are made the differential elimination routine `rifsimp`. At first sight, this is puzzling since the DE is not over-determined and has no non-trivial integrability conditions.

However, an important class of integration methods for differential equations is based on finding infinitesimal Lie symmetry vector fields $\xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$ leaving the DE invariant. For background on symmetry methods see [3] and [6]. Applying the Maple command `[PDEtools]DetermingPDE` with the option `integrabilityconditions := false` to the DE yields an over-determined system of 9 PDE for ξ, η, the coefficients of the symmetry vector fields leaving the DE invariant. The first 4 (shortest) equations of that system are:

$$\left[\frac{\partial}{\partial y}\xi = 0, \quad \frac{\partial^2}{\partial y^2}\xi = 0, \quad 2y\frac{\partial^3}{\partial y^3}\xi + 6\frac{\partial^2}{\partial y^2}\xi = 0, 6y\frac{\partial^2}{\partial x^2}\xi - 6y\frac{\partial^2}{\partial x \partial y}\eta - 6\frac{\partial}{\partial x}\eta = 0, \cdots\right] \tag{1.3}$$

Such symmetry determining systems are linear in their coefficients, and usually over-determined. Differential-elimination methods are natural for such problems and have proved to be standard tools for simplifying such systems. Applying `rifsimp` to the above system yields RIF form given by:

$$\left[\frac{\partial^3}{\partial x^3}\eta = \frac{y\left(\frac{\partial}{\partial y}\eta\right)}{2} - \frac{\eta}{2}, \frac{\partial^2}{\partial y \partial x}\eta = -\frac{\frac{\partial}{\partial x}\eta}{y}, \frac{\partial^2}{\partial y^2}\eta = \frac{-y\left(\frac{\partial}{\partial y}\eta\right) + \eta}{y^2}, \frac{\partial}{\partial x}\xi = 0, \frac{\partial}{\partial y}\xi = 0\right] \tag{1.4}$$

The `initialdata` algorithm yields that the solution space is finite-dimensional with 5 dimensional initial data given by

$$\left[\eta(x_0, y_0) = C_1, \eta_x(x_0, y_0) = C_2, \eta_y(x_0, y_0) = C_3, \eta_{xx}(x_0, y_0) = C_4, \xi(x_0, y_0) = C_5\right] \tag{1.5}$$

Additionally, the Lie Algebra of vector fields package (LAVF) enables the structure of the 5-dimensional Lie symmetry algebra to be computed directly from the RIF-form and the initial data. The Lie algebra structure is:

$$[X_1, X_2] = \frac{X_5}{2}, [X_1, X_3] = X_1 - \frac{X_3}{y}, [X_1, X_4] = -\frac{X_4}{y}, [X_1, X_5] = -\frac{X_5}{y}, [X_2, X_3] = \frac{yX_5}{2},$$
$$[X_2, X_4] = X_1 - \frac{X_3}{y}, [X_2, X_5] = X_4, [X_3, X_4] = -X_4, [X_3, X_5] = -X_5$$

where $y = y_0 \neq 0$ is a constant. Here we can use the commands to determine the dimension of the the derived algebra $\dim DerivedAlgebra(L) = 3$. In particular an r-th order ODE is linearizable iff $\dim DerivedAlgebra(L) = r$ and $DerivedAlgebra(L)$ is abelian. Applying these results and the algorithms given in Mohammadi, Reid and Huang [21] and Lyakhov, Gerdt and Michels [12] shows that the ODE is linearizable and with linearizing transformation $\hat{u} = u^2$, $\hat{x} = x$ and target linear ODE $\left(\frac{d}{d\hat{x}}\right)^3 \hat{u} + \hat{u} = 0$.

2 Reduction of systems of PDE with finite-dimensional solution spaces to parameterized ODE

The defining systems for symmetries of ODE and PDE are often over-determined as discussed in the introduction. Consequently as discussed in the introduction, differential-elimination algorithms have become essential tools for the determination of symmetries and mappings of differential equations. Such algorithms are also important in the analysis of differential equations with constraints (so-called DAE) which arise naturally from modeling environments such as `MapleSim` and `SystemModeler`.

For an algorithmic approach we need to exploit the algorithmic existence and uniqueness results from differential-elimination algorithms for polynomially nonlinear differential systems with rational coefficients. Here we exploit the results for the rifsimp algorithm. See Rust, Reid and Wittkopf [22] for details of the existence and uniqueness results.

We now give a brief outline of our algorithm and its justification for reducing systems of differential equations with finite-dimensional solution spaces. A more detailed exposition will be given elsewher. Let R denote an exact system of polynomially nonlinear PDE with independent variables x and dependent variables u and rational coefficients and let $\prec$ be a ranking of derivatives [16].

Given $\prec$ the RIF algorithm applies a finite number of differentiations and eliminations to R outputting a finite number of cases labeled by j with an associated local existence and uniqueness theorem. Each case consists of a system of equations E_j, and a system of inequations I_j. Let $Prin(E_j)$ be the leading derivatives of E_j with respect to $\prec$. Then the number of free parameters in solutions of E_j is finite, it is determined locally near x^0 by a finite list of parametric derivatives of u. Considering this list as new dependent variables v for R_j near x^0, R_j can be rewritten in first-order form $\frac{\partial}{\partial x_i} v = f_i(x, v)$, $h(x, v) = 0$. Thus, R is rewritten as a system of parametrized ODE on a constraint.

Example 1

If we apply the RIF algorithm to R, then provided $2\frac{\partial}{\partial y}u(x, y) + 1 \neq 0$, the leading linear system of RIF(R) is

$$LeadingLinear = \left[\frac{\partial^2}{\partial x^2}u(x, y) = \frac{\frac{\partial}{\partial x}u(x, y)}{2\frac{\partial}{\partial y}u(x, y) + 1}, \frac{\partial^2}{\partial y \partial x}u(x, y) = \frac{\frac{\partial}{\partial x}u(x, y)}{2\frac{\partial}{\partial y}u(x, y) + 1}, \frac{\partial^2}{\partial y^2}u(x, y) = \frac{\frac{\partial}{\partial y}u(x, y)}{2\frac{\partial}{\partial y}u(x, y) + 1}\right],$$

and the leading nonlinear system of RIF(R) is

$$LeadingNonlinear = \left[\left(\frac{\partial}{\partial x}u(x, y)\right)\left(\frac{\partial}{\partial y}u(x, y)\right) - \left(\frac{\partial}{\partial x}u(x, y)\right)^2 = 0, \left(\frac{\partial}{\partial y}u(x, y)\right)^2 + \frac{\partial}{\partial y}u(x, y) - u(x, y) = 0\right].$$

Note that RIF also computes all possible splitting on coefficients of the leading linear system. Here, that yielded two cases $2\frac{\partial}{\partial y}u(x, y) + 1 \neq 0$ and $2\frac{\partial}{\partial y}u(x, y) + 1 = 0$. The second leads to a branch with no solutions, so that case is discarded.

The key to decoupling the RIF form above into x derivatives and y derivatives is to compute the parametric derivatives of the leading linear PDE. First, the leading derivatives of the leading linear PDE are computed, yielding

$$LeadingDerivatives = \left[\frac{\partial^2}{\partial x^2}u(x, y), \frac{\partial^2}{\partial y \partial x}u(x, y), \frac{\partial^2}{\partial y^2}u(x, y)\right].$$

Next, the complementary set of parametric derivatives is computed. These are all the derivatives that are not derivatives of the leading derivatives. They are:

$$\mathscr{P} = \left[u(x, y), \frac{\partial}{\partial x}u(x, y), \frac{\partial}{\partial y}u(x, y)\right].$$

Relabelling these parametric derivatives as new dependent variables yields

$$\left[u = u(x, y), u_x = \frac{\partial}{\partial x}u(x, y), u_y = \frac{\partial}{\partial y}u(x, y)\right].$$

Notice for simplicity of notation; we have used u_x and u_y to denote new dependent variables. Computing RIF form with respect to the original system yields the ODE system with respect to x:

$$\left[\frac{\partial}{\partial x}u = u_x, \frac{\partial}{\partial x}u_x = \frac{u_x}{2u_y + 1}, \frac{\partial}{\partial x}u_y = \frac{u_x}{2u_y + 1}\right],$$

and the ODE system with respect to y:

$$\left[\frac{\partial}{\partial y}u = u_y, \frac{\partial}{\partial y}u_x = \frac{u_x}{2u_y + 1}, \frac{\partial}{\partial y}u_y = \frac{u_y}{2u_y + 1}\right],$$

where these ODE with respect to x and y share the constraint:

$$u_x u_y - u_x^2 = 0,$$

and the inequation $2u_y + 1 \neq 0$.

Example 2

We apply the algorithm to the RIF-form determining system given in (1.4) and corresponding finite-dimensional initial data given in (1.5). The list of parametric derivatives is

$$\xi, \eta, \eta_x, \eta_y, \eta_{xx}$$

Then taking x derivatives of this list and simplifying with respect to the RIF form (1.4) yields the system of ODE with respect to x with y regarded as a parameter:

$$\left[\frac{\partial}{\partial x}\xi = 0, \frac{\partial}{\partial x}\eta = \eta_x, \frac{\partial}{\partial x}\eta_x = \eta_{xx}, \frac{\partial}{\partial x}\eta_y = -\frac{\eta_x}{y}, \frac{\partial}{\partial x}\eta_{xx} = \frac{y\eta_y}{2} - \frac{\eta}{2}\right] \tag{2.1}$$

Similarly taking y derivatives of this list and simplifying with respect to the RIF form (1.4) yields the system of ODE with respect to y with x regarded as a parameter:

$$\left[\frac{\partial}{\partial y}\xi = 0, \frac{\partial}{\partial y}\eta = \eta_y, \frac{\partial}{\partial y}\eta_x = -\frac{\eta_x}{y}, \frac{\partial}{\partial y}\eta_y = \frac{-y\eta_y + \eta}{y^2}, \frac{\partial}{\partial y}\eta_{xx} = -\frac{\eta_{xx}}{y}\right] \tag{2.2}$$

3 Discussion and Applications

The above computations have been automated in the RIF package available in distributed Maple, and could also be implemented in other symbolic differential elimination packages.

3.1 Analytical solutions

Reduction of PDE to ODE has obvious advantages because of the breadth of available ODE methods. Indeed, early examples of such computations date back to classical differential geometers and, in particular, Cartan [9], whose exterior differential systems naturally express the ODE-like character of such overdetermined PDE systems.

3.2 Numerical solutions for exact polynomially nonlinear PDE

The numerical solution of such PDE systems for exact polynomially nonlinear PDE, via the reduction method we outlined above, involves the solution of ordinary differential equations on manifolds (or so-called Differential Algebraic Equation - DAE). The RIF algorithm reduces the involved DAE to the index one case. A point with the initial data $v(x^0) = v^0$ determines a unique local solution. Then a DAE solver can approximate this solution along a curve through this point. Points on this solution curve can then be used as initial values for the DAE to approximate solutions and build, by iteration, an approximation of the local solution to $v(x^0) = v^0$. Details will be given elsewhere.

3.3 Numerical Polynomial Algebra

A key area of the conference is polynomial and matrix algebra. Indeed suppose that we are considering polynomial systems in a ring $\mathbb{Q}$ in the indeterminates $x_1, x_2, \cdots, x_n$. Then ideal computations in the polynomial ring $\mathbb{Q}[x_1, \cdots, x_n]$ could equivalently be done in the differential ring $\mathbb{Q}[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}]$ via the map $x_j \leftrightarrow \frac{\partial}{\partial x_j}$. Indeed, using this mapping, the methods described in our article correspond to eigenvalue-matrix methods for solving 0-dimensional polynomial systems.

For example, Michałek and Sturmfels [4] consider the polynomial ideal $\langle x^3 - yz, y^3 - xz, z^3 - xy\rangle$ and associate this with the differential ideal $\langle \left(\frac{\partial}{\partial x}\right)^3 - \frac{\partial}{\partial y}\frac{\partial}{\partial z}, \left(\frac{\partial}{\partial y}\right)^3 - \frac{\partial}{\partial x}\frac{\partial}{\partial z}, \left(\frac{\partial}{\partial z}\right)^3 - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\rangle$. Following the algorithm of our article, yields a 27-dimensional ideal, and yields 3 ODE systems of the form:

$$\frac{\partial}{\partial x}\mathbf{v} = X\mathbf{v}, \frac{\partial}{\partial y}\mathbf{v} = Y\mathbf{v}, \frac{\partial}{\partial z}\mathbf{v} = Z\mathbf{v}$$

where X, Y, Z are sparse 27×27 matrices that mutually commute, and $\mathbf{v}$ is a 27-dimensional vector. Solving these elementary ODE systems yields the same result as in the text [4]. As the authors describe the different representations (here differential versus polynomial) can yield valuable insights.

3.4 Numerical solutions for approximate polynomially nonlinear PDE

Many applications for PDE involve approximate parameters. Thus applying exact differential-elimination algorithms such as the RIF algorithm, can be subject to issues such as pivoting on small coefficients. The strategy we outlined above enables the solution in terms of ODE prolongations. It is natural to ask whether we can exploit such ODE prolongations without using the RIF algorithm. Indeed we can take a system of PDE, and for example, substitute derivatives up to some given order as new dependent variables. Then, potentially, the ODE prolongation with respect to each independent variable could be computed using methods such as those of Pantiledes [24], Pryce [25, 26] and Yang, Wu and Reid [23]. If all these prolongations are finite-dimensional, then numerically, a bound can be found for in which to search for additional integrability conditions, in an incremental way, without the strong ordered (and unstable) elimination of the exact methods. Reduction to ODE on constraints also potentially enables more efficient prolongation methods to be developed for approximate systems of PDE, based on ODE prolongation structures.

Acknowledgements

We thank the referees for their helpful comments.

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Revision

1. reviewer 1:
   (a) The abstract and other parts of the paper have been rewritten, extended and examples added. In particular this clarifies the classes of PDE that can benefit from our approach.
   (b) Our revision addresses the comment about ”provided $2u_y + 1 \neq 0$ ”.
   (c) The first order form has integrability conditions that are satisfied. In the case concerned these imply the commutativity of the matrices for all systems of polynomials with finitely many solutions, regardless of whether they are invariant under permutation of the variables. The invariance under permutation is an interesting discrete symmetry - that merits further investigation.
2. reviewer 2:
   (a) Intuitively, one might expect that 2 PDEs for one unknown might have no solutions. However higher dimensional nature of PDE in their jet spaces $(x, y, u, u_x, ...)$ enables intersections to be nontrivial.
   (b) The abstract and other parts of the paper have been rewritten, extended, and examples added. This should address your comment about the RIF algorithm. The other comments have been addressed.
   (c) We explain what an exact system is.
   (d) We fix some typos.
3. reviewer 3:
   (a) We rewrite the abstract and give and include an introductory section, to address the above comments. Some brief comments about the RIF algorithm are also given, which is more accessible to a broad audience. In particular, it describes the advantages and some limitations of such algorithms.
   (b) We fix some typos.