A REAL MECHANICAL GEOMETER

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Abstract. We introduce, first, the current performance of the programs GeoGebra and GeoGebra Discovery regarding diverse automatic reasoning tools features, that rely on computational algebraic geometry methods, mostly in the complex setting. Then we focus on the pending theoretical and algorithmic issues to extend (in the same technological and educational framework) such methods to deal with statements in the real algebraic geometry context.

Celebrating Lalo's 60th birthday,
after so many years of close friendship.
In recognition of his successful scientific contributions
and generous dedication to the academic community.

The Automated Geometer in GeoGebra and GeoGebra Discovery: current performance

The Automated Geometer in GeoGebra

Along the past decade we have worked towards the development of a kind of *Automated Geometer*([1,2,14,3,15,17]), initially implemented on the popular program GeoGebra.

GeoGebra (www.geogebra.org) is a freely available software, operative over smartphones, tablets and computers, or just accessible through a web page. It has been translated to circa 100 languages, and is used by more than 100 million persons all over the world, sharing among them over 1 million educational resources. It includes Computer Algebra (CAS) and Dynamic Geometry (DGS) features, and can be also used as versatile numerical and statistical calculator. See the above mentioned web for detailed information.

Let us remark that the general version of GeoGebra includes already some *automated reasoning* tools (using and implementing partially some of the algorithms mentioned in the previous references). For example, GeoGebra is capable of

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– finding the relation between two given geometric objects in a construction (i.e. of formulating a thesis holding under the set of hypotheses describing the geometric construction),
– of checking the truth/failure of a conjectured (by the user) relation,
– or finding missing hypotheses for a given relation to hold true.

Thus, in Figure 1 it is displayed a cyclic quadrilateral $BCDE$, the sides f, g, h, i and diagonals j, k, and, in the input line, it is asked GeoGebra to verify the truth of Ptolemy's theorem (`https://en.wikipedia.org/wiki/Ptolemy%27s_theorem`) about the coincidence of the product of the lengths of the diagonals of the quadrilateral and the sum of the products of the lengths of the pairs of opposite sides.

Then, in Figure 2 we show the (practically immediate, on a personal laptop) answer: it is true. And in Figure 3 we see that, if we start with an arbitrary quadrilateral and we ask where to pose point C for Ptolemy's theorem to hold, GeoGebra outputs the circle (in red) going through the other three vertices of the quadrilateral: that is, the converse of Ptolemy's theorem is also true.

Moreover, in Figure 4 we describe how the user asks GeoGebra for any possible relation holding between the three altitudes of a triangle (using the *Relation* command). In Figure 5 GeoGebra replies that, at least numerically, visually, it seems that the three altitudes have a common intersection. Finally, after clicking on the *More* button, GeoGebra performs internally a symbolic calculation and concludes that this fact is mathematically true. that, as recently developed in [4],

> ... This paper describes the formalization of the arithmetization of Euclidean plane geometry... The arithmetization of geometry paves the way for the use of algebraic automated deduction methods in synthetic geometry. Indeed, without a back-translation from algebra to geometry, algebraic methods only prove theorems about polynomials and not geometric statements. However, thanks to the arithmetization of geometry, the proven statements correspond to theorems of any model of Tarski's Euclidean geometry axioms...

the symbolic computation algorithms (originally described in [20]), involved in the proving protocols implemented in GeoGebra, are formally valid.

The performance of these GeoGebra automated reasoning tools is quite impressive (as highlighted in the initially mentioned references), and could be very useful in the educational context, even if they do not provide readable arguments for the provided answers, by collaborating with students in the process of conjecturing and experimenting in geometric contexts. This is a topic that has already raised the attention of relevant experts in mathematics education (see [10], that refers specifically to GeoGebra, or [11]), but it is not what we want to pay attention to in this occasion.

Indeed, we want to focus here in the fact that GeoGebra commands, for proving or finding geometric properties, have some relevant limitations. For example, given a segment $f = AB$ and point B' in the perpendicular line h to AB through A, satisfying that $AB = AB'$, we ask for the *Relation(i,f)*, where $f = AC$ and where C lies on line h and is such that $B'B = B'C$. GeoGebra simply declares that both segments are not equal, but it is unable to find the ratio $1 + \sqrt(2)$ that holds between them. See Figures 7and 8.

Similarly, GeoGebra can not discover the basic inequality holding between $a + b$ and c, for the three sides of a triangle a, b, c, see Figures 9and 10.

The Automated Geometer in GeoGebra Discovery

These limitations have been addressed in the fork version of GeoGebra named GeoGebra Discovery, that has, moreover, enlarged the automated reasoning performance of GeoGebra with new tools.

GeoGebra Discovery (see `https://kovzol.github.io/geogebra-discovery/` for further details) includes several off-line releases, for different OS (Mac, Windows, Linux, RaspberryPi, see `https://github.com/kovzol/geogebra/releases`) as well as an on-line version: `https://www.autgeo.online/geogebra-discovery/`. Moreover, GeoGebra Discovery has a specific *url* for the *Automated Geometer* prototype version (`https://autgeo.online/ag/automated-geometer.html?\offline=1`), exclusively devoted to compute all possible statements (following a combinatorial algorithm) that hold true over a given figure.

GeoGebra Discovery enlarges GeoGebra tools by including new features (see [5,6,7,8,12,13,15,16,19] for details and examples):

- finding the relation between two given geometric objects in a construction, including, in the case of lengths, the *discovery of relations other than equality*: obtaining algebraic number ratios (see Figure 11), or inequality relations (see Figure 12), through comparison,
- of checking the truth/failure of a conjectured (by the user) relation *involving inequalities* (e.g. see Figure 13, related to an AMS-Monthly Problem, see [21])
- discovering automatically all possible relations of a certain kind (co-circularity, parallelism, perpendicularity, etc) among the elements of a construction involving a chosen element (see Figures 14,15) or all elements (see Figures 16,17,18).
- the possibility, through the *ShowProof* command [18], of displaying in the GeoGebra Discovery CAS (Computer Algebra System) window the different algebraic steps internally performed by the program to confirm or deny a given statement,
- as well as a *measure of the complexity* of the proved statement, defined (see [18] for details) as the maximum degree of the polynomials f_i, such that $t = f_1h_1 + \ldots + f_rh_r$, or such that $1 = f_1h_1 + \ldots + f_rh_r + f_{r+1} * (zt - 1)$, where t is the thesis and $h_1, \ldots, h_r$ are the hypotheses.

These improvements required, in particular, the porting, on a web platform and to GeoGebra Discovery, of the software program Tarski (`https://www.usna.`

edu/Users/cs/wcbrown/tarski/index.html, https://matek.hu/zoltan/tarski/webtarski.html), for performing the required real quantifier elimination algorithms, see [12] for details.

A real automated geometer

The previous, summary, description of the current automated reasoning features of GeoGebra and GeoGebra discovery, did not include any reference to some subtle, but core, issues ruling the implemented protocols and algorithms. Let us roughly declare (referring to [20,14,5] for details and references) that dealing with a statement such as $\{h_1 = 0, \ldots, h_r = 0\} \Rightarrow t = 0$

- can *not* be approached simply by considering that it is true iff t belongs to the ideal of the zeroes of $(h_1, \ldots, h_r)$ (using the corresponding Nullstellensatz, when working over the complex or over the real numbers). Indeed, most statements require to handle (to define, to get rid off) the *degenerate* instances that arise, unexpectedly, in the algebraic formulation of practically any statement. This involves
- selecting (perhaps by the user or automatically, following the steps of the geometric construction) a collection of variables equal to the (topological, Hilbert) dimension of the algebraic variety defined by the hypotheses,
- considering as non-degenerate components those where such variables remain independent
- labeling as *algebraically/geometrically/generally/partially* true those statements where the thesis *belongs to the ideal of hypotheses/vanishes over the whole hypotheses variety/or at least over all the non-degenerate components/or at most over some, but not all such components.* Let us remark that, in practice, it is quite common, and illustrative, to learn which of these cases holds for a given statement.
- detecting each of these cases without having to compute the ideal of the hypotheses variety or its components, but simply proceeding by checking if the output of some elimination algorithm is or not zero,
- obtaining, in the generally true case, some degeneracy conditions that should be avoided for the statement to be geometrically true,
- computing the *complexity* of a geometrically true statement by expressing 1 as an element of the ideal generated by the hypotheses and the negation $zt - 1$ of the thesis, after computing its normal form (0) with respect to a Gröbner basis of this ideal, and then expressing each element of this basis as a combination of the given generators.

It is well known (see [5]) that there are relevant coincidences (and differences) regarding the concept of truth for elementary geometry statements, if considered in the complex or real settings (for the zeroes of the involved hypotheses and thesis algebraic varieties). Many statements that hold true (in some of the senses defined above) over the reals hold as well over the complexes, so it is advisable to proceed, first, to verify its truth over the latter field, as it is easier, computationally speaking.

Yet, our on-going research work aims to develop a similar theoretical framework to the one described above,

- specifically adapted to the case of real zeroes, including the case of statements dealing with inequalities (in the hypotheses, in the thesis). This requires, in particular, thinking about the equivalent notion to *non-degenerate component* for semi-algebraic hypotheses sets.
- without having to compute the real radical, or the real irreducible components, for detecting the *generally true, etc.* case.
- replacing the *elimination* protocol of the complex case by a certain projection (i.e. a basic elimination of quantifiers), where we might be interested just in learning if such projection is contained, or not, on a proper, real hypersurface.
- keeping the (complex) notion of complexity, but this time considering, for its computation, the polynomials appearing in the real Nullstellsataz or Positivstellesatz, expressing that a given element belongs to the real radical of an ideal, or that it is zero, positive, strictly positive, etc. over a semialgebraic set.

We consider that this list of on-going tasks for developing a truly performing *real automated geometer* involve several, elementary, real algebraic geometry (RAG) issues that we are currently unable to address from a *practical* algorithmic perspective. Presenting the basic ideas of our on-going project, to get some feedback from the scientists gathering with occasion of prof. Laureano González-Vega's (aka Lalo) 60th birthday, someone who has contributed so much to the RAG and CAS communites, is the final goal of our communication.

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Appendix: Figures

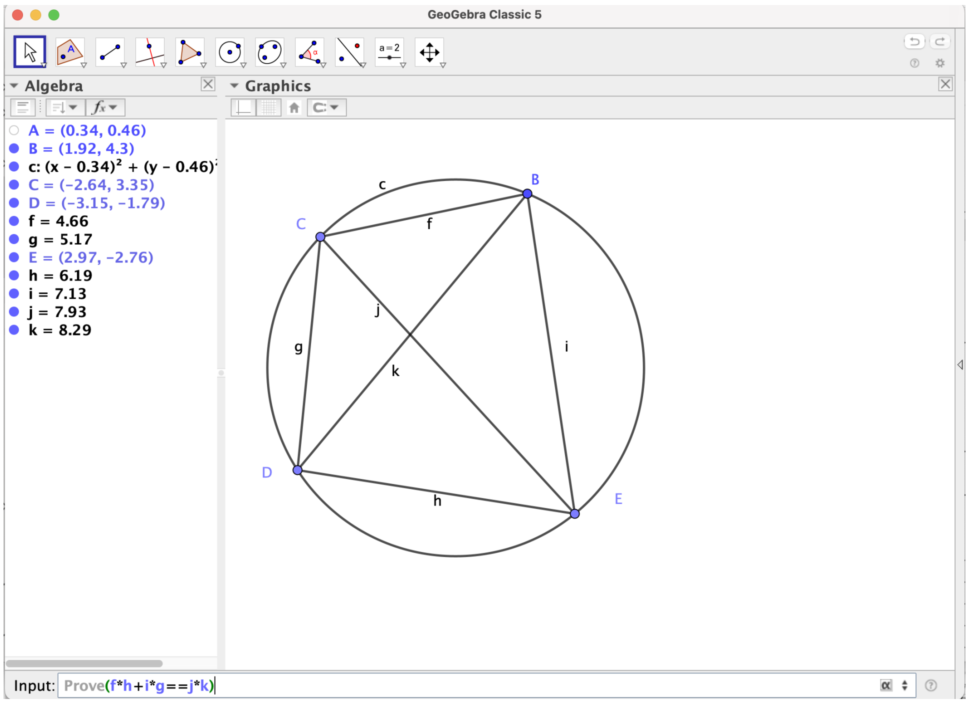

Fig. 1. Asking for the truth of Ptolomy's theorem

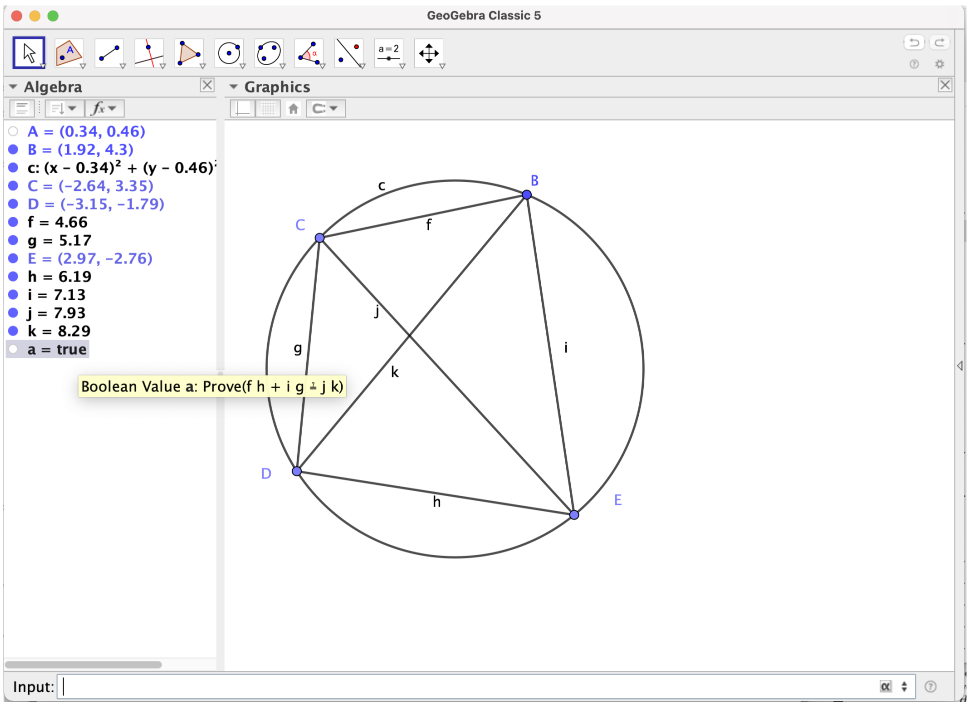

Fig. 2. Confirmation of Ptolomy's theorem

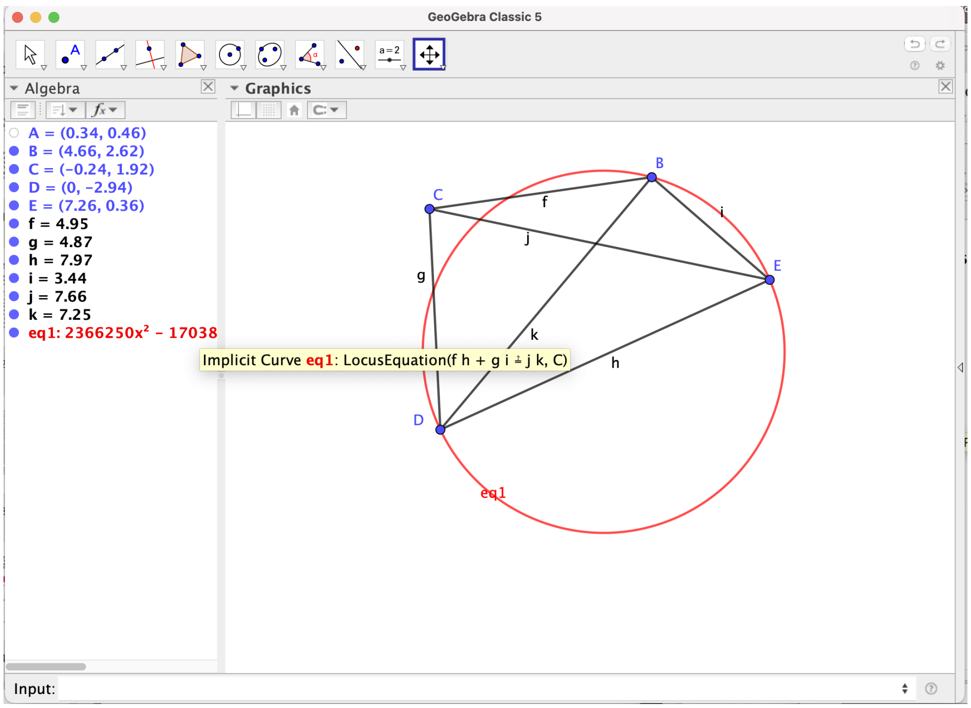

Fig. 3. Looking for missing hypotheses for Ptolomy's theorem

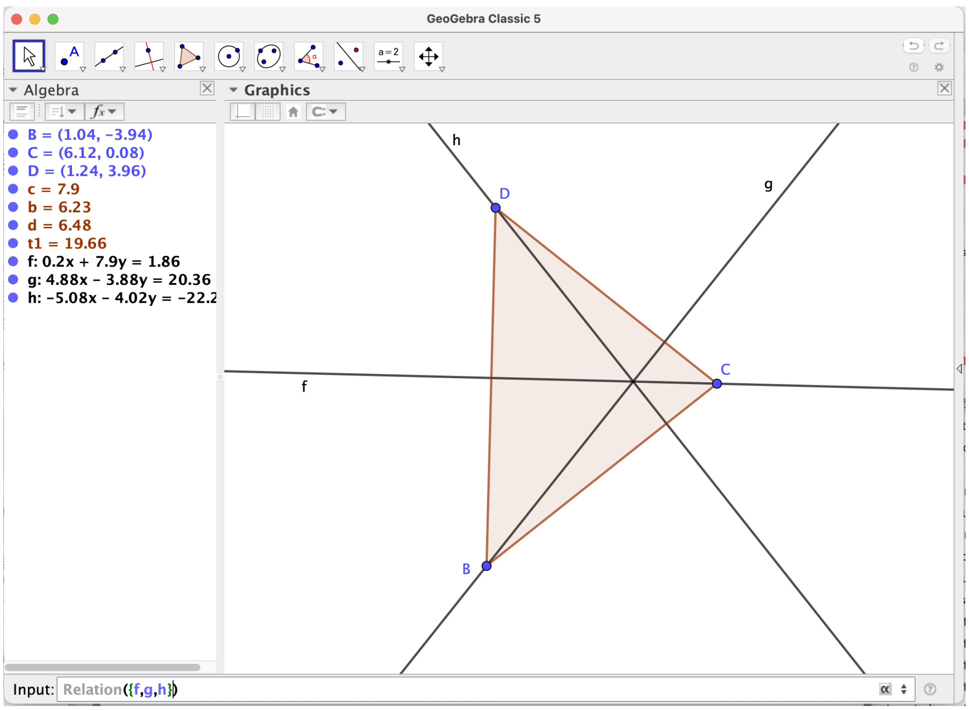

Fig. 4. Asking for the Relation holding on the altitudes of a triangle.

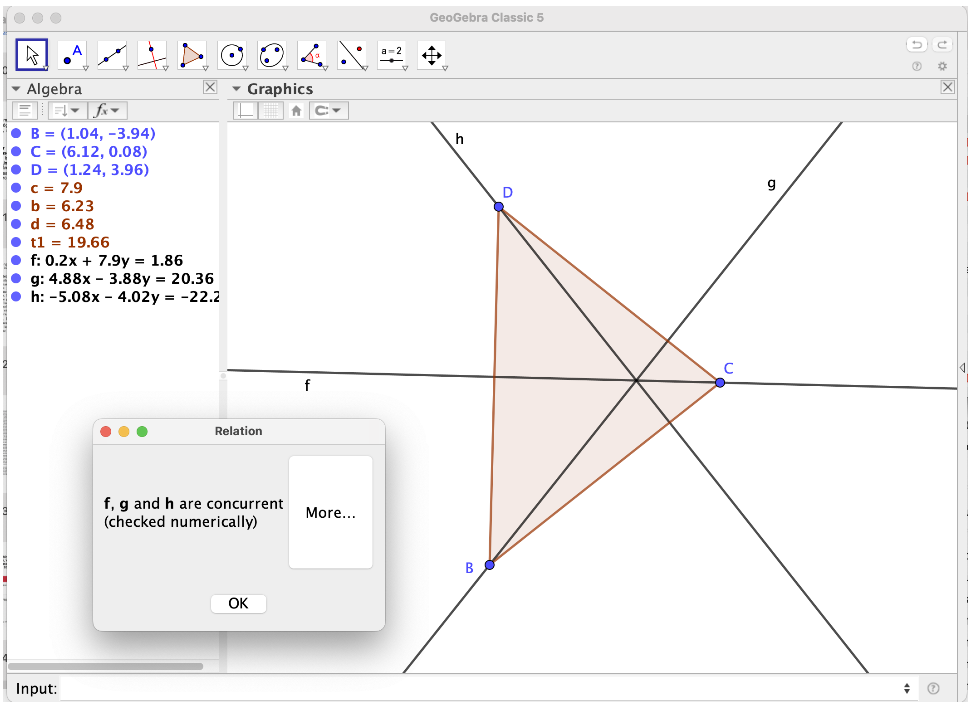

Fig. 5. GeoGebra observes that the three altitudes are, visually, having a common intersection

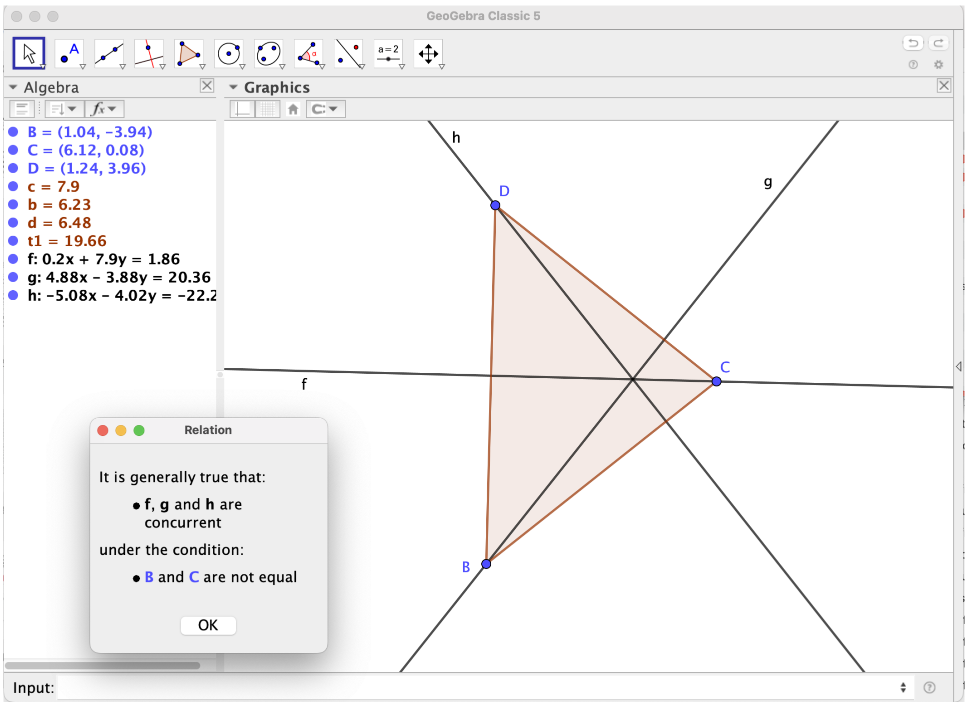

Fig. 6. Formally declaring that the three altitudes have a common intersection

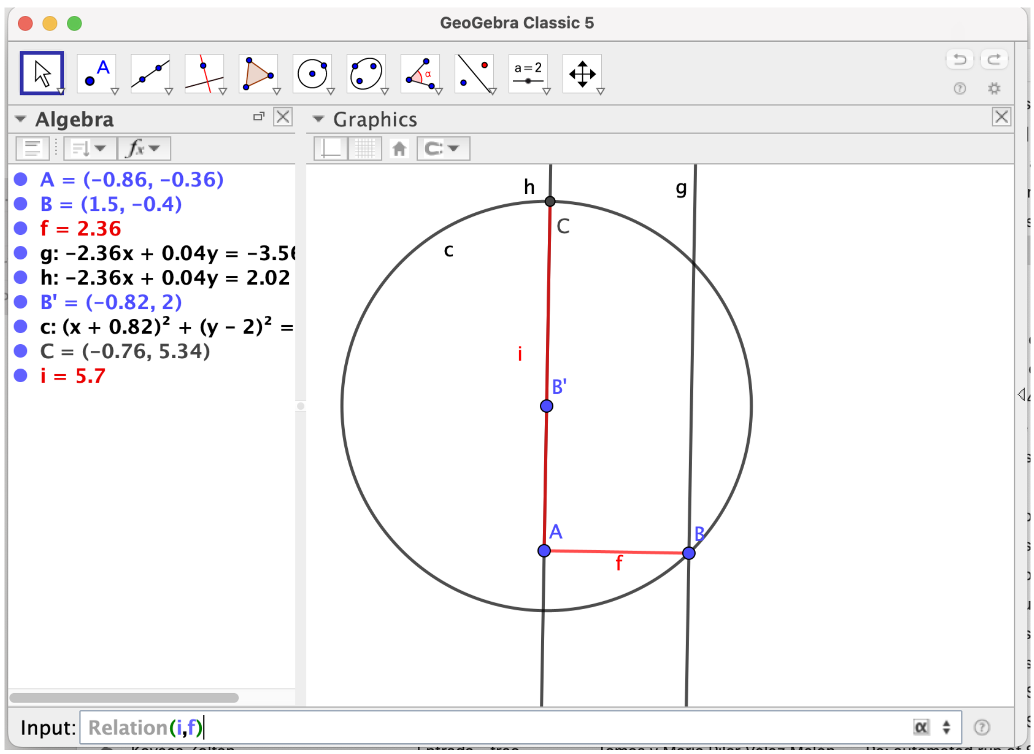

Fig. 7. Asking for the *Relation(i,f)*

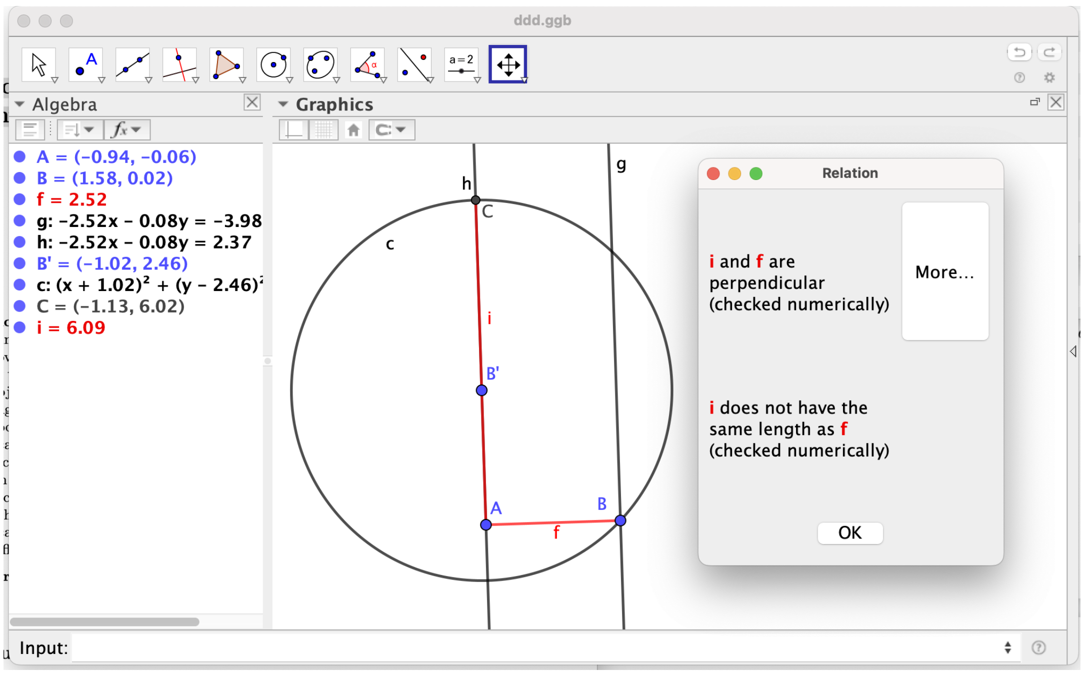

Fig. 8. Since $i \neq f$, GeoGebra does not find any Relation on the lengths of these segments

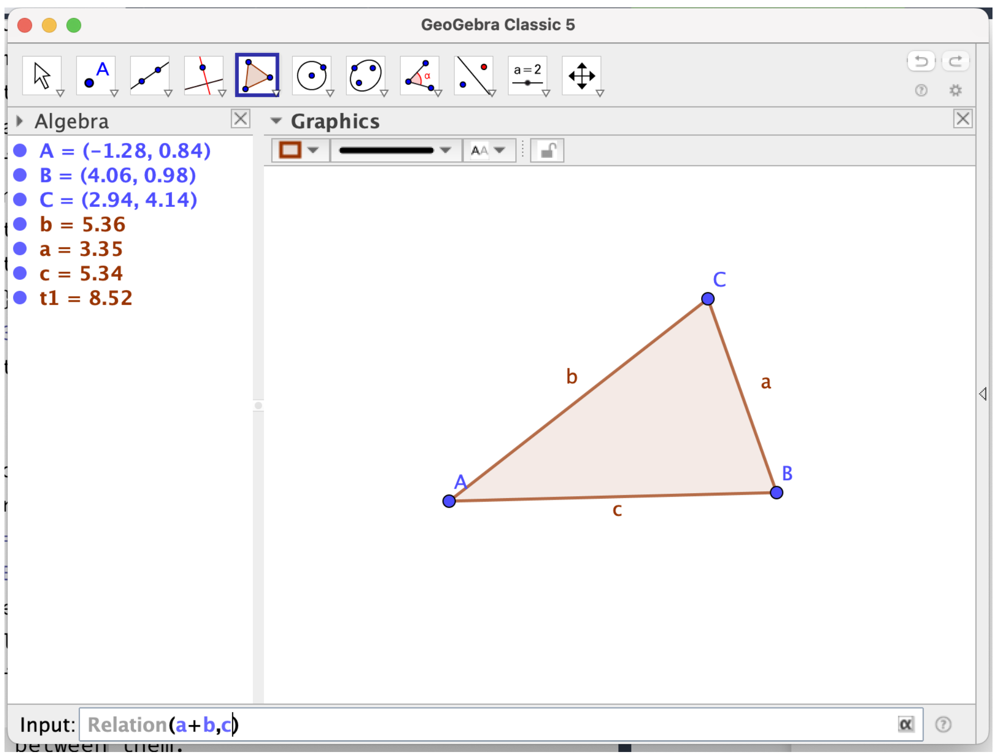

Fig. 9. Asking for the *Relation(a+b,c)*

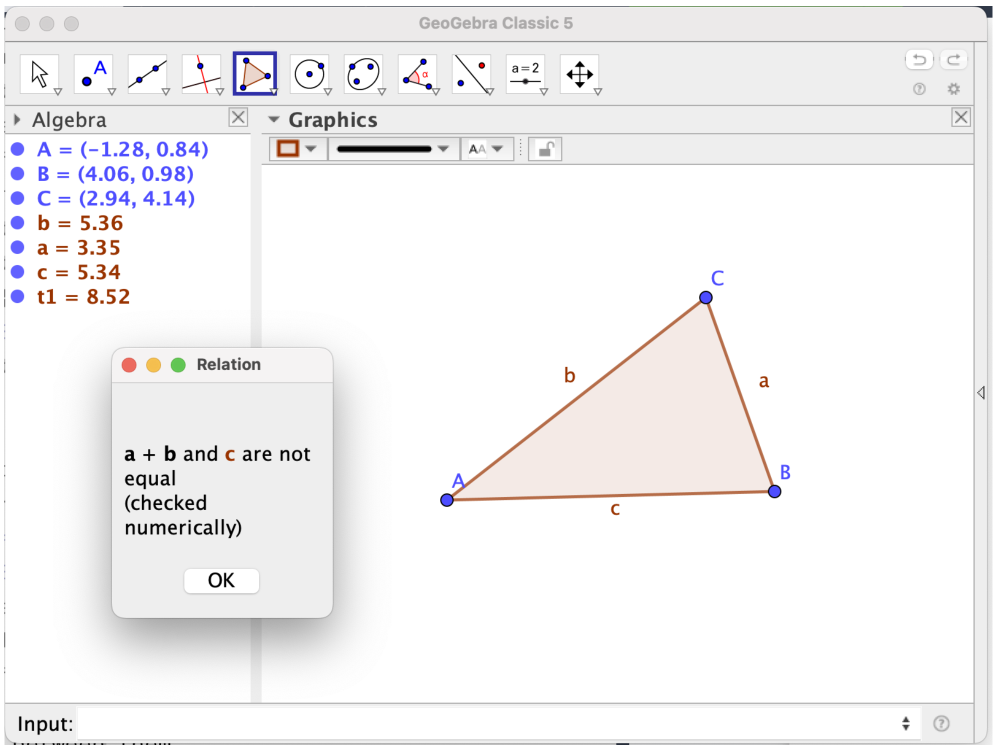

Fig. 10. GeoGebra can not find the classical inequality $a + b \leq c$

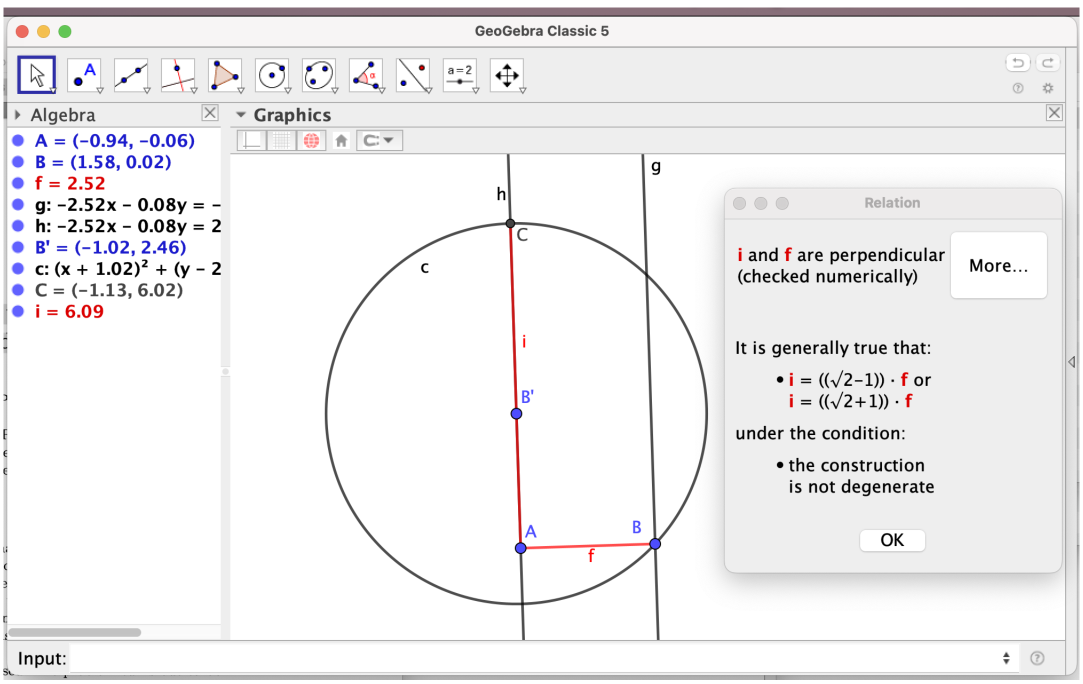

Fig. 11. Finding the relation between i, f.The two options correspond to the different definitions of i as the segment from A to the (two) possible intersections of circle c and line h.

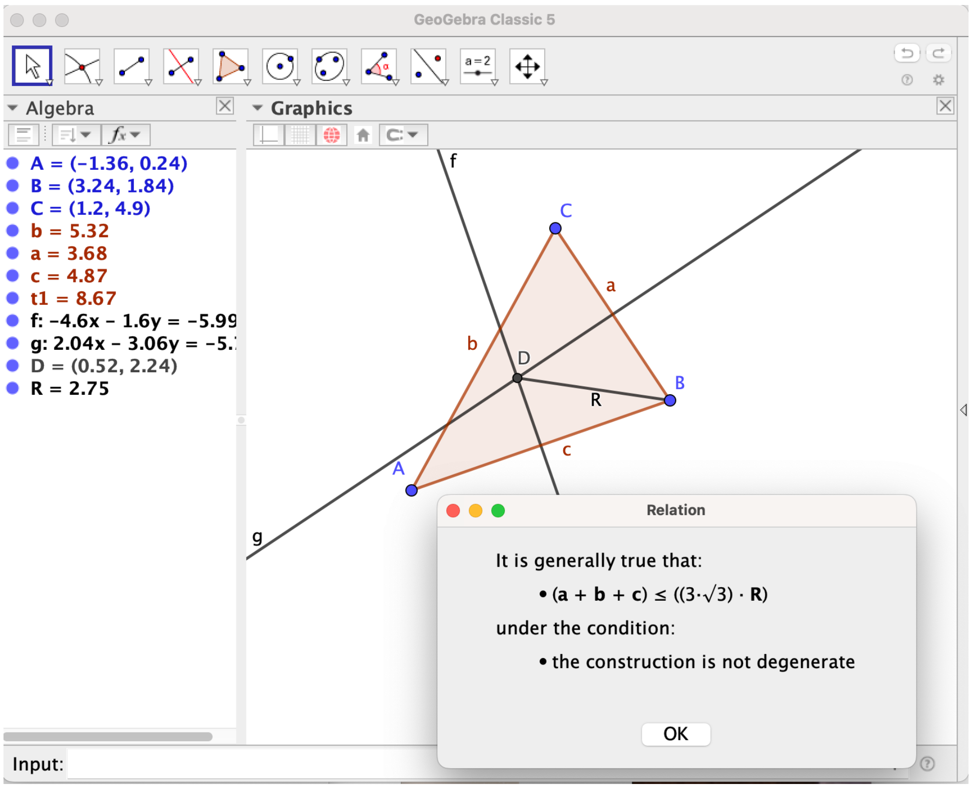

Fig. 12. Asking for the Relation between the perimeter of a triangle and the radius of the circumcircle.

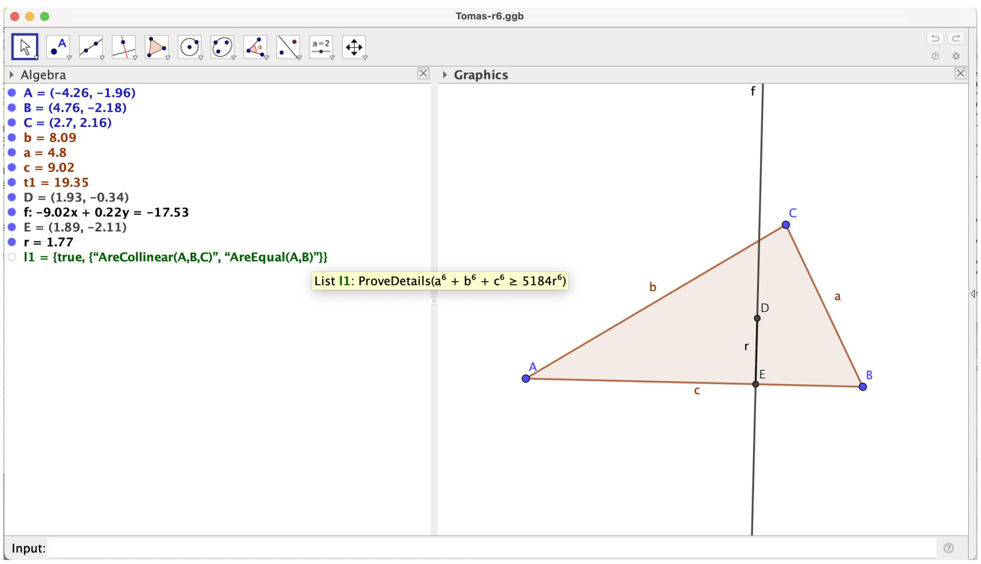

Fig. 13. Proving the inequality $a^6 + b^6 + c^6 \geq 5184r^6$ involving the sides of a triangle and the radius r of the incircle.

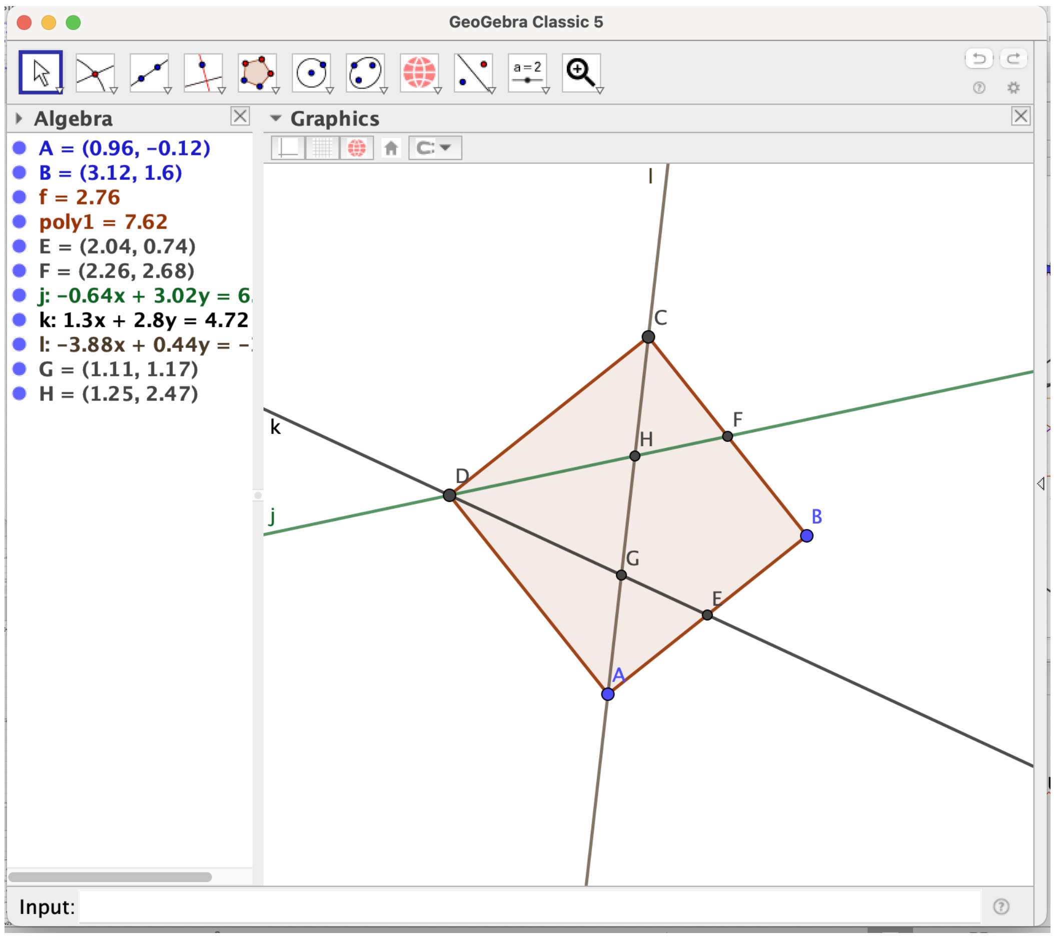

Fig. 14. A square, mid-points of sides AB, BC and lines from D to such mid-points. Points G, H are the intersection of these lines with diagonal AC

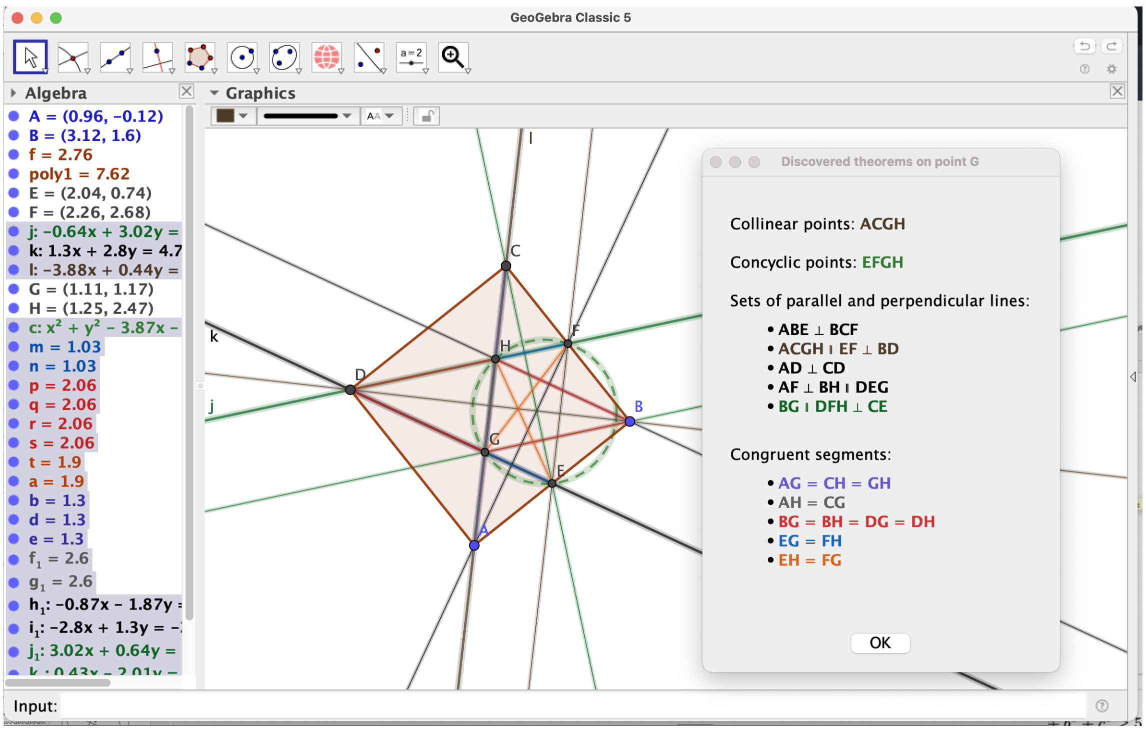

Fig. 15. Discovering automatically statements involving point G.

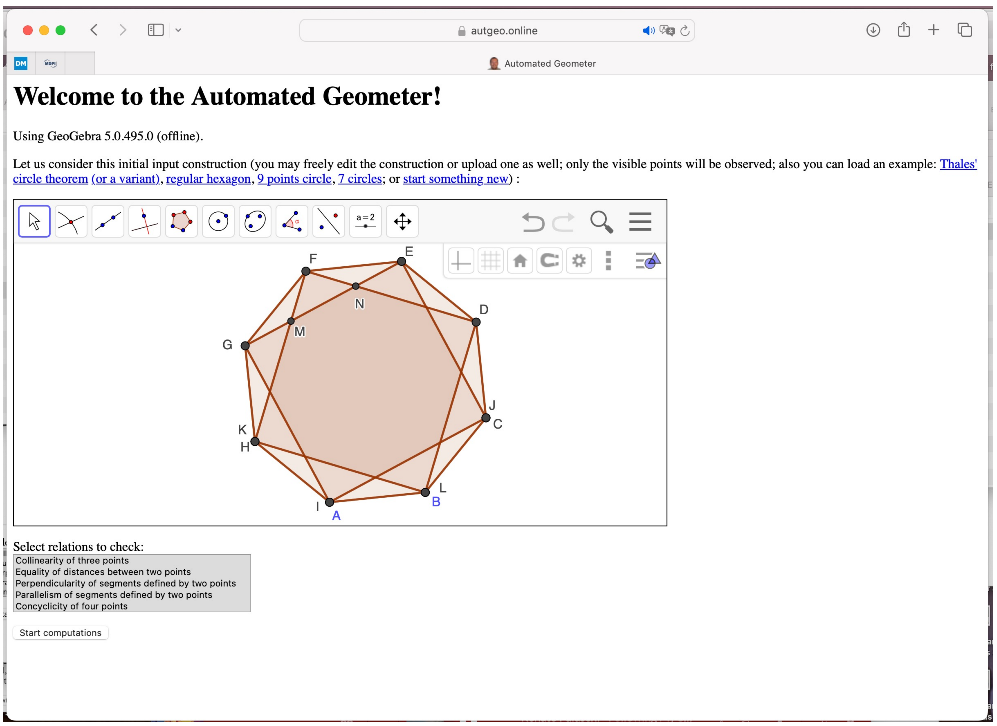

Fig. 16. A regular octagon, two inscribed squares, two points M, N of intersection from sides of the squares.

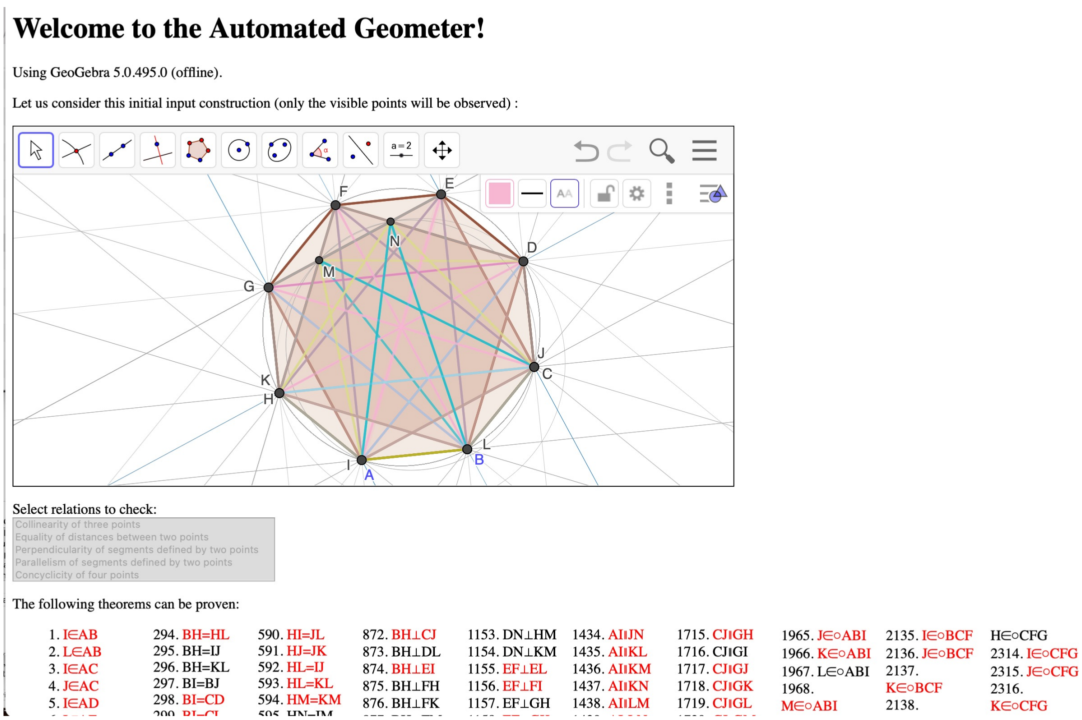

Fig. 17. Discovering automatically statements concerning all elements in the precedent figure. Initial screen.

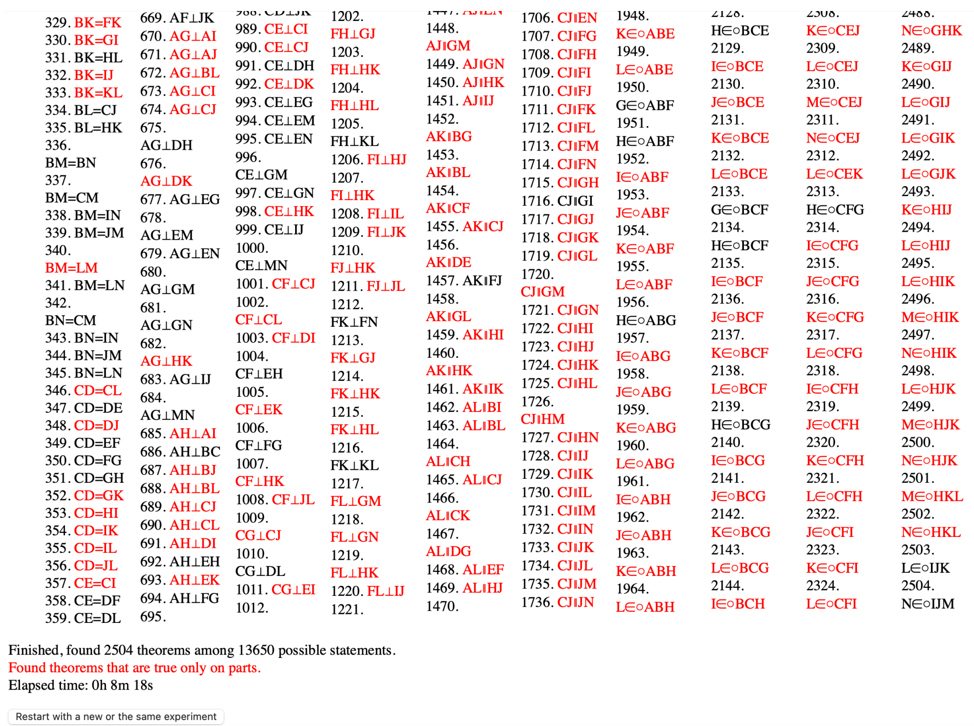

Fig. 18. Discovering automatically statements concerning all elements in Figure 16. Final screen, after several ones presenting thousand of results.