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INVOLUTIVE FIBONACCI WORDS

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ABSTRACT

"Fibonacci strings" were first defined by Knuth in his 1968 "The Art of Computer Programming," as being an infinite sequence of strings obtained from two initial letters $f_1 = a$ and $f_2 = b$, by the recursive definition $f_{n+2} = f_{n+1} \cdot f_n$, for all positive integers $n \geq 1$, where "$\cdot$" denotes word concatenation. In this paper, we first propose a unified terminology that allows readers to identify the different types of Fibonacci words, and corresponding results, that appear under the umbrella term "Fibonacci words" in the extensive literature on the topic. Motivated by ideas stemming from theoretical studies of DNA computing, we then define and explore involutive Fibonacci words (ϕ-Fibonacci words and indexed ϕ-Fibonacci words, where ϕ denotes either a morphic or an antimorphic involution), and study various properties of such words.

Keywords: DNA computing, Watson-Crick complementarity, antimorphic involution, Fibonacci words, involutive Fibonacci words

1. Introduction

Fibonacci words or Fibonacci strings were introduced as word counterparts of the Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and the recursion $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. "Fibonacci strings" were first defined by Knuth in his "The Art of Computer Programming" (volume 1, section 1.2.8, exercise 36, [29]), as being an infinite sequence of strings obtained from two initial letters $f_1 = a$ and $f_2 = b$ by the recursive definition $f_{n+2} = f_{n+1} \cdot f_n$, for all $n \geq 1$, where "$\cdot$" denotes word/string concatenation. Various other definitions of Fibonacci words have been proposed since, as detailed in the sequel. In this paper, we first propose a unified terminology for the purpose of clarification and comparison of the multiple variants of the definition of Fibonacci words that exists in the literature.

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In the following, an alphabet Σ is a finite non-empty set of symbols, and Σ^* denotes the set of all words over Σ including the empty word λ, while Σ^+ is the set of all non-empty words over Σ. The following definition proposes a uniform and intuitive terminology for the various types of Fibonacci words studied in the literature.

Definition 1. Let Σ be an alphabet with $|\Sigma| \geq 2$ and let $u, v \in \Sigma^+$. The n^{th} standard Fibonacci words are defined recursively as:

$$f_1(u,v) = u,\ f_2(u,v) = v,$$

$$f_n(u,v) = f_{n-1}(u,v) \cdot f_{n-2}(u,v),\ \ n \geq 3.$$

The sequence of standard Fibonacci words is defined as $F(u,v) = \{f_n(u,v)\}_{n\geq 1}$, that is, $F(u,v) = \{u, v, vu, vuv, vuvvu, vuvvuvuv, vuvvuvuvvuvvu, \ldots\}$. Similarly, the n^{th} reverse Fibonacci words are defined recursively as:

$$f'_1(u,v) = u,\ f'_2(u,v) = v,$$

$$f'_n(u,v) = f'_{n-2}(u,v) \cdot f'_{n-1}(u,v),\ \ n \geq 3,$$

and the sequence of reverse Fibonacci words is defined as $F'(u,v) = \{f'_n(u,v)\}_{n\geq 1}$, that is, $F'(u,v) = \{u, v, uv, vuv, uvvuv, vuvuvvuv, uvvuvvuvuvvuv, \ldots\}$.

If the initial words u and v are singleton letters, the resulting words will be called atom standard Fibonacci words, respectively atom reverse Fibonacci words.

Below is a summary - not necessarily exhaustive - of papers in the literature, grouped by the types of Fibonacci words they study, according to Definition 1:

(i) atom standard Fibonacci words: [4, 6, 9, 10, 11, 12, 13, 15, 16, 17, 28, 29, 30, 31, 32, 33, 34, 35, 36, 43, 44, 46, 47, 48, 49, 54],

(ii) atom reverse Fibonacci words: [10, 15, 19, 24, 47, 52, 53, 56],

(iii) (non-atomic) standard Fibonacci words: [8, 15, 56],

(iv) (non-atomic) reverse Fibonacci words: [8, 15, 52, 53, 56].

Indeed, according to Definition 1, the strings defined in [29] are atom standard Fibonacci words. They were extensively studied in, e.g., [4, 11, 12, 30, 43], with some slight modifications consisting in either changing the initial letters or slightly changing the indices, see, e.g., [4, 12, 43, 44]. Some properties involving the structure of such atom standard Fibonacci words were studied in [13, 17].

It was noted in [15], that every standard (reverse) Fibonacci sequence $F(u,v)$ (respectively $F'(u,v)$) is a homomorphic image of the atom standard (atom reverse) Fibonacci sequence $F(a,b)$ (respectively $F'(a,b)$), via the homomorphism $h(a) = u$ and $h(b) = v$, where $a \neq b$. Thus, properties of atom Fibonacci words are especially important.

In the remainder of this paper, if the first two Fibonacci words are obvious from the context, the argument (u,v) will be omitted, and we will write the n^{th} standard Fibonacci word as f_n, the n^{th} reverse Fibonacci word as f'_n, the standard Fibonacci sequence as F, and the reverse Fibonacci sequence as F'.

An equivalent definition of the sequence of atom standard Fibonacci words, using the iteration of a morphism, was given in, e.g., [4, 31], where a morphism $\nu : \Sigma^* \to \Sigma^*$ is defined by $\nu(b) = ba, \nu(a) = b, f_1 = a$, and $f_{n+1} = \nu^n(a)$ for all $n \geq 1$, which determines the sequence $a, b, ba, bab, babba, babbabab, babbababbabba, \ldots$. Properties of atom standard Fibonacci words and sequence generated by iterating the morphism ν were studied in [6, 31, 34, 35, 36, 46, 54]. It was shown in [35] that the length of the word f_n, defined by such a morphism is the n^{th} Fibonacci number. In [48], it was shown that the infinite atom standard Fibonacci sequence is an automatic sequence (a sequence computed by a deterministic finite automaton with output), and Mousavi, Schaeer, and Shallit [32] used the software Walnut to study properties of Fibonacci words by using such automata.

Yet another alternative definition of the sequence of atom standard Fibonacci words is based on the "golden mean" $\Phi = (1 + \sqrt{5})/2$, whereby $f_n = c_1 c_2 \cdots c_n$, with $c_i = a$ if $i \in \{\lfloor k\Phi \rfloor \mid k \geq 1\}$, and $c_i = b$ otherwise, for all $1 \leq i \leq n$, see [49].

The atom reverse Fibonacci words of Definition 1 were defined in [24] using an iterative morphism, as well as discussed by Higgins [19] under the name of *papal sequence*. Their properties were further studied in [47].

A generalization of atom standard and reverse Fibonacci words to the standard and reverse Fibonacci words of Definition 1 (wherein the first two words are non-atomic) was discussed in [8, 15, 52, 53, 56].

Another generalization was introduced in [8], which defined what we herein call indexed Fibonacci words. Under this definition, every indexed Fibonacci word is associated with a binary sequence whose last digit is 0 if the word was obtained by the standard concatenation order of the previous two words, and 1 if it was obtained by reverse concatenation order. Properties of indexed Fibonacci words were studied in [9, 10].

In this paper, we propose several generalizations of standard, reverse, and indexed Fibonacci words, motivated by an idea first advanced and studied in the context of DNA computing [25, 37], whereby the Watson-Crick DNA complementarity is formalized as an antimorphic involution function θ on Δ^*, where Δ is the DNA alphabet defined as $\Delta = \{A, C, G, T\}$. Indeed, a DNA strand can be viewed as a word over Δ, where in A is Watson-Crick complementary to T, and C to G, that is, $\theta(A) = T$, and $\theta(C) = G$. Two complementary DNA single strands of opposite orientation bind together to form a DNA double strand (intermolecular structure). Also, if non-overlapping subwords of a DNA strand are complementary, the strand may bind to itself forming intramolecular structures such as stem-loops, known more commonly as hairpins (Figure 1).

As such, hairpins tend to interfere with DNA computations, and therefore are usually explicitly avoided by DNA computing experimentalists when encoding information as DNA strands. See [1, 2, 21, 22, 38] about this problem and about some of the "good" designs of DNA strands that are free of hairpins. However, hairpins are not a structure to always be avoided in DNA computations. For example, hairpins are the main component of "hairpin engine" DNA computing techniques [18, 42, 45]. In [23, 50, 51] hairpins serve as a binary information medium for DNA-based Random Access Memory (RAM). Last, but not least, hairpins are the basic components

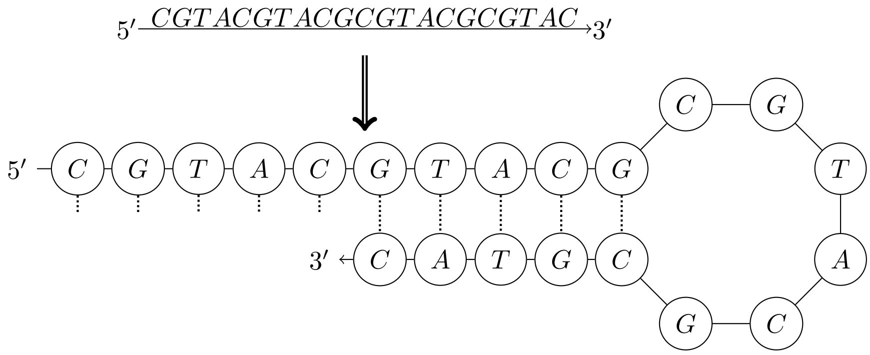

Figure 1: A single-stranded DNA molecule $CGTACGTACGCGTACGCGTAC$ forms a hairpin loop, due to Watson-Crick complementarity, pairing G with C, and A with T. The orientation of the DNA strand is denoted by its two ends labelled by $5'$ and $3'$ respectively, to indicate their different chemical characteristics.

of some DNA-based programmable "smart drugs" [3]. As it turns out, several of the generalizations of the Fibonacci words defined in this paper are guaranteed to form hairpins, which makes them a good candidate for encodings in hairpin-based DNA computations.

The remainder of the paper is organized as follows. Following the main definitions and notations in Section 2, in Section 3 we define several generalizations of Fibonacci words. Given an involution ϕ that is either morphic or antimorphic (termed "(anti)morphic" involution), we define, for $n \geq 1$, *standard* alternating ϕ-Fibonacci words g_n, standard palindromic ϕ-Fibonacci words w_n, and standard hairpin ϕ-Fibonacci words z_n, and their *reverse* counterparts (g'_n, w'_n, z'_n), as well as their atomic versions which correspond to the case where the first two words are singleton letters. Similar to the relationship between atom standard and atom reverse Fibonacci words (Lemma 5), we show that atom standard ϕ-Fibonacci words are the mirror image of the corresponding atom reverse ϕ-Fibonacci words (Lemma 6). However, for the ϕ-Fibonacci words that are not atomic, we prove that the above statement holds only under special circumstances (Corollary 9 and Corollary 11). We also show that the ϕ-Fibonacci words g_n and g'_n are conjugates (Corollary 9), and that if $\phi = \theta$ is an antimorphic involution, under certain conditions we have that z_n and z'_n are conjugates and w_n and w'_n are θ-conjugates (Theorem 12). In Subsection 3.1, we explore relations between various types of Fibonacci words and μ-Fibonacci words, where $\phi = \mu$ is a morphic involution. In particular we show that μ-Fibonacci words can be obtained by alternating two different μ-Fibonacci sequences (Theorem 15).

Section 4 proposes a generalization of indexed Fibonacci words to indexed ϕ-Fibonacci words, and discusses their interrelationships. Section 5 explores borders and ϕ-borders of ϕ-Fibonacci words. We show that if $\phi = \mu$ is a morphic involution then all μ-Fibonacci words are bordered, and discuss conditions under which the ϕ-Fibonacci words are bordered or ϕ-bordered (Theorem 27). Section 6 discusses conclusions and future work.

Note that other extensions of the atom Fibonacci sequence have been proposed and investigated in the literature, such as: The mapped shuffled Fibonacci languages defined as $F_{(u,v)} = \{h(w) \mid w \in F_{(a,b)}\}$, whereby $h(a) = u$, $h(b) = v$, $F_{(a,b)} = \bigcup_{i \geq 1} F_i$, and the languages $F_i, i \geq 1$, are obtained from $F_1 = \{a\}$, $F_2 = \{b\}$ by the recursive definition $F_{n+2} = F_n \diamond F_{n+1}$, for all $n \geq 1$, where $\diamond$ is the shuffle operation, see [20]; The sequence $\{s_n\}_{n \geq -1}$ defined by $s_{-1} = 1$, $s_0 = 0$ and $s_n = s_{n-1}^{d_n} s_{n-2}$ for $n \geq 1$, where $d_1 \geq 0$ and $d_n > 0$ for $n > 1$, [5]; The $k-$Fibonacci words, whereby $f_{k,0} = 0$, and $f_{k,1} = 0^{k-1}1$, and $f_{k,n} = f_{k,n-1}^k\ f_{k,n-2}$, $n \geq 2, k \geq 1$, [39]; The $(n,i)-$Fibonacci words whereby $f_0^{[i]} = 0$, $f_1^{[i]} = 0^{i-1}1$, and $f_n^{[i]} = f_{n-1}^{[i]}\ f_{n-2}^{[i]}$, $n \geq 2, i \geq 1$, [40, 41]; The $m-$bonacci words, whereby $f_i = \phi_m^i(0)$ and $\phi_m(m-1) = 0, \phi_m(i) = 0(i+1)$, for all $0 \leq i \leq m-2$, [7], to name just a few. These and other extensions of Fibonacci words, such as that in [14], are out of the scope of this paper.

2. Definitions and notations

The length of a word $u \in \Sigma^*$ (i.e., the number of symbols in u) is denoted by $|u|$. We denote by $|u|_a$ the number of occurrences of a letter a in u and denote by $alph(u)$ the set of all the letters that occur in u. For a positive integer m, we denote by Σ^m the set of all words of length m over Σ, with the empty word λ being of length 0. A language L is a subset of Σ^*. The complement of a language $L \subseteq \Sigma^*$ is $L^c = \Sigma^* \backslash L$, the concatenation of two languages L_1 and L_2 is defined as $L_1L_2 = \{uv \mid u \in L_1, v \in L_2\}$, and the power of a language L is defined as $L^0 = \lambda$, $L^1 = L$ and $L^i = L^{i-1}L$ for all integers $i \geq 2$.

A word $x \in \Sigma^*$ is said to be a prefix of $w \in \Sigma^+$ (suffix, respectively) if w has a decomposition $w = x\alpha$ ($w = \beta x$, respectively) where $\alpha, \beta \in \Sigma^*$. If $\alpha \in \Sigma^+$ ($\beta \in \Sigma^+$, respectively) then x is said to be a proper prefix (proper suffix, respectively) of w. The set of all prefixes (suffixes) of w is denoted by $\text{Pref}(w)$ ($\text{Suff}(w)$, respectively), and $\text{Pref}'(w)$ ($\text{Suff}'(w)$, respectively) denotes the set of non-empty prefixes (suffixes) of w.

An *involution* is a function f that is its own inverse, i.e., for all x in the domain of f we have $f(f(x)) = x$. A function $h : \Sigma^* \to \Sigma^*$ is a *morphism on* Σ^* (respectively an *antimorphism on* Σ^*) if $h(\lambda) = \lambda$ and we have that $h(uv) = h(u)h(v)$ (respectively $h(uv) = h(v)h(u)$) for all $u, v \in \Sigma^*$. Note that if h is a morphism on the language Σ^* then $h(a_1a_2 \cdots a_n) = h(a_1)h(a_2) \cdots h(a_n)$, and if h is an antimorphism on Σ^* then $h(a_1a_2 \cdots a_n) = h(a_n) \cdots h(a_2)h(a_1)$, for all $a_i \in \Sigma, 1 \leq i \leq n$. A function $\phi : \Sigma^* \to \Sigma^*$ is called a *morphic involution on* Σ^* (respectively an *antimorphic involution on* Σ^*) if it is an involution on Σ extended to a morphism on Σ^* (respectively to an antimorphism on Σ^*). For convenience, in the remainder of this paper we use the convention that the letter ϕ denotes an involution that is either morphic or antimorphic (such a function will be termed *(anti)morphic involution*), that the letter θ denotes an antimorphic involution, and that the letter μ denotes a morphic involution.

Given an (anti)morphic involution ϕ, a word $x \in \Sigma^*$ is called a ϕ-border of $w \in \Sigma^+$ if $w = x\alpha = \beta\phi(x)$ for some $\alpha, \beta \in \Sigma^*$, and a proper ϕ-border if in addition, $|x| \neq |w|$, see [26]. The empty word λ is a ϕ-border of every $w \in \Sigma^+$. A non-empty word is said to be ϕ-bordered if it has a proper non-empty ϕ-border, and ϕ-unbordered otherwise. If ϕ is the identity on Σ extended to a morphism on Σ^*, then the ϕ-bordered words coincide with the classical bordered words, and the ϕ-unbordered words coincide with the classical unbordered words [55]. We recall:

Lemma 2. *[26] Let θ be an antimorphic involution on Σ^*. Then, for all $x \in \Sigma^+$ we have that x is θ-bordered iff $x = ay\theta(a)$ for some $a \in \Sigma$ and $y \in \Sigma^*$.*

A word $u \in \Sigma^*$ is called a *conjugate* of $v \in \Sigma^*$ if there exists a word $w \in \Sigma^*$ such that $uw = wv$ or, equivalently, if $u = xy$ and $v = yx$ for $x, y \in \Sigma^*$. In [27] the concept of conjugacy of words was extended to incorporate the notion of an (anti)morphic involution: A word $u \in \Sigma^*$ is a ϕ-conjugate of $v \in \Sigma^*$ if there exists a word $w \in \Sigma^*$ such that $uw = \phi(w)v$. In [27] it was shown that ϕ-conjugacy on words is not an equivalence relation when $\phi \neq I$. If ϕ is the identity on Σ extended to a morphism on Σ^*, this notion becomes the classic conjugacy on words.

A word $w \in \Sigma^*$ is called a palindrome if $w = w^R$, where the reverse, or mirror image operator is defined as $\lambda = \lambda^R$ and $(a_1a_2 \ldots a_n)^R = a_n \ldots a_2a_1$, when $a_i \in \Sigma$ for all $1 \leq i \leq n$. A word $w \in \Sigma^*$ is called a ϕ-palindrome if $w = \phi(w)$, and the set of all ϕ-palindromes is denoted by P_ϕ. If $\phi = \mu$ is a morphic involution on Σ^* then the only μ-palindromes are the words over Σ', where $\Sigma' \subseteq \Sigma$, and μ is the identity on Σ'. Lastly, if $\phi = \theta$ is the identity function on Σ extended to an antimorphism on Σ^*, then a θ-palindrome is a classical palindrome, while if $\phi = \mu$ is the identity function on Σ extended to a morphism on Σ^*, then every word is a μ-palindrome.

3. Involutive Fibonacci words

In this section, we generalize Definition 1 to define the n^{th} standard and reverse ϕ-Fibonacci words/sequences in several ways, where ϕ is an (anti)morphic involution. We consider special cases of such ϕ-Fibonacci words where the initial two words u and v have various properties: u and v are both palindromes, u and v are both ϕ-palindromes, or $u = \phi(v)$. As before, if the first two ϕ-Fibonacci words u and v are obvious from the context, the argument (u, v) can be omitted.

Definition 3. Let Σ be an alphabet with $|\Sigma| \geq 2$, let ϕ be an (anti)morphic involution on Σ^*, and $u, v \in \Sigma^+$. If the first two ϕ-Fibonacci words are u, respectively v, three types of n^{th} standard ϕ-Fibonacci words, $g_n(u, v)$, $w_n(u, v)$, $z_n(u, v)$, $n \geq 3$, are defined recursively as follows:

$$\begin{aligned} g_n &= \phi(g_{n-1})g_{n-2} && \text{(standard alternating } \phi\text{-Fibonacci words)},\\ w_n &= \phi(w_{n-1})\phi(w_{n-2}) && \text{(standard palindromic } \phi\text{-Fibonacci words)},\\ z_n &= z_{n-1}\phi(z_{n-2}) && \text{(standard hairpin } \phi\text{-Fibonacci words)}. \end{aligned}$$

Similarly, three types of n^{th} reverse ϕ-Fibonacci words, $g'_n(u, v)$, $w'_n(u, v)$, $z'_n(u, v)$, for $n \geq 3$, are defined recursively as follows:

$$g'_n = g'_{n-2}\phi(g'_{n-1}) \qquad \text{(reverse alternating } \phi\text{-Fibonacci words)},$$
$$w'_n = \phi(w'_{n-2})\phi(w'_{n-1}) \qquad \text{(reverse palindromic } \phi\text{-Fibonacci words)},$$
$$z'_n = \phi(z'_{n-2})z'_{n-1} \qquad \text{(reverse hairpin } \phi\text{-Fibonacci words)}.$$

The sequence of standard alternating ϕ-Fibonacci words $G(u,v)$ can now be defined as $G(u,v) = \{g_n(u,v)\}_{n\geq 1}$, and the sequences $W(u,v)$, $Z(u,v)$, $G'(u,v)$, $W'(u,v)$ and $Z'(u,v)$ can be similarly defined.

If the first two ϕ-Fibonacci words in the sequence are singleton letters in Σ, the respective ϕ-Fibonacci words will be called atom (standard or reverse) ϕ-Fibonacci words. If the involution ϕ is clear from the context, we will sometimes call ϕ-Fibonacci words simply involutive Fibonacci words.

Note that, in the particular case when ϕ is the identity function on Σ extended to a morphism on Σ^*, the words $g_n = w_n = z_n$ all coincide with the standard Fibonacci words f_n, while $g'_n = w'_n = z'_n$ all coincide with f'_n, for all $n \geq 1$. Thus, f_n and f'_n can also be called standard, respectively reverse, ϕ-Fibonacci words, for all $n \geq 1$, with ϕ being the identify function extended to a morphism.

We now illustrate the definitions with the following examples. Consider the quarternary alphabet $\Delta = \{A, C, G, T\}$, $\phi(A) = T$, $\phi(G) = C$, and vice versa. Assume that the first two ϕ-Fibonacci words are A, respectively C. Table 1 describes the first of the various atom standard and reverse ϕ-Fibonacci words, where ϕ is either a morphic involution (MI) or an antimorphic involution (AMI) on Δ^*. Note that, since Δ denotes the DNA alphabet, if the involution $\phi = \theta$ defined as above on Δ is extended to an antimorphism on Δ^*, then it models the DNA Watson-Crick complementarity of DNA strands. In this case, the standard palindromic θ-Fibonacci words w_n and the reverse palindromic θ-Fibonacci words w'_n form hairpin structures with partially double-stranded stems (Figure 1 depicts the word w_8), while the standard and reverse hairpin θ-Fibonacci words z_n, and z'_n form hairpins with fully double-stranded stems.

It was first shown in [11] that the prefix of the atom standard Fibonacci word f_n of length $|f_n| - 2$ is a palindrome, for all $n \geq 3$. Later in [15], the authors proved that the prefix of length $|f_n| - 2$ of f_n is also the suffix of the atom reverse Fibonacci word f'_n, for all $n \geq 3$. These results from [11, 15] can be combined in the following lemma.

Lemma 4. *Let $\Sigma = \{a, b\}$, with $a \neq b$, and let $f_1 = a$ and $f_2 = b$. Then, for $n \geq 3$, we have that $f_n = s_n d_n$ and $f'_n = d'_n s_n$ where $d'_n = d_n^R$ such that s_n is a palindrome, and $d_n = ab$ if n is even, while $d_n = ba$ if n is odd.*

One can easily observe from Lemma 4 that $s_n = s_n^R$, for all $n \geq 3$. Hence, for $n \geq 3$, we have that $d'_n s_n = f'_n = f_n^R = d_n^R s_n^R$. Thus, we conclude the following.

Lemma 5. *Let $\Sigma = \{a, b\}$, with $a \neq b$, and let $f_1 = f'_1 = a$, $f_2 = f'_2 = b$. Then we have $f'_n = f_n^R$, for all $n \geq 1$.*

One can easily show that a result similar to Lemma 5 also holds for atom standard and atom reverse ϕ-Fibonacci words.

In the remainder of this paper, we will often have to make statements that hold for several types of ϕ-Fibonacci words. For brevity, we will use the notational convention that a statement of the type "$\alpha_n \in \{f_n, g_n, w_n, z_n\}$ for all $n \geq 1$" means that either we have $\alpha_n = f_n$ for all $n \geq 1$, or that $\alpha_n = g_n$ for all $n \geq 1$, or that $\alpha_n = w_n$ for all $n \geq 1$, or that $\alpha_n = z_n$ for all $n \geq 1$. Moreover, we will use the notational convention that a statement of the type "$\alpha_n = g_n$, for all $n \geq 1$" also implies the statement "$\alpha'_n = g'_n$, for all $n \geq 1$."

Lemma 6. *Let $\Sigma = \{a, b, c, d\}$, let ϕ be an (anti)morphic involution on Σ^*, and let the initial ϕ-Fibonacci words be $\alpha_1 = \alpha'_1 = a$, $\alpha_2 = \alpha'_2 = b$. If $\alpha_n \in \{f_n, g_n, w_n, z_n\}$ for all $n \geq 1$, then $\alpha'_n = \alpha_n^R$ for all $n \geq 1$.*

Proof. By strong induction on n. □

	$g_n = \phi(g_{n-1})g_{n-2}$		$g'_n = g'_{n-2}\phi(g'_{n-1})$	
n	MI	AMI	MI	AMI
3	GA	GA	AG	AG
4	CTC	TCC	CTC	CCT
5	$GAGGA$	$GGAGA$	$AGGAG$	$AGAGG$
6	$CTCCTCTC$	$TCTCCTCC$	$CTCTCCTC$	$CCTCCTCT$

	$w_n = \phi(w_{n-1})\phi(w_{n-2})$		$w'_n = \phi(w'_{n-2})\phi(w'_{n-1})$	
n	MI	AMI	MI	AMI
3	GT	GT	TG	TG
4	CAG	ACG	GAC	GCA
5	$GTCCA$	$CGTAC$	$ACCTG$	$CATGC$
6	$CAGGTGTC$	$GTACGCGT$	$CTGTGGAC$	$TGCGCATG$

	$z_n = z_{n-1}\phi(z_{n-2})$		$z'_n = \phi(z'_{n-2})z'_{n-1}$	
n	MI	AMI	MI	AMI
3	CT	CT	TC	TC
4	CTG	CTG	GTC	GTC
5	$CTGGA$	$CTGAG$	$AGGTC$	$GAGTC$
6	$CTGGAGAC$	$CTGAGCAG$	$CAGAGGTC$	$GACGAGTC$

Table 1: The n^{th} atom ϕ-Fibonacci words g_n, g'_n, w_n, w'_n, z_n, and z'_n with initial words A and C, $3 \leq n \leq 6$, where $\phi(A) = T$, $\phi(C) = G$, is an involution extended to either a morphism (MI), or an antimorphism (AMI).

Note that Lemma 5 and Lemma 6 justify our terminology, of calling f_n, g_n, w_n and z_n "standard" ϕ-Fibonacci words, while calling f'_n, g'_n, z'_n and w'_n "reverse" ϕ-Fibonacci words. Lemma 5, which holds for atom Fibonacci words, can now be generalized to Fibonacci words with $f_1 = u$ and $f_2 = v$, provided that u and v are (non-empty) palindromes. Indeed, the following result is a direct corollary of Theorem 4 in [8], which stated that $f_n(u, v)$ and $f'_n(u, v)$ are conjugates for any $u, v \in \Sigma^+$.

Corollary 7. *Let $f_1 = f'_1 = u$ and $f_2 = f'_2 = v$, such that u and v in Σ^+ are palindromes. Then, for all $n \geq 3$, there exist non-empty palindromes $x_n, y_n \in \Sigma^+$ such that $f_n = x_n y_n$ and $f'_n = y_n x_n$ and hence $f'_n = f_n^R$.*

Note that, in general, the decomposition of f_n as a product of two non-empty palindromes x_n and y_n (Corollary 7), is not necessarily unique. For example, if we have $f_1 = bab$ and $f_2 = aba$, then $f_3 = ababab = (a)(babab) = (ababa)(b) = (aba)(bab)$ has three different decompositions into a product of two palindromes.

In the following, we will prove a result similar to Corollary 7, for the ϕ-Fibonacci words g_n and g'_n, for both morphic as well as antimorphic involutions. The following result is a generalization of Lemma 4, to the case of g_n and g'_n.

Proposition 8. *Let ϕ be an (anti)morphic involution on Σ^*, and let $g_1 = g'_1 = u$ and $g_2 = g'_2 = v$, where $u, v \in \Sigma^+$. Then, for $n \geq 3$, we have:*

$$g_n = \begin{cases} s_n xy & : n \text{ is even}, \\ s_n pq & : n \text{ is odd}, \end{cases} \qquad g'_n = \begin{cases} yxs'_n & : n \text{ is even}, \\ qps'_n & : n \text{ is odd}, \end{cases}$$

where $s_3 = s'_3 = \lambda$, and

(I) *If $\phi = \mu$ is a morphic involution, then $x = \mu(u)$, $y = v$, $p = \mu(v)$ and $q = u$ and for all $n \geq 4$, $s_n = s'_n = \mu(s_{n-1})g'_{n-2} = g_{n-2}\mu(s_{n-1})$. In addition, for $n \geq 4$, there exists word y_n such that $y_n g_n = g'_n y_n$ and $y_n = g'_{n-2}\mu(u)v$ when n is even and $y_n = g'_{n-2}\mu(v)u$ otherwise.*

(II) *If $\phi = \theta$ is an antimorphic involution, then $x = y = v$, $p = \theta(v)$, $q = u$ and for all $n \geq 4$, $s_n = \theta(y_{n-1})\theta(s'_{n-1})$ and $s'_n = \theta(s_{n-1})\theta(y_{n-1})$. In addition, for $n \geq 4$ there exists a word y_n such that $y_n g_n = g'_n y_n$ and $y_n = y_{n-2}g_{n-2}$, where $y_3 = u$ and $y_2 = v$.*

Proposition 8 can now be used to prove Corollary 9, which generalizes Theorem 4 in [8] and Corollary 7 to the case of g_n and g'_n.

Corollary 9. *Let ϕ be an (anti)morphic involution on Σ^*, and let the two initial words $g_1 = g'_1 = u$ and $g_2 = g'_2 = v$ be words in Σ^+. Then the words g_n and g'_n are conjugates for all $n \geq 1$. If in addition, u and v are palindromes, then for all $n \geq 3$, there exists palindromes x_n and y_n such that $g_n = x_n y_n$, $g'_n = y_n x_n$, and hence also $g'_n = g_n^R$.*

Given an (anti)morphic involution ϕ and initial words g_1 and g_2, the decomposition of g_n into palindromes is not necessarily unique. Consider for example the (anti)morphic involution ϕ such that $\phi(a) = b$ and $\phi(b) = a$. If $g_1 = bab$ and $g_2 = aba$, then $g_4 = abaabaaba$ which can expressed as $g_4 = (aba)(abaaba) = (abaaba)(aba)$.

Theorem 4 of [8] showed that the Fibonacci words f_n and f'_n are conjugates of each other. Similarly, Corollary 9 shows that the ϕ-Fibonacci words g_n and g'_n are conjugates of each other, for both morphic and antimorphic ϕ. This does not hold for ϕ-Fibonacci words z_n and w_n, as shown by checking the case $n = 4$ in Table 1.

Even though we cannot prove conjugacy of z_n and z'_n (respectively w_n and w'_n) for all $n \geq 1$ in the general case, the following proposition holds, implying that if the first two Fibonacci words are palindromes, then $z'_n = z_n^R$ and $w'_n = w_n^R$ (Corollary 11).

Proposition 10. *Let ϕ be an (anti)morphic involution on Σ^*, and let $\alpha_1 = \alpha'_1 = u$ and $\alpha_2 = \alpha'_2 = v$, where $u, v \in \Sigma^+$. If $\alpha_n \in \{z_n, w_n\}$ for all $n \geq 1$, then, for all $n \geq 3$, we have that $\alpha_n = s_n d_n$ and $\alpha'_n = d'_n s'_n$, where:*

(I) *If $\alpha_n = z_n$ and $\phi = \mu$ is a morphic involution, then*

$$d_n = \begin{cases} \mu(u)\mu(v) \ : \ n \bmod 4 = 0, \\ \mu(v)u \ : \ n \bmod 4 = 1, \\ uv \ : \ n \bmod 4 = 2, \\ v\mu(u) \ : \ n \bmod 4 = 3, \end{cases} \qquad d'_n = \begin{cases} \mu(v)\mu(u) \ : \ n \bmod 4 = 0, \\ u\mu(v) \ : \ n \bmod 4 = 1, \\ vu \ : \ n \bmod 4 = 2, \\ \mu(u)v \ : \ n \bmod 4 = 3. \end{cases}$$

(II) *If $\alpha_n = z_n$ and $\phi = \theta$ is an antimorphic involution, then we have $d_n = u\theta(v)$ and $d'_n = \theta(v)u$ for all $n \geq 5$.*

(III) *If $\alpha_n = w_n$ and $\phi = \mu$ is a morphic involution, then*

$$d_n = \begin{cases} u\mu(v) \ : \ n \bmod 4 = 0, \\ vu \ : \ n \bmod 4 = 1, \\ \mu(u)v \ : \ n \bmod 4 = 2, \\ \mu(v)\mu(u) \ : \ n \bmod 4 = 3, \end{cases} \qquad d'_n = \begin{cases} \mu(v)u \ : \ n \bmod 4 = 0, \\ uv \ : \ n \bmod 4 = 1, \\ v\mu(u) \ : \ n \bmod 4 = 2, \\ \mu(u)\mu(v) \ : \ n \bmod 4 = 3. \end{cases}$$

(IV) *If $\alpha_n = w_n$ and $\phi = \theta$ is an antimorphic involution, then*

$$d_n = \begin{cases} uv \ : \ n \bmod 3 = 1, \\ \theta(v)\theta(u) \ : \ n \bmod 3 = 2, \\ v\theta(v) \ : \ n \bmod 3 = 0, \end{cases} \qquad d'_n = \begin{cases} vu \ : \ n \bmod 3 = 1, \\ \theta(u)\theta(v) \ : \ n \bmod 3 = 2, \\ \theta(v)v \ : \ n \bmod 3 = 0. \end{cases}$$

Proof. We only prove the case when $\alpha_n = w_n$ for all $n \geq 1$ and $\phi = \mu$ is a morphic involution, by strong induction on n. The base cases for $n = 4, 5, 6, 7$ can be verified directly where $\mu(u) = u'$ and $\mu(v) = v'$. Assume the statement true for all $k \leq n$, and consider w_{n+1}. If $n + 1 = 4k$, then we have that

$$w_{n+1} = \mu(w_n)\mu(w_{n-1}) = \mu(s_n v' u' s_{n-1} u' v) = \mu(s_n) vu\mu(s_{n-1}) uv' = s_{n+1} uv',$$

where $s_{n+1} = \mu(s_n)vu\mu(s_{n-1})$, and at the same time

$$w'_{n+1} = \mu(w'_{n-1})\mu(w'_n) = \mu(vu's'_{n-1})\mu(u'v's'_n) = v'u\mu(s'_{n-1})uv\mu(s'_n) = v'us'_{n+1},$$

where $s'_{n+1} = \mu(s'_{n-1})uv\mu(s'_n)$. If $u = u^R$ and $v = v^R$, then we have a decomposition whereby $s'_{n+1} = \mu(s^R_{n-1})uv\mu(s^R_n) = s^R_{n+1}$. Also, note that $w'_{n+1} = w^R_{n+1}$. The cases where $n + 1 = 4k + 1$, $n + 1 = 4k + 2$, or $n + 1 = 4k + 3$ can be proved similarly. $\square$

Corollary 11. *Let ϕ be an (anti)morphic involution on Σ^*, let $u, v \in \Sigma^+$ and let $\alpha_1 = \alpha'_1 = u$ and $\alpha_2 = \alpha'_2 = v$. If $\alpha_n \in \{z_n, w_n\}$ for all $n \geq 1$ then, for all $n \geq 3$, we have that $\alpha_n = s_n d_n$ and $\alpha'_n = d'_n s'_n$ as in Proposition 10. If in addition, u and v are palindromes then, for all $n \geq 5$, we have that $\alpha'_n = \alpha_n^R$, $s'_n = s_n^R$ and $d'_n = d_n^R$.*

Corollaries 7, 9 and 11 show that in the special case of u and v being non-empty palindromes, if $\alpha_n \in \{f_n, g_n, w_n, z_n\}$ for all $n \geq 1$, then α'_n is the reverse of α_n for all $n \geq 5$. We now consider other special cases by imposing other constraints on the initial words u and v. We first consider the case when the initial words u and v are (non-empty) ϕ-palindromes. If $\phi = \mu$ is a morphic involution, then μ is the identity mapping on $alph(u) \cup alph(v)$, and a newly defined ϕ-Fibonacci word g_n, w_n, or z_n coincides with f_n, for all $n \geq 1$ (the case of f_n was discussed in Corollary 7). The following proposition considers the case when $\phi = \theta$ is an antimorphic involution.

Theorem 12. *Let $f_i = f'_i$, $g_i = g'_i$, $w_i = w'_i$ and $z_i = z'_i$ for $i = 1, 2$, and let θ be an antimorphic involution on Σ^*. If f_i, g_i, w_i, z_i are non-empty θ-palindromes for $i = 1, 2$ then:*

(I) *For all $n \geq 1$, $f_n = \theta(f'_n)$, $g_n = \theta(g'_n)$, $w_n = \theta(w'_n)$ and $z_n = \theta(z'_n)$.*

(II) *For all $n \geq 5$, g_n and g'_n are conjugates of each other, namely $g_n = x_n y_n$ and $g'_n = y_n x_n$, where x_n, y_n are θ-palindromes. In addition, x_{2n} can be decomposed as $x_{2n} = y_{2n-1} = g_1 g_3 \cdots g_{2n-5} g_{2n-3}$, while x_{2n+1} can be decomposed as $x_{2n+1} = y_{2n} = g_2^2 g_4 g_6 \cdots g_{2n-4} g_{2n-2}$.*

(III) *For $n \neq 3k, k \geq 1$, we have that $z_n = z'_n$, and for $n = 3k, k \geq 1$, we have that z_n and z'_n are conjugates, that is, $z_n = x_n y_n$, $z'_n = y_n x_n$ where both $x_n = z_{n-1}$ and $y_n = z_{n-2}$ are θ-palindromes.*

(IV) *For all $n \geq 5$, w_n and w'_n are θ-conjugates, that is, we have $w_n = x_n y_n$ and $w'_n = \theta(y_n) x_n$ where x_n is a θ-palindrome and we have decompositions whereby $x_n = w_{n-3} w'_{n-3}$ and $y_n = w'_{n-4} w'_{n-2}$.*

Proof. By strong induction on n. □

It was observed, e.g., in [19], that the length of the atom reverse Fibonacci word f'_n is the Fibonacci number F_n. For other ϕ-Fibonacci words $\alpha_n \in \{g_n, z_n, w_n, g'_n, z'_n, w'_n\}$ for all $n \geq 1$, if either $\alpha_1 = \alpha_2$ or $\phi(\alpha_1) = \alpha_2$ then $\alpha_n = u_1 u_2 u_3 \cdots u_k$, where, for $1 \leq i \leq k$, $u_i \in \{\alpha_1, \phi(\alpha_1)\}$, and therefore the length of the n^{th} ϕ-Fibonacci word equals $|\alpha_1| \times F_n$.

We end this section by considering another special case of ϕ-Fibonacci words, where the initial words u and v satisfy the condition $\phi(u) = v$.

Proposition 13. *Let $u, v \in \Sigma^+$, let $g_1 = w_1 = z_1 = g'_1 = w'_1 = z'_1 = u$ and let $g_2 = w_2 = z_2 = g'_2 = w'_2 = z'_2 = v$. Let ϕ be an (anti)morphic involution on Σ^* such that $\phi(u) = v$. Then, for all $n \geq 3$, we have:*

(I) *If n is odd, $g_n = g'_n = u^{F_n}$, and if n is even, $g_n = g'_n = v^{F_n}$.*

(II) *If $\phi = \mu$ is a morphic involution and u and v are palindromes, then*

$$\begin{cases} w_n = x_n y_n & w'_n = y_n^R x_n \ : \ n \text{ is odd}, \\ w_n = x_n y_n & w'_n = y_n x_n^R \ : \ n \text{ is even}. \end{cases}$$

(III) *If $\phi = \theta$ is an antimorphic involution and u and v are palindromes, then*

$$\begin{cases} w_n = w'_n \ : \ n \neq 3k, k \geq 1, \\ w_n = (uv)^i \ , \ w'_n = (vu)^i \ : \ n = 3k \ , k \geq 1, i = \frac{|F_n|}{2}. \end{cases}$$

Proof.

(I) Follows from the definition of g_n, g'_n, and the hypothesis that $\phi(u) = v$.

(II) Follows by induction on n, from the definition of w_n, w'_n, and the assumptions that $\mu(u) = v$ and that u and v are palindromes.

(III) Proof similar to that of (II).

□

3.1. Relations between Fibonacci words and μ-Fibonacci words

In this section we find some relationships that exist between alternating μ-Fibonacci words g_n and standard Fibonacci words f_n for morphic involutions μ (see Theorem 15). We namely prove that, as suggested by Table 1, in the case of a morphic involution μ, the words in the sequence of μ-Fibonacci words $\{g_n(u, v)\}_{n\geq 1}$ can be obtained by alternating the words from two different sequences of standard Fibonacci words, $\{f_n(u, \mu(v))\}_{n\geq 1}$ and $\{f_n(\mu(u), v)\}_{n\geq 1}$, as follows: $g_n(u, v)$ coincides with $f_n(u, \mu(v))$ for odd n, and with $f_n(\mu(u), v)$ for even n. This property was the rationale for calling g_n "alternating Fibonacci words." A similar relationship holds between sequences $\{g'_n\}_{n\geq 1}$ and $\{f'_n\}_{n\geq 1}$, and between sequences $\{w_n\}_{n\geq 1}$ (respectively $\{w'_n\}_{n\geq 1}$) and $\{z_n\}_{n\geq 1}$ (respectively $\{z'_n\}_{n\geq 1}$). The following lemma is used to prove these relationships.

Lemma 14. *Let $\phi = \mu_2$ be morphic involution on Σ^*, let μ_1 be a morphic involution on Σ^* such that $\mu_1\mu_2 = \mu_2\mu_1$, and let $u, v \in \Sigma^+$. If $\alpha_n \in \{f_n, g_n, w_n, z_n\}$ for all $n \geq 1$ are μ_2-Fibonacci words then $\mu_1(\alpha_n(u, v)) = \alpha_n(\mu_1(u), \mu_1(v))$, and we have $\mu_1(\alpha'_n(u, v)) = \alpha'_n(\mu_1(u), \mu_1(v))$, for all $n \geq 1$.*

Proof. We consider the standard μ_2-Fibonacci words g_n. The proof is by strong induction on n. By definition, we have $\mu_1(g_1(u, v)) = \mu_1(u) = g_1(\mu_1(u), \mu_1(v))$, and $\mu_1(g_2(u, v)) = \mu_1(v) = g_2(\mu_1(u), \mu_1(v))$, so the base case holds. Assume now that $\mu_1(g_i(u, v)) = g_i(\mu_1(u), \mu_1(v))$, for all $1 \leq i \leq k$. Using the definition of g_n, for $n \geq 1$, the fact that μ_1 is a morphism, and the induction hypothesis, we have that

$$\begin{aligned}\mu_1(g_{k+1}(u,v)) &= \mu_1(\mu_2(g_k(u,v)) \cdot g_{k-1}(u,v)) \\ &= \mu_2(\mu_1(g_k(u,v))) \cdot \mu_1(g_{k-1}(u,v)) \\ &= \mu_2(g_k(\mu_1(u),\mu_1(v))) \cdot g_{k-1}(\mu_1(u),\mu_1(v)) \\ &= g_{k+1}(\mu_1(u),\mu_1(v)).\end{aligned}$$

The proofs for other μ_2-Fibonacci words are similar. □

Lemma 14 can be used to prove the following result.

Theorem 15. *Let μ be a morphic involution on Σ^*, let $u, v \in \Sigma^+$, and let*

$$(\alpha_n, \beta_n) \in \{(f_n, g_n), (g_n, f_n), (z_n, w_n), (w_n, z_n), (f'_n, g'_n), (g'_n, f'_n), (z'_n, w'_n), (w'_n, z'_n)\},$$

for all $n \geq 1$. The following relations hold for all $n \geq 1$:

$$\alpha_n(u,v) = \begin{cases} \beta_n(u,\mu(v)) & : \ n \text{ is odd,} \\ \beta_n(\mu(u),v) & : \ n \text{ is even.} \end{cases}$$

Proof. We prove for the pair (g_n, f_n) by using strong induction on n. By definition, we have $g_1(u,v) = u = f_1(u,\mu(v))$, $g_2(u,v) = v = f_2(\mu(u),v)$.

Assume now that for $1 \leq i \leq k$, we have that $g_i(u,v) = f_i(u,\mu(v))$ if i is odd, and that $g_i(u,v) = f_i(\mu(u),v)$ if i is even.

If $k+1$ is odd (the case of $k+1$ even is similar), then by the definition of g_{k+1} and f_{k+1}, the induction hypothesis, and Lemma 14 with $\mu_1 = \mu_2 = \mu$, we have that:

$$\begin{aligned}g_{k+1}(u,v) =&\mu(g_k(u,v)) \cdot g_{k-1}(u,v) = \mu(f_k(\mu(u),v)) \cdot f_{k-1}(u,\mu(v)) \\ =&f_k(u,\mu(v)) \cdot f_{k-1}(u,\mu(v)) = f_{k+1}(u,\mu(v)).\end{aligned}$$

The proofs for other cases are similar. □

In the case of an antimorphic involution, a relation like that of Theorem 15 does not hold. Indeed, for example, in Table 1 we have that

$$g_4(A,C) = TCC \neq CTC = f_4(\theta(A),C) = f_4(T,C).$$

We will end this subsection with some observations on iterated morphisms generating certain types of involutive Fibonacci words. Let $\Delta = \{A,C,G,T\}$ be the DNA alphabet and let θ be the Watson-Crick antimorphic involution on Δ^* that maps A to T, and C to G. Then, assuming that A and C are the first two Fibonacci words, we have that:

- The word g_n can be obtained by iterating on A the morphism h_g defined as $h_g(A) = C$, $h_g(C) = GA$, $h_g(G) = TC$, and $h_g(T) = G$.
- The word w_n can be obtained by iterating on A the morphism h_w defined as $h_w(A) = C$, $h_w(C) = GT$, $h_w(T) = G$, and $h_w(G) = AC$.
- The word z_n can be obtained by iterating on A the morphism h_z defined as $h_z(A) = C$, $h_z(C) = CT$, $h_z(T) = G$ and $h_z(G) = AG$.

4. Indexed involutive Fibonacci words

In this section we show that in the case of both a morphic and an antimorphic involution, the ϕ-Fibonacci words are connected to the indexed Fibonacci words defined and studied in [8]. We also define indexed ϕ-Fibonacci words (Definition 22), which are a generalization of indexed Fibonacci words, and find relationships between various types of such words.

We first show the relations between the θ-Fibonacci words g_n and indexed Fibonacci words. Recall the notion of indexed Fibonacci words, defined and investigated in [8] (note that [8] used a different notation):

Definition 16. Let Σ be an alphabet, and let $u, v \in \Sigma^+$. The indexed Fibonacci words are defined recursively as

$$f^0(u,v) = u, f^{00}(u,v) = v,$$

and, for all $n \geq 2$,

$$f^{r_1 r_2 \ldots r_n 0}(u,v) = f^{r_1 r_2 \ldots r_n}(u,v) \cdot f^{r_1 r_2 \ldots r_{n-1}}(u,v),$$
$$f^{r_1 r_2 \ldots r_n 1}(u,v) = f^{r_1 r_2 \ldots r_{n-1}}(u,v) \cdot f^{r_1 r_2 \ldots r_n}(u,v),$$

where $r_1 = r_2 = 0$ and $r_i \in \{0,1\}, 3 \leq i \leq n$.

Informally, in the construction of an indexed Fibonacci word $f^{00 r_3 r_4 \ldots r_n r_{n+1}}(u,v)$, we use the digit $r_{n+1} = 0$ to denote concatenating the last word with the second last word in the sequence (according to the standard Fibonacci concatenation order), and digit $r_{n+1} = 1$ to denote concatenating the second last word with the last word (according to the reverse Fibonacci concatenation order). Note that the standard (respectively reverse) Fibonacci words now become particular cases of indexed Fibonacci words, in the construction of which the standard Fibonacci concatenation order (respectively reverse Fibonacci concatenation order) is always used, that is, $r_n = 0$ for all $n \geq 3$ (respectively $r_n = 1$ for all $n \geq 3$), as follows:

$$f_1(u,v) = f_1'(u,v) = f^0(u,v) = u, \; f_2(u,v) = f_2'(u,v) = f^{00}(u,v) = v,$$
$$f_n(u,v) = f^{00\,0^{n-2}}(u,v), \; f_n'(u,v) = f^{00\,1^{n-2}}(u,v), \quad n \geq 3.$$

As before, when the initial words u, v are clear for the context, the argument (u, v) will be omitted.

The derivation of a sequence of indexed Fibonacci words can be represented by a path from the root (f^0 and f^{00}) to a leaf $f^{00 r_3 \ldots r_n}$, $n \geq 3$, in a tree-like structure, as follows:

$$f^0 = u, f^{00} = v \rightarrow \begin{cases} f^{000} = vu \rightarrow \begin{cases} f^{0000} = vuv \rightarrow \ldots \\ f^{0001} = vvu \rightarrow \ldots \end{cases} \\ f^{001} = uv \rightarrow \begin{cases} f^{0010} = uvv \rightarrow \ldots \\ f^{0011} = vuv \rightarrow \ldots \end{cases} \end{cases}$$

We now recall a result from [8].

Proposition 17. *If u and v are non-empty palindromes then, for all $n \geq 3$, we have that $f^{00r_3r_4\ldots r_n}(u,v) = (f^{00s_3s_4\ldots s_n}(u,v))^R$, where $s_j = 1 - r_j$ for $3 \leq j \leq n$.*

The following Lemma generalizes the above result and will aid in the proof of Proposition 19.

Lemma 18. *Let ϕ be an (anti)morphic involution on Σ^* and $u, v \in \Sigma^+$. Then, for all $n \geq 3$, we have that $\phi(f^{00r_3\ldots r_n}(u,v)) = f^{00s_3\ldots s_n}(\phi(u),\phi(v))$, where $r_i \in \{0,1\}$, and for all $3 \leq i \leq n$*

$$\begin{cases} s_i = r_i \ : \ \phi \textit{ is a morphic involution,} \\ s_i = 1 - r_i \ : \ \phi \textit{ is an antimorphic involution.} \end{cases}$$

Proof. We only prove for the case when $\phi = \theta$ is an antimophic involution by strong induction on n. The base case ($n = 3$) follows by the definition of indexed Fibonacci words, and the fact that θ is antimorphic. Indeed, we have that

$$\theta(f^{000}(u,v)) = \theta(v \cdot u) = \theta(u) \cdot \theta(v) = f^{001}(\theta(u),\theta(v)),$$

$$\theta(f^{001}(u,v)) = \theta(u \cdot v) = \theta(v) \cdot \theta(u) = f^{000}(\theta(u),\theta(v)).$$

The case $n = 4$ can be proven similarly.

Assume now that $\theta(f^{00r_3\ldots r_j}(u,v)) = f^{00s_3\ldots s_j}(\theta(u),\theta(v))$, where $r_i \in \{1,0\}$ and $s_i = 1 - r_i$, $3 \leq i \leq j$, for all $3 \leq j \leq k$, and $k \geq 4$. Consider $f^{00r_3\ldots r_k r_{k+1}}(u,v)$, and assume that $r_{k+1} = 0$. Using the definition of indexed Fibonacci words and the induction hypothesis, we have:

$$\begin{aligned} \theta(f^{00r_3\ldots r_k 0}(u,v)) =& \theta(f^{00r_3\ldots r_k}(u,v) \cdot f^{00r_3\ldots r_{k-1}}(u,v)) \\ =& \theta(f^{00r_3\ldots r_{k-1}}(u,v)) \cdot \theta(f^{00r_3\ldots r_k}(u,v)) \\ =& f^{00s_3\ldots s_{k-1}}(\theta(u),\theta(v)) \cdot f^{00s_3\ldots s_k}(\theta(u),\theta(v)) \\ =& f^{00s_3\ldots s_{k-1}s_k 1}(\theta(u),\theta(v)). \end{aligned}$$

The case $r_{k+1} = 1$ can be proved similarly. Thus, the inductive step and the proof hold. □

The next result shows that for an antimorphic involution θ, the sequence of alternating θ-Fibonacci words g_n consists of interleaving words from two sequences: If n is odd, it takes the word from the sequence of indexed Fibonacci words $f^{00010101010\ldots}$ (which alternates between the standard and reverse concatenation in its construction); If n is even it takes the word from the sequence of indexed Fibonacci words $f^{0010101010\ldots}$ (which alternates between the reverse and standard concatenation in its construction).

Proposition 19. *Let θ be an antimorphic involution on Σ^*, let u, v be two words in Σ^+, and let $\alpha_n \in \{g_n, g'_n\}$ for all $n \geq 1$. The following relations hold for all $n \geq 3$:*

$$\alpha_n(u,v) = \begin{cases} f^{00r\{sr\}^i}(u,\theta(v)) \ : \ n \textit{ is odd, } i = \frac{n-3}{2}, \\ f^{00\{sr\}^i}(\theta(u),v) \ : \ n \textit{ is even, } i = \frac{n-2}{2}, \end{cases}$$

where $r = 0, s = 1$ *if* $\alpha_n = g_n$*, and* $r = 1, s = 0$ *if* $\alpha_n = g'_n$*.*

Proof. It follows by strong induction on n, using Lemma 18. $\square$

A relationship similar to that of Proposition 19 can be obtained for the case of morphic involutions as stated in Proposition 21, the immediate proof of which uses Lemma 18 and Lemma 20.

Lemma 20. *Let* u, v *be two words in* Σ^+*. Then,* $f^{\gamma 00}(u, v) = f^{\gamma 11}(u, v)$ *for all words* $\gamma = r_1 r_2 \cdots r_n$*,* $n \geq 2$*, where* $r_1 = r_2 = 0$*,* $r_i \in \{0, 1\}$ *for* $3 \leq i \leq n$*.*

Proof. It follows $f^{\gamma 00}(u, v) = f^{\gamma}(u, v) f^{r_1 r_2 \cdots r_{n-1}}(u, v) f^{\gamma}(u, v) = f^{\gamma 11}(u, v)$. $\square$

Proposition 21. *Let* μ *be a morphic involution on* Σ^**, let* u, v *be two words in* Σ^+*, and let* $\alpha_n \in \{g_n, g'_n\}$ *for all* $n \geq 1$*. Then,* $\alpha_3(u, v) = f^{00r}(u, \mu(v))$ *and the following relations hold for* $n \geq 4$*:*

$$\alpha_n(u, v) = \begin{cases} f^{00r^i ss}(\mu(u), v) & : \ i = n - 4, \ \ n \text{ is even,} \\ f^{00r^i ss}(u, \mu(v)) & : \ i = n - 4, \ \ n \text{ is odd,} \end{cases}$$

where $r = 0, s = 1$ *if* $\alpha_n = g_n$*, and* $r = 1, s = 0$ *if* $\alpha_n = g'_n$*.*

We now extend the concept of indexed Fibonacci words defined and studied in [8] to indexed ϕ-Fibonacci words.

Definition 22. Let ϕ be an (anti)morphic involution on Σ^*, let $u, v \in \Sigma^+$, and let $\alpha \in \{f, g, w, z\}$. The indexed ϕ-Fibonacci words are defined recursively as

$$\alpha^0(u, v) = u, \alpha^{00}(u, v) = v,$$

and for all $n \geq 2$ we have that $r_1 = r_2 = 0$, $r_i \in \{0, 1\}$ for $3 \leq i \leq n$, and:

$$\alpha^{r_1 r_2 \cdots r_n 0}(u, v) = \begin{cases} \alpha^{r_1 r_2 \cdots r_n}(u, v) \ \cdot \ \alpha^{r_1 r_2 \cdots r_{n-1}}(u, v) & : \ \alpha = f, \\ \phi(\alpha^{r_1 r_2 \cdots r_n}(u, v)) \ \cdot \ \alpha^{r_1 r_2 \cdots r_{n-1}}(u, v) & : \ \alpha = g, \\ \phi(\alpha^{r_1 r_2 \cdots r_n}(u, v)) \ \cdot \ \phi(\alpha^{r_1 r_2 \cdots r_{n-1}}(u, v)) & : \ \alpha = w, \\ \alpha^{r_1 r_2 \cdots r_n}(u, v) \ \cdot \ \phi(\alpha^{r_1 r_2 \cdots r_{n-1}}(u, v)) & : \ \alpha = z, \end{cases}$$

$$\alpha^{r_1 r_2 \cdots r_n 1}(u, v) = \begin{cases} \alpha^{r_1 r_2 \cdots r_{n-1}}(u, v) \ \cdot \ \alpha^{r_1 r_2 \cdots r_n}(u, v) & : \ \alpha = f, \\ \alpha^{r_1 r_2 \cdots r_{n-1}}(u, v) \ \cdot \ \phi(\alpha^{r_1 r_2 \cdots r_n}(u, v)) & : \ \alpha = g, \\ \phi(\alpha^{r_1 r_2 \cdots r_{n-1}}(u, v)) \ \cdot \ \phi(\alpha^{r_1 r_2 \cdots r_n}(u, v)) & : \ \alpha = w, \\ \phi(\alpha^{r_1 r_2 \cdots r_{n-1}}(u, v)) \ \cdot \ \alpha^{r_1 r_2 \cdots r_n}(u, v) & : \ \alpha = z. \end{cases}$$

Note that for a morphic involution μ, the results in Lemma 20 hold also for the indexed μ-Fibonacci words g, while Proposition 21 holds also if the roles of f and g are swapped. However, one can easily verify that Lemma 20 does not hold for the indexed μ-Fibonacci words z or w, and Proposition 21 does not hold in the case where $\alpha_n \in \{z_n, z'_n\}$ for all $n \geq 1$ and f is the indexed μ-Fibonacci word w, or in the case where $\alpha_n = \{w_n, w'_n\}$ for all $n \geq 1$ and f is the indexed μ-Fibonacci word z.

However, the following results hold, which extends Theorem 15 to the case of indexed Fibonacci words.

Proposition 23. *Let μ be a morphic involution on Σ^*, let $u, v \in \Sigma^+$, and let $(\alpha_n, \beta) \in \{(f_n, g), (g_n, f), (z_n, w), (w_n, z), (f'_n, g'), (g'_n, f'), (z'_n, w'), (w'_n, z')\}$ for all $n \geq 1$. Then the following relations hold for all $n \geq 1$:*

$$\alpha_n(u,v) = \begin{cases} \beta^{00r^{n-2}}(u, \mu(v)) \ : \ n \text{ is odd,} \\ \beta^{00r^{n-2}}(\mu(u), v) \ : \ n \text{ is even,} \end{cases}$$

where $r = 0$ if $\alpha_n \in \{g_n, f_n, z_n, w_n\}$, and $r = 1$ if $\alpha_n \in \{g'_n, f'_n, z'_n, w'_n\}$.

Proof. It follows by strong induction on n. □

We now generalize Lemma 18 to indexed ϕ-Fibonacci words as follows.

Lemma 24. *Let $\phi = \phi_2$ be an (anti)morphic involution on Σ^*, and let ϕ_1 be an (anti)morphic involution on Σ^* such that $\phi_1\phi_2 = \phi_2\phi_1$. Let u, v be two words in Σ^+, and let $\alpha \in \{f, g, w, z\}$ be constructed using the (anti)morphic involution ϕ_2. Then, for all $n \geq 1$, we have that $\phi_1(\alpha^{r_1 r_2 r_3 \ldots r_n}(u, v)) = \alpha^{s_1 s_2 s_3 \ldots s_n}(\phi_1(u), \phi_1(v))$, where $r_1 = r_2 = s_1 = s_2 = 0$, and for all $3 \leq i \leq n$, we have:*

$$\begin{cases} s_i = r_i, \ \text{if } \phi_1 \text{ is a morphic involution,} \\ s_i = 1 - r_i, \ \text{if } \phi_1 \text{ is an antimorphic involution.} \end{cases}$$

Proof. It follows by induction on n. □

Using Lemma 24 one can now show relations between various indexed θ-Fibonacci words, similar to those of Proposition 19.

Proposition 25. *Let θ be an antimorphic involution on Σ^*, let $u, v \in \Sigma^+$, and let $(\alpha_n, \beta) \in \{(f_n, g), (g_n, f), (z_n, w), (w_n, z), (f'_n, g'), (g'_n, f'), (z'_n, w'), (w'_n, z')\}$, for all $n \geq 1$. The following relations hold for all $n \geq 3$:*

$$\alpha_n(u,v) = \begin{cases} \beta^{00r\{sr\}^i}(u, \theta(v)) \ : \ i = \frac{n-3}{2}, \ n \geq 3, \ n \text{ is odd,} \\ \beta^{00\{sr\}^i}(\theta(u), v) \ : \ i = \frac{n-2}{2}, \ n \geq 4, \ n \text{ is even,} \end{cases}$$

where $r = 0, s = 1$ if $\alpha_n \in \{g_n, f_n, z_n, w_n\}$, and $r = 1, s = 0$ if $\alpha_n \in \{g'_n, f'_n, z'_n, w'_n\}$.

Proof. The statement follows from Lemma 24, by induction on n. □

5. Borders and ϕ-borders of ϕ-Fibonacci words

It is well-known that both the standard and reverse Fibonacci words are bordered for all $n \geq 3$. In this section, we investigate the borderedness and ϕ-borderedness of ϕ-Fibonacci words. As seen in the next example, the borderdness of ϕ-Fibonacci words depends on the two initial Fibonacci words, as well as on the involution under consideration.

Example 26. Let ϕ be an (anti)morphic involution on Δ^*, where $\Delta = \{A, C, G, T\}$, defined as $\phi(A) = T$, $\phi(C) = G$ and vice versa, and let $g_1 = w_1 = z_1 = AC$ and $g_2 = w_2 = z_2 = T$. Consider the ϕ-Fibonacci words w_n, g_n, z_n, for $n \geq 3$.

If $\phi = \theta$ is an antimorphic involution:

- The first standard palindromic θ-Fibonacci words are

$$w_3 = AGT, w_4 = ACTA, w_5 = TAGTACT, w_6 = AGTACTATAGT,$$

  and $w_7 = ACTATAGTACTAGTACTA$. Thus, the word w_6 is bordered as well as θ-bordered, but the word w_7 is not θ-bordered.
- The first standard alternating θ-Fibonacci words are

$$g_3 = AAC, g_4 = GTTT, g_5 = AAACAAC.$$

  Note that the words g_i, for $3 \leq i \leq 5$, are neither bordered nor θ-bordered.
- The first standard hairpin θ-Fibonacci words are

$$z_3 = TGT, z_4 = TGTA, z_5 = TGTAACA, z_6 = TGTAACATACA.$$

  Note that z_3 is bordered but, z_4, z_5 and z_6 are not bordered. Also, z_4, z_5 and z_6 are θ-bordered.

If, on the other hand, $\phi = \mu$ is a morphic involution:

- The first standard palindromic μ-Fibonacci words are

$$w_3 = ATG, w_4 = TACA, w_5 = ATGTTAC, w_6 = TACAATGATGT.$$

  Thus, w_6 is both bordered as well as μ-bordered word.
- The first standard alternating μ-Fibonacci words are

$$g_3 = AAC, g_4 = TTGT, g_5 = AACAAAC, g_6 = TTGTTTGTTGT.$$

  Note that the words g_5 and g_6 are bordered but not μ-bordered.
- The first standard hairpin μ-Fibonacci words are

$$z_3 = TTG, z_4 = TTGA, z_5 = TTGAAAC, z_6 = TTGAAACAACT.$$

  Note that z_4, z_5 and z_6 are μ-bordered but, but z_4 and z_5 are not bordered.

Example 26 suggests the following result.

Theorem 27. *Let ϕ be an (anti)morphic involution on Σ^*. Then, for all $n \geq 6$, we have:*

- *If $\phi = \mu$, then the μ-Fibonacci words g_n, g'_n, w_n, w'_n, and z_n, z'_n are bordered.*
- *If $\phi = \mu$ then the μ-Fibonacci words w_n, w'_n are μ-bordered.*
- *If $\phi = \theta$ then the θ-Fibonacci words w_n, w'_n are bordered.*
- *The ϕ-Fibonacci words z_n, z'_n are ϕ-bordered.*

Proof. The statement follows from Definition 3, as one can easily infer the following. For a morphic involution μ, we have:

- For all $n \geq 4$, $g_n = g_{n-2}\mu(g_{n-3})g_{n-2}$ and $g'_n = g'_{n-2}\mu(g'_{n-3})g'_{n-2}$.
- For all $n \geq 4$, $w_n = w_{n-2}w_{n-3}\ \mu(w_{n-2})$ and $w'_n = \mu(w'_{n-2})w'_{n-3}w'_{n-2}$.
- For all $n \geq 6$, $z_n = z_{n-4}\mu(z_{n-5})\mu(z_{n-4})\mu(z_{n-3})\mu(z_{n-3})z_{n-4}$, and similarly, for all $n \geq 6$, $z'_n = z'_{n-4}\mu(z'_{n-3})\mu(z'_{n-3})\mu(z'_{n-4})\mu(z'_{n-5})z'_{n-4}$.
- For all $n \geq 6$, $w_n = w_{n-4}w_{n-5}\mu(w_{n-4})w_{n-3}w_{n-3}w_{n-4}$, and similarly, for all $n \geq 6$, $w'_n = w'_{n-4}w'_{n-3}w'_{n-3}\mu(w'_{n-4})w'_{n-5}w'_{n-4}$.

For an antimorphic involution θ, we have $w_n = w_{n-3}w_{n-2}w_{n-4}w_{n-3}$ and similarly $w'_n = w'_{n-3}w'_{n-4}w'_{n-2}w'_{n-3}$ for $n \geq 5$. For an (anti)morphic involution ϕ, we have $z_n = z_{n-2}\phi(z_{n-3})\phi(z_{n-2})$ and $z'_n = \phi(z'_{n-2})\phi(z'_{n-3})z'_{n-2}$ for $n \geq 4$. □

The results are summarized in Table 2.

	$g_n = \phi(g_{n-1})g_{n-2}$		$z_n = z_{n-1}\phi(z_{n-2})$		$w_n = \phi(w_{n-1})\phi(w_{n-2})$	
	MI	AMI	MI	AMI	MI	AMI
bordered	True	False	True	False	True	True
ϕ-bordered	False	False	True	True	True	False

Table 2: The ϕ-borderedness of ϕ-Fibonacci words.

From Theorem 27 we see that z_n is ϕ-bordered. In fact, for the case of an antimorphic involution θ, the following relations hold for all $n \geq 6$: the θ-borders of z_n are longer than z_{n-2} and $z_n = A_n z_r \theta(A_n)$ when n is odd, while $z_n = A_n\theta(z_r)\theta(A_n)$ if n is even, where $A_n = z_{i_1}z_{i_2}\cdots z_{i_{k-1}}z_{i_k}$, with $r = (n \mod 3)+3$, $i_1 = n-2$, $i_t = i_{t-1}-3$, and $i_k = r+1$. A similar decomposition holds for z'_n. Since, when θ is the Watson-Crick antimorphic involution on DNA strings, the property of θ-borderedness results in the DNA strings binding to themselves and forming so-called *hairpin* structures (with fully double-stranded stems) [45], we call z_n and z'_n "hairpin ϕ-Fibonacci words."

In the cases where the borderedness or ϕ-borderedness does not hold in general, the following examples suggest that placing additional constraints on the initial words may ensure borderedness or ϕ-borderedness of ϕ-Fibonacci words.

Example 28. Let ϕ be defined as in Example 26. We have the following:

- If $z_1 = C$ and $z_2 = AT$ and $\phi = \theta$, then $z_3 = ATG$, $z_4 = ATGAT$, and also $z_5 = ATGATCAT$ and $z_6 = ATGATCATATCAT$. Note that z_4, z_5 and z_6 are bordered. $(\mathrm{Pref}'(z_2) \cap \mathrm{Suff}'(\theta(z_2)) \neq \emptyset)$.

- If $g_1 = A$ and $g_2 = CT$ and $\phi = \theta$, then $g_3 = AGA$, $g_4 = TCTCT$, and also $g_5 = AGAGAAGA$ and $g_6 = TCTTCTCTTCTCT$. Note that g_3, g_4, g_5 and g_6 are bordered. $(\mathrm{Pref}'(\theta(g_1)) \cap \mathrm{Suff}'(g_2) \neq \emptyset)$.
- If $g_1 = A$ and $g_2 = CA$ and $\phi = \theta$, then $g_3 = TGA$, $g_4 = TCACA$, and also $g_5 = TGTGATGA$ and $g_6 = TCATCACATCACA$. Note that g_3, g_4, g_5 and g_6 are θ-bordered. $(\mathrm{Suff}'(g_1) \cap \mathrm{Suff}'(g_2) \neq \emptyset)$.
- If $g_1 = AT$ and $g_2 = TGA$ and $\phi = \mu$, then $g_3 = ACTAT$, $g_4 = TGATATGA$, and $g_5 = ACTATACTACTAT$ and $g_6 = TGATATGATGAT\ ATGATATGA$. Note that g_2, g_3, g_4, g_5 and g_6 are μ-bordered. $(\mathrm{Suff}'(g_1) \cap \mathrm{Pref}'(g_2) \neq \emptyset)$.
- If $w_1 = CA$ and $w_2 = CTG$ and $\phi = \theta$, then

$$w_3 = CAGTG, w_4 = CACTGCAG, w_5 = CTGCAGTGCACTG,$$

and $w_6 = CAGTGCACTGC\ AGCTGCAGTG$. Note that w_2, w_3, w_4, w_5 and w_6 are θ-bordered. Note that the language $\mathrm{Pref}'(w_2) \cap \mathrm{Suff}'(\theta(w_1))$ is non-empty, and so are $\mathrm{Pref}'(w_1) \cap \mathrm{Pref}'(w_2)$ and $(\mathrm{Pref}'(\theta(w_1)) \cap \mathrm{Suff}'(w_2)$.

We will now prove that the observations inferred from Example 28 hold in general (Proposition 30). We use the following Lemma.

Lemma 29. *Let u, v be two words in Σ^+, let $g_1 = g'_1 = w_1 = w'_1 = z_1 = z'_1 = u$ and let $g_2 = g'_2 = w_2 = w'_2 = z_2 = z'_2 = v$. If $\phi = \theta$ is an antimorphic involution on Σ^* then the following relations hold, for all $n \geq 3$ and some $t_n, t'_n \in \Sigma^*$:*

$$g_n = \begin{cases} \theta(v)t_n u & : \ n \text{ is odd}, \\ \theta(u)t_n v & : \ n \text{ is even}, \end{cases} \qquad g'_n = \begin{cases} ut'_n\theta(v) & : \ n \text{ is odd}, \\ vt'_n\theta(u) & : \ n \text{ is even}, \end{cases}$$

$$w_n = \begin{cases} \theta(v)t_n\theta(u) & : \ n \bmod 3 = 0, \\ ut_n\theta(v) & : \ n \bmod 3 = 1, \\ vt_n v & : \ n \bmod 3 = 2, \end{cases} \qquad w'_n = \begin{cases} \theta(u)t'_n\theta(v) & : \ n \bmod 3 = 0, \\ \theta(v)t'_n u & : \ n \bmod 3 = 1, \\ vt'_n v & : \ n \bmod 3 = 2, \end{cases}$$

$$z_n = vt_n\theta(v), \qquad z'_n = \theta(v)t'_n v \ : \quad n \geq 4.$$

If, on the other hand, $\phi = \mu$ is a morphic involution on Σ^* then the following relations hold, for all $n \geq 3$ and some $t_n, t'_n \in \Sigma^*$:

$$g_n = \begin{cases} \mu(v)t_n u & : \ n \text{ is odd}, \\ vt_n v & : \ n \text{ is even}, \end{cases} \qquad g'_n = \begin{cases} ut'_n\mu(v) & : \ n \text{ is odd}, \\ vt'_n v & : \ n \text{ is even}. \end{cases}$$

Proof. We only prove for the θ-Fibonacci word g_n, for n odd. Note that the first standard alternating θ-Fibonacci words are $g_1 = u$, $g_2 = v$, $g_3 = \theta(v)u$, $g_4 = \theta(u)vv$. Assume true for all $k \leq n$. Consider the word g_{n+1} for $n+1$ an even number. Then by strong induction we have

$$g_{n+1} = \theta(g_n)g_{n-1} = \theta(\theta(v)t_n u)\theta(u)t_{n-1}v = \theta(u)\theta(t_n)v\theta(u)t_{n-1}v = \theta(u)t_{n+1}v,$$

hence the result. The other cases can be proved in a similar fashion. □

It is clear from Theorem 4 in [8] that if $f_1 = u$ and $f_2 = v$ then the Fibonacci words f_n are of the form vut_nvu when n is odd and of the form vs_nv when n is even, and hence are bordered for all $n \geq 4$. One can clearly see that if ϕ is an (anti)morphic involution such that $\phi(\mathrm{Pref}'(y)) \cap \mathrm{Suff}'(x) \neq \emptyset$ and $\phi(\mathrm{Pref}'(y)) \cap \mathrm{Suff}'(y) \neq \emptyset$ then each f_n is ϕ-bordered.

We now use Lemma 29 to provide conditions on the initial words which are sufficient to ensure that the θ-Fibonacci words g_n and w_n are θ-bordered, that the μ-Fibonacci words g_n are μ-bordered, and that the θ-Fibonacci words z_n and g_n are bordered, for all $n \geq 4$.

Proposition 30. *Let $\phi = \theta$ be an antimorphic involution on Σ^*, let $u, v \in \Sigma^+$, and let $g_1 = w_1 = z_1 = u$ and $g_2 = w_2 = z_2 = v$. The following relations hold for all $n \geq 3$:*

(I) *If $\mathrm{Pref}'(v) \cap \mathrm{Suff}'(\theta(v)) \neq \emptyset$, then z_n is bordered.*

(II) *If $\mathrm{Pref}'(\theta(u)) \cap \mathrm{Suff}'(v) \neq \emptyset$, then g_n is bordered.*

(III) *If $\mathrm{Suff}'(u) \cap \mathrm{Suff}'(v) \neq \emptyset$, then g_n is θ-bordered.*

(IV) *If $\mathrm{Pref}'(\theta(u)) \cap \mathrm{Suff}'(v) \neq \emptyset$, $\mathrm{Pref}'(v) \cap \mathrm{Suff}'(\theta(u)) \neq \emptyset$, $\mathrm{Pref}'(u) \cap \mathrm{Pref}'(v) \neq \emptyset$, then w_n is θ-bordered.*

If, on the other hand, $\phi = \mu$ is a morphic involution on Σ^* and $\mathrm{Suff}'(u) \cap \mathrm{Pref}'(v) \neq \emptyset$ and v is μ-bordered, then g_n is μ-bordered for $n \geq 4$.

Proof. It follows directly from Lemma 29 and the definition of ϕ-borders. $\square$

Note that the initial words α_1 and α_2 given in Example 28 satisfy the conditions in Proposition 30. Also, Proposition 30 implies the following, for the case of the atom θ-Fibonacci words.

Corollary 31. *Let $\phi = \theta$ be an antimorphic involution on Σ^*, let a, b be two letters in Σ, and let $g_1 = z_1 = w_1 = a$ and $g_2 = z_2 = w_2 = b$. Then for all $n \geq 4$, we have:*

(I) *If $b = \theta(b)$, then z_n is θ-bordered.*

(II) *If $\theta(a) = b$, then g_n is bordered.*

(III) *If $a = b$, then g_n is θ-bordered and in addition if $\theta(a) = b$, then w_n is θ-bordered.*

Lastly, we present a property of w_n for the case of an antimorphic involution, which justifies their being called "palindromic Fibonacci words".

Definition 32. Let θ be an antimorphic involution on Σ^*, and define

$$P_\theta = \{w \in \Sigma^+ \mid w = \theta(w)\}, \text{ and } P_{2\theta} = \{w\theta(w) \mid \ w \in \Sigma^+\}.$$

We call a string x a *θ-palstar* if it belongs to $P_{2\theta}^*$. A non-empty θ-palstar is said to be *prime θ-palstar* if it cannot be written as a concatenation of two or more θ-palstars.

Note that in the particular case when θ is the mirror image, the definition above becomes the well-known definition of palstar and prime palstar, that were introduced in [30]. Note that each string in $P_{2\theta}$ is a θ-palindrome of even length, and conversely, that is $P_{2\theta} = \{x \in P_\theta \mid 2k = |x| \text{ for some } k\}$. By repeated decompositions, one can show that every θ-palstar is expressible as a concatenation of prime θ-palstars. One can also prove that such a decomposition of a θ-palstar into prime θ-palstars is unique, which is a consequence of the following Lemma.

Lemma 33. *For an antimorphic involution θ on Σ^*, a prime θ-palstar cannot begin with another prime θ-palstar.*

Proof. It can be proved similarly to the known fact that a prime palstar cannot begin with another prime palstar [30]. □

The following result shows that θ-Fibonacci words w_n and w'_n can be expressed in two different ways as a θ-palstar concatenated with a θ-Fibonacci word w_i (respectively w'_i). This justifies the ϕ-Fibonacci words w_n being called "palindromic-Fibonacci words."

Proposition 34. *Given an antimorphic involution $\phi = \theta$ on Σ^*, for all $n \geq 6$, the word w_n can be decomposed as $w_n = A_n w_r = w_r B_n$, where A_n, B_n are the θ-palstars*

$$A_n = p_{i_1} q_{j_1} p_{i_2} q_{j_2} \cdots p_{i_{k-1}} q_{j_{k-1}} p_{i_k} q_{j_k}, \quad B_n = p_{s_1} p_{s_2} \cdots p_{s_{l-1}} p_{s_l},$$

such that $p_i = w_i \theta(w_i)$, $q_i = \theta(w_i) w_i$, $j_t = i_t - 1$, $i_t = i_{t-1} - 3$, $i_1 = n - 3$, $i_k = r$ and $s_1 = r + 1$, $s_l = n - 2$, $s_t = s_{t-1} + 3$ and $r = (n \mod 3) + 3$.

Proof. We prove the result by induction on n. For the base case, let $n = 6$ and we have, $w_6 = \theta(w_5)\theta(w_4) = w_3 w_4 w_2 w_3 = w_3 \theta(w_3) \theta(w_2) w_2 w_3$. Assume that, $w_k = p_{k-3} q_{k-4} p_{k-6} q_{k-7} \cdots p_{r+3} q_{r+2} p_r q_{r-1} w_r$, for $r = (k \mod 3) + 3$ and for all $k \leq n$. Consider w_{k+1}. Then, by definition, we have

$$w_{k+1} = \theta(w_k)\theta(w_{k-1}) = w_{k-2} w_{k-1} w_{k-3} w_{k-2} = w_{k-2} \theta(w_{k-2}) \theta(w_{k-3}) w_{k-3} w_{k-2}.$$

Hence, we have by induction that, $w_{k+1} = p_{k-2} q_{k-3} p_{k-5} q_{k-6} \cdots p_r q_{r-1} w_r$. The proof for the other equality is similar. □

For the reverse palindromic θ-Fibonacci words, the following result, similar to Proposition 34, holds.

Proposition 35. *Let $\phi = \theta$ be an antimorphic involution on Σ^*. Then, for all $n \geq 6$ the word w'_n can be decomposed as $w'_n = C_n w'_r = w'_r D_n$, where C_n, D_n are the θ-palstars*

$$C_n = q'_{i_1} q'_{i_2} \cdots q'_{i_{k-1}} q'_{i_k}, \quad D_n = p'_{j_1} q'_{s_1} p'_{j_2} q'_{s_2} \cdots p'_{j_{k-1}} q'_{s_{k-1}} p'_{j_k} q'_{s_k},$$

such that $p'_i = w'_i \theta(w'_i)$, $q'_i = \theta(w'_i) w'_i$, $i_1 = n - 2$, $i_t = i_{t-1} - 3$, $i_k = r + 1$, and $j_1 = r - 1$, $j_k = n - 3$, $s_t = j_t + 1$, $j_t = j_{t-1} + 3$, and $r = (n \mod 3) + 3$.

Proof. Similar to that of Proposition 34, by induction on n. □

6. Conclusions and future work

This paper proposes a unified terminology (Definition 1) for the various definitions of Fibonacci words that exist in the literature. It also defines and investigates two generalizations of Fibonacci words, namely the ϕ-Fibonacci words (Definition 3) and the indexed ϕ-Fibonacci words (Definition 22), where ϕ is a morphic or an antimorphic involution on Σ^*.

An antimorphic involution θ on Δ^*, where Δ is the DNA alphabet $\Delta = \{A, C, G, T\}$ is a mathematical model of the Watson-Crick complementarity of DNA strands, and the implications of the new concepts and of some of the results in this paper are as follows. According to Theorem 27, the standard and reverse hairpin θ-Fibonacci words z_n and z'_n are θ-bordered for $n \geq 6$, and thus form hairpins with fully double-stranded stems. One can easily observe that the standard palindromic θ-Fibonacci words w_n and the reverse palindromic θ-Fibonacci words w'_n form hairpin structures with partially double-stranded stems, as illustrated in Figure 1. In addition, according to Proposition 30, if the initial θ-Fibonacci words u and v satisfy some additional conditions, then the standard alternating θ-Fibonacci words $g_n(u, v)$ and the standard palindromic θ-Fibonacci words $w_n(u, v)$ are θ-bordered for $n \geq 3$, and thus form hairpins with fully double-stranded stems. Lastly, according to Propositions 34 and 35, the standard and reverse palindromic θ-Fibonacci words w_n and w'_n contain θ-palstars for $n \geq 6$, and thus can self-assemble into DNA secondary structures containing multiple hairpin structures.

Future topics of research include, e.g., investigating relations between various types of ϕ-Fibonacci words for the antimorphic case (similar to the results obtained in Subsection 3.1 for the morphic case), and the study of properties of the ϕ-Fibonacci words z_n in the special case when the first two ϕ-Fibonacci words satisfy the relation $\phi(u) = v$. Other areas of investigation include the primitivity of ϕ-Fibonacci words, as well as other combinatorial properties of ϕ-Fibonacci words (counting the number of distinct factors, squares, ϕ-squares, cubes, ϕ-cubes, palindromes, ϕ-palindromes).

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