# Canonical forms for Polynomial Matrices

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May 20, 2024

In recent decades, the study of polynomial, rational matrix functions has become a prolific area of active an innovative research, and much of this involves computing the "structural data" (eigenvalues and minimal indices). Current studies of the structural data of polynomial and rational matrices are very much motivated by nonlinear eigenvalue problems (NEPs): given a matrix-valued function $F\colon \mathbb{C} \to \mathbb{C}^{n\times n}$, find $\lambda \in \mathbb{C}$ and $\mathbf{v} \in \mathbb{C}^n$ such that $F(\lambda)\mathbf{v} = \mathbf{0}$. Nonlinear eigenvalue problems have become increasingly relevant as new applications in physics, engineering, and systems theory (to name a few) have given rise to NEPs with interesting spectral structure [7, 9].

Some NEPs that are of particular interest include polynomial and rational eigenvalue problems (PEPs and REPs, resp.), and quadratic eigenvalue problems (QEPs) which have a multitude of applications on their own [12]. One reason for the special interest in polynomial and rational NEPs stems from the following method for solving NEPs. First approximate the matrix function $F(\lambda)$ in a compact domain $\Omega \subset \mathbb{C}$ by a polynomial matrix $P(\lambda)$ or a rational matrix $R(\lambda)$. Then, by a process called linearization, a matrix pencil $L(\lambda)$ is constructed with the same structural data as $P(\lambda)$ (or $R(\lambda)$). Finally, the structural data of $L(\lambda)$ is computed numerically and then translated to approximate solutions to the original NEP.

The *structural data* of a polynomial or rational matrix consists of zeros with multiplicities, poles with multiplicities (only for rational), and minimal indices. The zeros can be finite or infinite, and for a polynomial matrix, they are the eigenvalues. The poles of a rational matrix can also be finite or infinite, and for both polynomial and rational matrices, the minimal indices are non-negative integers associated with the left and right null spaces.

A natural question to ask in the study of polynomial and rational matrices is the *inverse problem*:

Given a list $\mathcal{L}$ of structural data, is there a polynomial matrix $P(\lambda)$ (or rational matrix $R(\lambda)$ if the structural data includes poles) with structural data given by $\mathcal{L}$, and if there is, can one be readily computed?

With all the new and evolving algorithms for computing the structural data of polynomial and rational matrices, it has become important to be able to build test matrices with prescribed structural data in order to stress test these algorithms. For generalized eigenvalue problems (GEPs), the Kronecker canonical form provides a starting point for building test pencils with known structural data. Thus the Holy Grail for researchers working on polynomial and rational inverse problems is a Kronecker-like canonical form for polynomial and rational matrices.

Canonical forms are an integral part of matrix theory. From the Jordan canonical form for square matrices under similarity and the Kronecker canonical form for matrix pencils under strict equivalence to the Smith canonical form for matrix polynomials under unimodular equivalence, the discovery of new canonical forms can lead to significant advances in new theory. We propose a new canonical form for strictly regular matrix polynomials under unimodular equivalence that not only has many of the features of existing canonical forms, but also has the property that the degree of the matrix polynomial is preserved. In addition, as a corollary, we get a new canonical form for skew-symmetric matrix polynomials under unimodular congruence.

It was established in [10] that the structural data of a polynomial matrix satisfies the equation:

$$\sum\{\text{e-val multiplicities}\} + \sum\{\text{minimal indices}\} = dr, \tag{1}$$

where $r$ is the rank and $d$ is the degree. This equation was dubbed in [5] the *index sum equation*, and it was shown that it is the only constraint for solving the polynomial inverse problem over $\mathbb{C}$. In 2019, we showed in [1] that the older rational index sum equation [14] is equivalent to its polynomial counterpart (1) and is the only constraint for solving the rational inverse problem over $\mathbb{C}$. While the results in [1, 5] provide a mechanism for producing solutions to the polynomial and rational inverse problems, the question of a Kronecker-like realization was left unanswered. The Kronecker canonical form has many valuable qualities, one of which is that it not only explicitly solves the generalized inverse problem, but it does so by producing a realization that is a direct sum of simple blocks from which the original structural data can be easily recovered without doing any numerical computations

Attempts have been made to emulate the Kronecker form for quadratic polynomial matrices in [3] and [4]; however, the number of required block types increases substantially, and it is no longer as straightforward to recover the partial multiplicities of the eigenvalues and minimal indices. Additionally, the structural data in the Kronecker form is separated into distict block types, one for finite eigenvalues, one for infinite eigenvalues, one for left minimial indices, and one for right minimal indices. However, the Kronecker-like quadratic forms fail to achieve this separation, and, in fact, the authors prove that such a separation is impossible, which is what leads to the dramatic increase in the number of block types.

In a parallel track, attempts have been made to emulate the Schur triangularization theorem for quadratic matrix polynomials in [13] and for general matrix polynomials in [11]. In 2021, we showed that every regular matrix polynomial over an *arbitrary field* can be block-triangularized with diagonal blocks that have an upper bound on their size [2].

In this talk we present a new strategy for constructing canonical forms for matrix polynomials that does not rely on the direct-sum-of-blocks approach of Kronecker. This new approach can also be seen as building a solution to the inverse problem that focuses on invariant polynomials instead of elementary divisors. When the original matrix polynomial has only finite eigenvalues (which we dubbed strictly-regular) we obtain a new canonical form under unimodular equivalence. In the general regular case, we obtain a new canonical form, but the equivalence transformations are unknown. When the original matrix polynomial is singular, our construction takes the form of a product of factors instead of direct sums of blocks.

These "product realizations", introduced in my doctoral thesis [8] but developed by Dopico, Makcey, and Van Dooren in an unpublished manuscript, solve the polynomial inverse problem as a product of three factors

$$P(\lambda) = L(\lambda)M(\lambda)R(\lambda), \tag{2}$$

where the left and right minimal indices are encoded in $L(\lambda)$ and $R(\lambda)$ respectively, while the partial multiplicities of the eigenvalues are encoded in the regular middle factor $M(\lambda)$. These factors are also constructed in such a way that the original data can be recovered from their respective factors without doing any numerical computations, one of the desirable properties of the Kronecker canonical form. These factors also benefit from "data sparsity"; that is, if the overall realization is $m \times n$, then each factor can be stored as and recovered from a data vector of length $\mathcal{O}(k)$ where $k = \max\{m, n\}$. When the strucutal data contains only eigenvalues and their multiplicities, the middle factor is the previously mentioned regular (or strictly regular when only finite eigenvalues) canonical form.

This work is also extended in [8] to solve the rational inverse problem as a similar product realization, but now of five factors

$$R(\lambda) = Z_L(\lambda)D_L(\lambda)M(\lambda)D_R(\lambda)Z_R(\lambda), \qquad (3)$$

where $Z_L(\lambda)$ and $Z_R(\lambda)$ encode the left and right minimal indices, $M(\lambda)$ encodes the finite poles and zeros along with their partial multiplicities, and $D_L(\lambda)M(\lambda)D_R(\lambda)$ encodes the infinite pole and zero multiplicities. The matrices $D_L(\lambda)$ and $D_R(\lambda)$ are both diagonal, and this realization benefits from the same nice properties as those of its polynomial counterpart in (2). As in the polynomial case, the middle factor $M(\lambda)$ can be interpreted as a canonical form under unimodular equivalence when the structural data contains only finite poles and zeros.

Another consideration that is made is that of structured matrix polynomials, e.g. skew-symmetric, alternating, palindromic. A Kronecker-like canonical form for palindromic quadratic matrix polynomials was presented in [4]. In the present work, a canonical form for skew-symmetric matrix polynomials of arbitrary degree is given, while alternating, symmetric, and palindromic are still under investigation.

Due to limited time, this talk will only discuss the polynomial inverse problem when the structural data contains only finite eigenvalues and their multiplicities. For such structural data, the product realization (2) simplifies to just the middle factor $M(\lambda)$. In this case, the realization is *strictly regular*; that is, regular with nonsingular lead coefficient. The realization $M(\lambda)$ can be constructed algorithmically, the amount of data needed to be stored is $\mathcal{O}(n)$ where $n$ is the size, and the original partial multiplicities of the eigenvalues can be recovered without any numerical computation. Such a realization is an ideal choice to use as a starting point to build test matrices for computational algorithms.

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