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Sparse Interpolation: from de Prony to Froissart and beyond

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In 1795 the French mathematician de Prony [6] published a method to fit a real-valued exponential model to some uniformly collected samples, making use of a linear recurrence connecting the data.

In the beginning of the 20-th century the famous Nyquist constraint [10] was formulated, which is the digital signal processing equivalent of stating that the argument of a complex exponential $\exp(\phi\Delta)$ with $\phi \in \mathbb{C}$ and $\Delta \in \mathbb{R}^+$ can only be retrieved uniquely under the condition that $|\Im(\phi)| < \pi/\Delta$.

Both methods are closely connected to sparse interpolation from computer algebra.

In the past two decades the Nyquist constraint was first broken when using randomly collected signal samples [7, 2] and later for use with uniformly collected samples [4]. Besides discussing how to avoid the Nyquist constraint, we also explain how to solve a number of remaining open problems in exponential analysis using sparse interpolation results.

We start from the most general problem statement. In the identification, from given values $f_k \in \mathbb{C}$, of the nonlinear parameters $\phi_1, \ldots, \phi_n \in \mathbb{C}$, the linear coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and of the sparsity $n \in \mathbb{N}$ in the fitting problem

$$\sum_{j=1}^{n} \alpha_j \exp(\phi_j k\Delta) = f_k, \qquad k = 0, \ldots, 2n-1, \ldots \quad f_k \in \mathbb{C}, \quad \Delta \in \mathbb{R}^+, \tag{1}$$

several cases are considered to be hard [4, 1]:

- When some of the ϕ_j cluster, the identification and separation of these clustered ϕ_j becomes numerically ill-conditioned. We show how the problem may be reconditioned.
- From noisy f_k samples, retrieval of the correct value of n is difficult, and more so in case of clustered ϕ_j. Here, decimation of the data offers a way to obtain a reliable estimate of n automatically.

- Such decimation allows to divide and conquer the inverse problem statement. The smaller subproblems are largely independent and can be solved in parallel, leading to an improved complexity and efficiency.
- At the same time, the sub-Nyquist Prony method proves to be robust with respect to outliers in the data. Making use of some approximation theory results [8, 9], we can also validate the computation of the ϕ_j and α_j.
- The Nyquist constraint effectively restricts the bandwidth of the $\Im(\phi_j)$. Therefore, avoiding the constraint offers so-called superresolution, or the possibility to unearth higher frequency components in the samples.

All of the above can be generalized in several ways, on the one hand to [5] the use of more functions besides the exponential, and on the other hand [3] to the solution of multidimensional inverse problems as in (1).

Keywords
sparse interpolation, exponential analysis, Nyquist, Padé, Prony, Froissart

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