Maple Transactions

Computing the bisector of two low degree curve segments

IBRAHIM ADAMOU, Université Dan Dicko Dankoulodo, Niger
MARIO FIORAVANTI, Universidad de Cantabria, Spain
LAUREANO GONZALEZ-VEGA, CUNEF, Spain
SEYDOU MOUSSA, Université Dan Dicko Dankoulodo, Niger

Abstract

The bisectors are geometric constructions which are common in several fields of applications such as CAD, CAGD, path generation, movement planning, the median axis, the computation of the Voronoi Diagram, structural biology, etc. In this work we present a new algorithmic approach for computing the exact representation for the bisector of two planar curve segments of degree 1 or 2. For two parametric curve segments $s_i(u)$, $u \in [0,1]$ and $s_j(t)$, $t \in [0,1]$ with their respective endpoints $A_i = s_i(0)$, $B_i = s_i(1)$, $A_j = s_j(0)$ and $B_j = s_j(1)$, and open segments c_i and c_j, such that $s_i = \{A_i, c_i, B_i\}$ and $s_j = \{A_j, c_j, B_j\}$, a process to compute the exact representation of their bisector will be established. This representation will derive from a combination of point-point bisector, point-curve bisector and curve-curve bisector representations.

The process involves, for each endpoint of one curve segment, determine, if possible, the position of the corresponding footpoint on the other curve segment and the corresponding bisector point. Then by fixing the curve segment s_j and sweeping each element of the curve segment s_i with the elements of s_j. This allows us to identify and properly store all components to obtain the exact bisector of two curve segments.

The resulting bisector may combine rational and semi-algebraic representations.

Keywords: Computer aided geometric design, Bisector curve, Curve segment, Parametrization, Algebraic representation.

Recommended Reference Format:

Ibrahim Adamou, Mario Fioravanti, Laureano Gonzalez-Vega, and Seydou Moussa. 2024. Computing the bisector of two low degree curve segments . *Maple Trans.* X, Y, Article Z (2024), 5 pages. https://doi.org/XXXXXXXXX

1 Introduction

Consider two geometric objects O_1 and O_2 in a metric space $\mathbb{R}^d, d \in \{2,3\}$, with respective parameterizations $\mathbf{O}_1(u), u \in I_1$ and $\mathbf{O}_2(v), v \in I_2$, where I_1 and I_2 are intervals or rectangles (eventually reduced to a point). Their bisector $\mathcal{B}(O_1, O_2)$ is defined as the equidistant set of points from the two objects, i.e.:

$$\mathcal{B}(O_1, O_2) = \left\{ B \in \mathbb{R}^d : \min_{u \in I_1} \|B - \mathbf{O_1}(u)\| = \min_{v \in I_2} \|B - \mathbf{O_2}(v)\| \right\}. \tag{1}$$

The computation of an algebraic representation for the bisector is not an obvious task from the definition (1). Indeed, even if the parametric object is given with a regular and proper parametrization, it should be noted that the distance function

$$d : (B, O) \mapsto \inf_{u \in I} \|B - \mathbf{O}(u)\|, \tag{2}$$

is not always differentiable with respect to the point B, and a minimum of the distance function could be achieved at more than one parameter value. To overcome these difficulties, the notion of untrimmed bisector is introduced (see [5, 6]).

Definition 1. *The untrimmed bisector of O_1 and O_2 is defined as the set of centres of spheres which are tangent to O_1 and O_2 simultaneously.*

Authors' addresses: Ibrahim Adamou, Université Dan Dicko Dankoulodo, Maradi, Niger, adamou.ibrahim@uddm.edu.ne; Mario Fioravanti, Universidad de Cantabria, Av los Castros, 48, Santander, Spain, mario.fioravanti@unican.es; Laureano Gonzalez-Vega, CUNEF, Madrid, Spain, laureano.gonzalez@cunef.edu; Seydou Moussa, Université Dan Dicko Dankoulodo, Maradi, Niger, adamou.ibrahim@uddm.edu.ne.

https://doi.org/XXXXXXXXX

This definition does not imply the same minimum distances measured from the two objects, in the presence of critical shapes on the objects (singular point, inflection point, self-intersection point). There are some extraneous parts that should be trimmed in order to achieve the true bisector.

For two regular plane rational curves s and r, with their parametrizations, respectively $s(u)$ and $r(t)$, the untrimmed bisector can be described as follow. A point $\mathbf{B} = (X, Y)^T \in \mathbb{R}^2$ is in the untrimmed bisector of the curves s and r if it satisfies the following system of equations (see [1, 4]):

- the point $\mathbf{B}$ is in the normal lines of s and r, at $s(u)$ and $r(t)$, respectively:
$$\langle (X,Y) - s(u), s'(u)\rangle = 0,\ \langle (X,Y) - r(t), r'(t)\rangle = 0, \tag{3}$$
where s' and r' denote derivatives.
- the point $\mathbf{B}$ is at equal distance from $s(u)$ and $r(t)$:
$$\langle (X,Y), 2\,(r(t) - s(u))\rangle + \|s(u)\|^2 - \|r(t)\|^2 = 0. \tag{4}$$

The Equations (3) can be written in matrix form as follows:
$$\mathbf{A}\,\mathbf{B} = \mathbf{V},\ \mathbf{A} = \begin{bmatrix} \mathbf{s}'_x(u) & \mathbf{s}'_y(u) \\ \mathbf{r}'_x(t) & \mathbf{r}'_y(t) \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \langle \mathbf{s}(u), \mathbf{s}'(u)\rangle \\ \langle \mathbf{r}(t), \mathbf{r}'(t)\rangle \end{bmatrix}. \tag{5}$$

Our goal is to compute a parametrization of the bisector of the curves s and r in terms of one parameter, either u or t. Our approach consists in:

- First solve the system (5) for $\mathbf{B}$ in terms of u, t, using Cramer's rule: For $\det(\mathbf{A}) \neq 0$, we have:
$$\mathbf{B}(u,t) = \mathbf{A}^{-1}\mathbf{V}, \tag{6}$$
and substituting $\mathbf{B}(u,t)$ in (4), we obtain the equation:
$$F(u,t) = \langle \mathbf{B}(u,t), 2(r(t) - s(u))\rangle + \|s(u)\|^2 - \|r(t)\|^2 = 0. \tag{7}$$
- Then, express one of the parameters, say u in terms of t, in the best possible way, from the equation (7). Since there might be more than one solution, we get
$$u_i = u_i(t),\ i = 1, \ldots, m.$$
- Finally, substitute u by $u_i(t)$ in $\mathbf{B}(u,t)$, for each solution, and obtain the parametrization of the untrimmed bisector of the form:
$$\mathbf{b}_i(t) = \mathbf{B}(u_i(t), t) = [x_i(t), y_i(t)]^T, \tag{8}$$
where $x_i(t), y_i(t)$ are in general non-rational.

For given values of the parameters u and t, a bisector point can be computed from the formula (6). Then for two corresponding footpoints $p_1(u_0) \in s$ and $p_2(t_0) \in r$, their corresponding bisector point is given by $B(u_0, t_0)$.

The bisector of a point and rational curve has a rational parametrization and the process of the computation and trimming extraneous part is given in ([5]).

The bisector of two planar rational curves is not a rational curve in general (see [6]). The algebraic representation is of high degree, the trimming extraneous process is not evident. An approximate representation can also be used in the nonrational case (see [4]). An algebraic approach is given in ([1, 2]) to compute an algebraic (rational and non rational) parametrization for the bisector for some particular curves. The trimming process is also shown.

For this work we propose an algorithmic method to calculate the exact representation for the bisector of two curve segments. The latter can be applied, among other things, two the characterization of the medial axis and Voronoi diagram for objects constructed from segments of curves see [3, 7].

The output of the method includes:

(1) The exact representation for the bisector of an endpoint A_i and open curve segment c_j denoted by $\mathcal{B}(A_i, c_j)$ is computed from the process in ([5]).

(2) The bisector of endpoints A_i and A_j denoted by $\mathcal{B}(A_i, A_j)$ is a half line or line segment which representation is trivial.

(3) The exact representation for the bisector of two open curve segments c_i and c_j denoted by $\mathcal{B}(c_i, c_j)$ is computed either from the formula (8), or from an implicit reprensentation $F(x, y) = 0$ (see [1]) of the bisector of the two corresponding curves:

$$\mathcal{B}(c_i, c_j) = \left\{(x, y) \in \mathbb{R}^2 : F(x, y) = 0, x_1 \leq x \leq x_2 \text{ and } y_1 \leq y \leq y_2\right\}$$

where $\mathcal{B}(c_i, c_j)$ is delimited by $B_1 = (x_1, y_1)$ and $B_2 = (x_2, y_2)$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

2 Bisector of two curve segments

In this section we consider s_i and s_j two curve segments of degree 1 or 2 and by $\star$ we denote 0 or 1 (i.e. $\star = 0$, or 1).

We have two possible configurations: a couple of curve segments with the same endpoints and one of disjoint curve segments.

The process to compute the bisector of a couple of curve segments involves, for each endpoint of s_i, to determine, if possible, the corresponding footpoint on s_j. These allow us to identify and properly store the various components to achieve the bisector. The determination of a footpoint will be done by computing the corresponding parametric values through the equation (7).

Proposition 1. *Let $s_i(u)$, $u \in [0, 1]$ and $s_j(t)$, $t \in [0, 1]$ be two curve segments. The set of solutions of the equation (7) on the border of the parameters domain $[0, 1] \times [0, 1]$, if non empty, corresponds to the couples of parameter values of an endpoint of s_i and its corresponding footpoint (might be more than one) on s_j.*

Let us introduce some technical definitions.

Definition 2. *(1) An endpoint of a segment is said Free-Point (FP) if it has no corresponding footpoint on the other segment sharing curve-curve bisector points.*

(2) An endpoint of a segment is said Non-Free-Point-type1 (NFP1) if it corresponds to solution couple $(u_0, \star)$ (resp. or $(\star, t_0)$) where u_0 (resp. or t_0), is a unique value corresponding to a unique footpoint on the other segment sharing curve-curve bisector points.

(3) An endpoint of a segment is said Non-Free-Point-type2 (NFP2) if it corresponds to solution couple $(u_0, \star)$ (resp. or $(\star, t_0)$) where u_0 (resp. or t_0) has two values, i.e. an endpoint with exactly two corresponding footpoints on the other segment sharing curve-curve bisector points.

We have the following theorem.

Theorem 1. *Let s_i and s_j be two disjoint curve segments of degree 1 or 2. The bisector of s_i and s_j is composed of at least one component and not more than 7 curve segments components.*

The following lemma is necessary in the proof of the theorem.

Lemma 1. *Let $s_i(u)$, $u \in [0, 1]$ and $s_j(t)$, $t \in [0, 1]$ be two curve segments with their respective endpoints $A_i = s_i(0)$, $B_i = s_i(1)$, $A_j = s_j(0)$, $B_j = s_j(1)$.*

(1) A FP point A_i of a curve segment s_i generates:

- *1 bisector component $\mathcal{B}(A_i, A_j)$, if the two curve segments are entirely on one side (of s_j) with respect to the line (A_i, A_j).*
- *2 bisector components $\mathcal{B}(A_i, A_j)$ and $\mathcal{B}(A_i, c_j)$, if a portion of c_j is on the other side of the line (A_i, A_j).*
- *3 bisector components $\mathcal{B}(A_i, A_j)$, $\mathcal{B}(A_i, c_j)$ and $\mathcal{B}(A_i, B_j)$, if a portion of c_j and the endpoint B_j are on the opposite side of s_j with respect to the line (A_i, A_j). The Figure 1 gives an illustration.*

(2) A NFP1 endpoint A_i of a curve segment s_i generates up to 3 bisector components: $\mathcal{B}(A_i, A_j)$, $\mathcal{B}(A_i, c_j)$ and $\mathcal{B}(A_i, B_j)$, where c_j is an open segment, and B_j is an endpoint of s_j.

(3) A NFP2 endpoint A_i of a curve segment s_i generates

(a) a single bisector component $\mathcal{B}(A_i, c_j)$ for portion of segment c_j bounded by the two corresponding footpoints of A_i,

(b) or the components of bisector $\mathcal{B}(A_i, c_j)$, $\mathcal{B}(A_i, A_j)$ and $\mathcal{B}(A_i, B_j)$, where c_j is an open segment, and B_j is an endpoint of s_j.

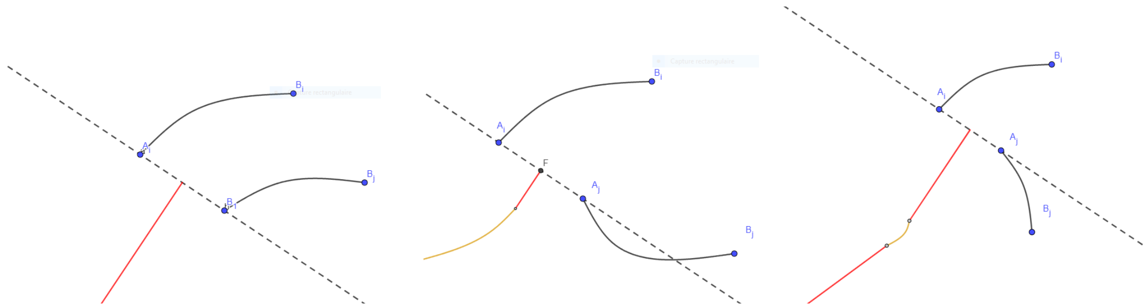

Fig. 1. FP configuration with 1, 2 and 3 component bisectors: point-point bisector in red color point-curve bisector in orange color

3 Effective computation of bisector of two curve segments

3.1 Stopping Condition

The Stopping Condition (see [7]) is the criterion giving the end of process in the search for components of the bisector. For a FP A_i of a curve segment s_i it allows to determine the elements of curve segment s_j that interact with it.
The criterion is detailed at (1) of the Lemma 1.

3.2 Computing process of the bisector of two curve segments

The process starts by computing the solutions of equation (7) w.r.t. $\partial\left([0,1]\times[0,1]\right)$:

$$\{(\star, t_0), (u_0, \star),\ u_0, t_0\ \in [0,1]\}.$$

(1) If the solutions of equation (7) is not empty then:

(a) If s_i and s_j have the same endpoints ($A_i = A_j$), then the endpoints B_i and B_j are either respectively FP and NFP1, or both FP, or NFP1, and $\mathcal{B}(s_i, s_j)$ is consisting respectively of:

(i) the components $\mathcal{B}(c_i, c_j)$, $\mathcal{B}(B_j, c_i)$ and $\mathcal{B}(B_i, B_j)$,

(ii) the components $\mathcal{B}(c_i, c_j)$ and $\mathcal{B}(B_i, B_j)$, or

(iii) the single component $\mathcal{B}(c_i, c_j)$. The Figure 2 is an illustration of this case.

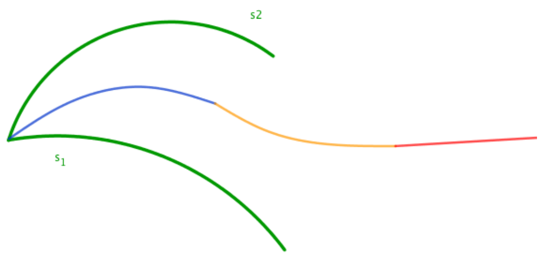

Fig. 2. The bisector of two curve segments with one common endpoint: $\mathcal{B}(c_i, c_j)$ in blue color, $\mathcal{B}(B_i, c_j)$ in orange color and $\mathcal{B}(B_i, B_j)$ in red color.

(b) If s_i and s_j are disjoint curve segments, we fix $s_i = \{A_i, c_i, B_i\}$ such that $x_{A_i} \le x_{B_i}, x_{A_j}, x_{B_j}$.

(i) We compute first $B(\star, t_0)$ and $B(u_0, \star)$ (delimiting the components $\mathcal{B}(c_i, c_j)$) from (6).

(ii) Then we scan s_j from the elements of s_i: successively we determine the bisector components linked to A_i, then to c_i and finally to B_i.

The linked components of each element of s_i are respectively connected in the order of the distance to the corresponding element of s_j starting with the farthest or the nearest. The Figure 3 gives an illustration.

(2) If the solutions of equation (7) is empty the bisector of the two curve segments is essentially made up of point-curve and point-point bisector components. We determine the bisectors components linked to A_i, then to c_i and finally to B_i. The Figure 4 gives an illustration.

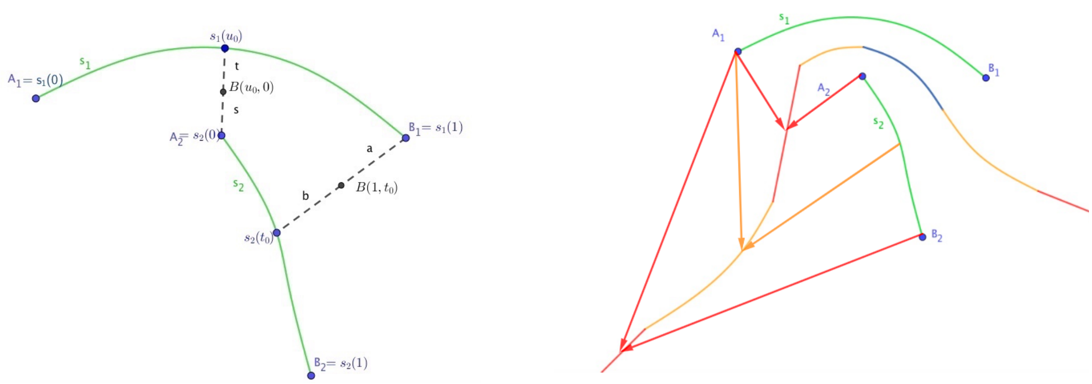

Fig. 3. The bisector points $B(u_0, 0)$ and $B(1, t_0)$ are computed. The scanning process from A_1: $\mathcal{B}(A_1, B_2)$, then $\mathcal{B}(A_1, c_2)$ and $\mathcal{B}(A_1, A_2)$

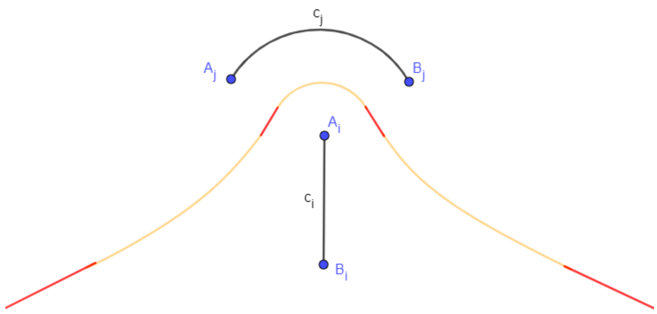

Fig. 4. The two endpoints of s_i are FP: point-point bisector in red color and point-curve in orange color

4 Conclusion

An algorithmic approach is presented to compute the exact representation of the bisector of two curve segments of degree 1 or 2. The computed representation is either a rational parametrization, or the combination of a rational parametrization and a semi-algebraic representation.

References

[1] Ibrahim Adamou: *Curvas y Superficies Bisectrices y Diagrama de Voronoi de una familia finita de semirrectas paralelas en* $\mathbb{R}^3$. PhD Thesis, Universidad de Cantabria, 2013.

[2] Ibrahim Adamou, Mario Fioravanti, Laureano Gonzalez-Vega and Bernard Mourrain: *Bisectors and Voronoi Diagram of a Family of Parallel Half-Lines.* In: Dokken, T., Muntingh, G. (eds) SAGA – Advances in ShApes, Geometry, and Algebra. Geometry and Computing, vol. 10 241–279, 2014 Springer, Cham.

[3] Oswin Aichholzer, Wolfgang Aigner, Franz Aurenhammer, Thomas Hackl, and Bert Jüttler and Margot Rabl: *Medial axis computation for planar free–form shapes.* CAD 41, 339–349, 2009.

[4] Gershon Elber and Myung–Soo Kim: *Computational Model for Nonrational Bisector Surfaces: Curve-Surface and Surface-Surface Bisectors.* Proc. GMP 2000. Theory and Applications, Hong Kong; China, pp 364-372, April 2000.

[5] Rida T. Farouki and John K. Johnstone: *The bisector of a point and a plane parametric curve.* CAGD 11, 117–151, 1994.

[6] Rida T. Farouki and John K. Johnstone: *Computing point/curve and curve/curve bisectors.* Proceedings of the 5th IMA Conference on the Mathematics of Surfaces, 327–354, Clarendon Press, 1994.

[7] Der Tsai Lee: *Medial Axis Transformation of a Planar Shape.* IEEE Transactions on Pattern Analysis and Machine Intelligence, PAMI–4, pp. 363–369, July 1982 , ISSN: 0162-8828, doi:10.1109/TPAMI.1982.4767267.