Improved Analysis of Higher Order Random Walks and Applications

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Abstract

The motivation of this work is to extend the techniques of higher order random walks on simplicial complexes to analyze mixing times of Markov chains for combinatorial problems. Our main result is a sharp upper bound on the second eigenvalue of the down-up walk on a pure simplicial complex, in terms of the second eigenvalues of its links. We show some applications of this result in analyzing mixing times of Markov chains, including sampling independent sets of a graph and sampling common independent sets of two partition matroids.

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1 Introduction

Consider the following random walks [KM17, DK17, KO18, DDFH18] defined[1] on a simplicial complex X. Initially, the random walk starts from an arbitrary face α_1 of dimension k in X.

- Down-Up Walk: In each step $t \geq 1$, we choose a uniform random element $i \in \alpha_t$ and delete i from α_t, and set α_{t+1} to be a uniform random face of dimension k in X that contains $\alpha_t \setminus \{i\}$. This is called the k-th down-up walk of X, and its transition matrix is denoted by $\mathsf{P}_k^{\triangledown}$.
- Up-Down Walk: In each step $t \geq 1$, we choose a uniform random face β of dimension $k+1$ in X that contains α_t, and choose a uniform random element $i \in \beta$ and set $\alpha_{t+1} = \beta \setminus \{i\}$. This is called the k-th up-down walk of X, and its transition matrix is denoted by $\mathsf{P}_k^{\triangle}$.

The stationary distribution of these random walks is the uniform distribution on the faces of dimension k in the simplicial complex X. The question of interest is the mixing time of these random walks, i.e. how many steps it takes for the distribution of α_t to be close to the uniform distribution.

A graph is a simplicial complex of dimension 1. The transition matrix of the lazy random walk on a graph is $\mathsf{P}_0^{\triangle}$. Fundamental results in spectral graph theory state that (i) the mixing time of the lazy random walk is small, if and only if (ii) the second eigenvalue of $\mathsf{P}_0^{\triangle}$ is small, if and only if (iii) the graph is an expander graph. See [HLW06, WLP09] for surveys on this topic.

Since the theory of expander graphs has many applications, there are various motivations in generalizing these results for graphs to simplicial complexes. Several definitions of high-dimensional expanders have been studied in the literature (e.g. [LM06, Gro10, PRT16, DKW16, KM17, Opp18]), and these results have found interesting applications in discrete geometry, complexity theory, coding theory, and property testing (e.g. [LM06, MW09, FGL+11, KL14, EK16, KM16, KKL16, DK17, DHK+19]).

Local Spectral Expanders

In this paper, we consider the definition of γ-local-spectral expanders developed in [KM17, DK17, KO18, Opp18, DDFH18] for the study of random walks on simplicial complexes. The local structures of a simplicial complex are described by its links. The link X_α of a face $\alpha \in X$ is defined as the simplicial complex $X_\alpha = \{\beta \setminus \alpha : \beta \in X, \beta \supset \alpha\}$. The graph $G_\alpha = (V_\alpha, E_\alpha)$ of the link X_α is defined as follows: (i) each vertex i in V_α corresponds to a singleton $\{i\}$ in X_α, (ii) two vertices $i, j \in V_\alpha$ have an edge in E_α if and only if $\{i, j\}$ is contained in some face of X_α, (iii) the weight w_{ij} of an edge $ij \in E_\alpha$ is proportional to the number of maximal faces in X_α that contains $\{i, j\}$. Informally, a simplicial complex X is a γ-local-spectral expander if G_α is an expander graph for every $\alpha \in X$. In the following, we say X is a pure simplicial complex if every maximal face of X is of the same dimension, and we call this the dimension of X.

Definition 1.1 (γ-local-spectral expanders [Opp18, KO18])**.** A d-dimensional pure simplicial complex X is a γ-local-spectral expander if $\lambda_2(G_\alpha) \leq \gamma$ for every face $\alpha \in X$ of dimension up to $d-2$, where $\lambda_2(G_\alpha)$ denotes the second largest eigenvalue of the random walk matrix of G_α (where the transition probabilities are proportional to the edge weights).

[1] All the definitions in the introduction will be formally defined again in a more general setting in Section 2.

Kaufman-Oppenheim Theorem

Kaufman and Oppenheim [KO18] proved that the k-th down-up walk and the $(k-1)$-th up-down walk have a non-trivial spectral gap as long as the simplicial complex is a γ-local-spectral expander for $\gamma < 2/k^2$.

Theorem 1.2 ([KO18])**.** *Let X be a pure d-dimensional simplicial complex. Suppose X is a γ-local-spectral expander. Then, for every $0 \le k \le d$,*

$$\lambda_2(\mathsf{P}_k^{\bigtriangledown}) = \lambda_2(\mathsf{P}_{k-1}^{\bigtriangleup}) \le 1 - \frac{1}{k+1} + \frac{k\gamma}{2},$$

Theorem 1.2 states that the spectral gap of $\mathsf{P}_k^{\bigtriangledown}$ is at least $g := 1 - \lambda_2(\mathsf{P}_k^{\bigtriangledown}) \ge \frac{1}{k+1} - \frac{k\gamma}{2}$, which implies by a standard argument (see Theorem 2.8) that the mixing time of these walks is at most $O(\frac{(k+1)\log(n)}{g})$ where n is the size of the ground set of X. For example, if $\gamma \le 0$, then the mixing time of $\mathsf{P}_k^{\bigtriangledown}$ is at most $O(k^2 \log(n))$.

Theorem 1.2 can also be used to bound the spectral gap of certain "longer" random walks on simplicial complexes (see Corollary 1.11 and Section 1.2.5). Dinur and Kaufman [DK17] use these results with the Ramanujan complexes of [LSV05] to construct efficient agreement testers, which have applications to PCP constructions. Recently, these ideas have also found applications in coding theory [DHK+19].

Oppenheim's Trickling Down Theorem

Kaufman-Oppenheim Theorem 1.2 provides a way to bound the mixing time of the down-up walks and up-down walks. To apply the theorem, however, one needs to check that $\lambda_2(G_\alpha) \le \gamma$ for every face $\alpha \in X$ of dimension at most $d-2$. This is not an easy task. There are exponentially many graphs G_α to check, and these graphs are defined implicitly where computing the edge weights involve non-trivial counting problems. A very useful result by Oppenheim [Opp18] makes this task easier, by relating the second eigenvalue of the graph of a lower-dimensional link to that of a higher-dimensional link.

Theorem 1.3 ([Opp18])**.** *Let X be a pure d-dimensional simplicial complex. Suppose $\lambda_2(G_\beta) \le \gamma \le \frac{1}{2}$ for every face β of dimension k, and G_α is connected for every face α of dimension $k-1$. Then, for every face α of dimension $k-1$, it holds that*

$$\lambda_2(G_\alpha) \le \frac{\gamma}{1-\gamma}.$$

Applying this theorem inductively, we can reduce the problem of bounding $\lambda_2(G_\alpha)$ for every α to bounding $\lambda_2(G_\beta)$ for only those faces β of highest dimension.

Corollary 1.4 ([Opp18])**.** *Let X be a pure d-dimensional simplicial complex. Suppose $\lambda_2(G_\beta) \le \gamma \le \frac{1}{d}$ for every face β of dimension $d-2$, and G_α is connected for every face α. Then, for every $k \le d-2$, and for every face α of dimension k, it holds that*

$$\lambda_2(G_\alpha) \le \frac{\gamma}{1-(d-2-k)\gamma}.$$

Corollary 1.4 is useful for two reasons: First, note that the weight of every edge in G_β for face β of dimension $d-2$ is either zero or one, which makes the task of bounding its second eigenvalue more tractable. Second, if one can prove that $\lambda_2(G_\beta) = O(\frac{1}{d^2})$ for every face β of dimension $d-2$ and G_α is connected for every face α, then one can conclude that $\lambda_2(G_\alpha) = O(\frac{1}{d^2})$ for every face α and hence the simplicial complex is a $O(\frac{1}{d^2})$-local-spectral expander. So, the reduction of Oppenheim is basically lossless in the regime when Kaufman-Oppenheim's Theorem 1.2 applies.

Analyzing Mixing Times of Markov Chains

Recently, Anari, Liu, Oveis Gharan, and Vinzant [ALOV19] found a striking application of Theorem 1.2 and Corollary 1.4 in proving the matroid expansion conjecture of Mihail and Vazirani [MV87], answering a long standing open question in Markov chain Monte Carlo methods.

To illustrate their result, consider the special case of sampling a random spanning tree from a graph $G = (V, E)$. Let X be the simplicial complex where the ground set is E and each acyclic subgraph of G is a face of X. Then X is a pure d-dimensional simplicial complex, where $d = |V| - 2$ and the spanning trees of G are the maximal faces of X. Note that $\mathsf{P}_d^{\triangledown}$ in X is exactly the natural Markov chain on the spanning trees of G, where in each step we delete a uniformly random edge e from the current spanning tree T and add a uniformly random edge f so that $T - e + f$ is a spanning tree. So, the problem of proving the Markov chain on spanning trees is fast mixing is equivalent to upper bounding $\lambda_2(\mathsf{P}_d^{\triangledown})$ of the simplicial complex X.

Using the nice structures of matroids, Anari, Liu, Oveis Gharan, and Vinzant [ALOV19] showed that the graph G_β is a complete multi-partite graph for every face β of dimension $d-2$, and this implies that $\lambda_2(G_\alpha) \leq 0$ for every face β of dimension $d-2$. Thus, it follows from Oppenheim's Corollary 1.4 that $\lambda_2(G_\alpha) \leq 0$ for every face α.[2] Then Kaufman-Oppenheim's Theorem 1.2 implies that $\lambda_2(\mathsf{P}_d^{\triangledown}) \leq 1 - \frac{1}{d+1}$, and thus the mixing time of the Markov chain of sampling matroid bases is at most $O(d^2 \log n)$. This provides the first FPRAS for counting the number of matroid bases, and also proves that the basis exchange graph of a matroid is an expander graph.

The proof of the matroid expansion conjecture shows that the techniques developed in higher order random walks provide a new simplicial complex approach to analyze mixing times of Markov chains. It is thus natural to investigate whether this approach can be extended to other problems. Here we would like to discuss some limitations of the current techniques. It can be shown that $\lambda_2(G_\beta) \leq 0$ only if G_β is a complete multi-partite graph [God] and more generally a 0-local-expander is a weighted matroid complex [BH19], and so the same analysis as in [ALOV19] only works for matroids. Note that Kaufman-Oppenheim Theorem 1.2 only applies when $\lambda_2(G_\alpha) \leq O(\frac{1}{d^2})$ for every face α up to dimension $d-2$. For many problems that we have considered, it does not hold that $\lambda_2(G_\beta) \leq O(\frac{1}{d^2})$ even when restricted to faces β of dimension $d-2$.

1.1 Main Result

The main motivation of this work is to extend this simplicial complex approach to analyze mixing times of more general Markov chains. Our main result is the following improved eigenvalue bound for higher order random walks.

Theorem 1.5. *Let X be a pure d-dimensional simplicial complex. Define*

$$\gamma_j := \max_{\alpha} \{\lambda_2(G_\alpha) : \alpha \in X \text{ and } \alpha \text{ is of dimension } j\},$$

For any $0 \leq k \leq d$,

$$\lambda_2(\mathsf{P}_k^{\triangledown}) = \lambda_2(\mathsf{P}_{k-1}^{\triangle}) \leq 1 - \frac{1}{k+1} \prod_{j=-1}^{k-2} (1 - \gamma_j).$$

The following are some remarks about Theorem 1.5.

[2] The result that every matroid complex is a 0-local-spectral expander was also proved by Huh and Wang [HW17], using techniques from Hodge theory for matroids [AHK18] instead of Oppenheim's theorem.

1. A basic result is that a simplicial complex X is gallery connected (i.e. $\lambda_2(\mathsf{P}_d^{\triangledown}) < 1$) if G_α is connected (i.e. $\lambda_2(G_\alpha) < 1$) for every face α of dimension up to $d-2$. Theorem 1.5 provides a quantitative generalization of this result.
2. A corollary of Theorem 1.5 is that the spectral gap $1 - \lambda_2(\mathsf{P}_k^{\triangledown})$ of the k-th down-up walk is at least $\Omega(1/k)$ if X is a $O(\frac{1}{k})$-local-spectral expander. This is an improvement of Theorem 1.2 where it requires the simplicial complex X to be a $O(\frac{1}{k^2})$-local-spectral expander to conclude that $\mathsf{P}_k^{\triangledown}$ has a non-zero spectral gap.
3. It can be shown that the spectral gap $1 - \lambda_2(\mathsf{P}_k^{\triangledown})$ of the k-th down-up walk is at most $O(\frac{1}{k})$ for any simplicial complex (see Proposition 3.3), so Theorem 1.5 shows that any $O(\frac{1}{k})$-local-spectral expander has the optimal spectral gap for the k-th down-up walk up to a constant factor.
4. The refinement of having a different bound γ_j for links of different dimension is very useful for analyzing Markov chains. We will see some applications in Section 4.
5. Theorem 1.5 can be used to provide a tighter bound on the spectral gap of certain "longer" random walks (see Corollary 1.12) which were known to be useful in coding theory and agreement testing (see Section 1.2.5).

Combining with Oppenheim's Theorem 1.3, Theorem 1.5 provides the following bound for the second eigenvalue of higher order random walks in a black box fashion. See Section 3 for the proof.

Corollary 1.6. *Let X be a pure d-dimensional simplicial complex. For any $0 \le k \le d$, suppose $\gamma_{k-2} \le \frac{1}{k+1}$ and G_α is connected for every face α up to dimension $k-2$, then*

$$\lambda_2(\mathsf{P}_k^{\triangledown}) = \lambda_2(\mathsf{P}_{k-1}^{\triangle}) \le 1 - \frac{1}{(k+1)^2}.$$

This provides a convenient way to bound the mixing time of Markov chains. Recall that the edge weights in G_β for face β of dimension $d-2$ are either zero or one, and so it is easier to bound their second eigenvalue. Corollary 1.6 states that as long as we can prove $\lambda_2(G_\beta) \le 1/(d+1)$ for these unweighted graphs in the highest dimension, then we can conclude that $\mathsf{P}_d^{\triangledown}$ is fast mixing.

1.2 Applications

We present several applications of Theorem 1.5 and Corollary 1.6, in analyzing mixing times of Markov chains (Section 1.2.1, Section 1.2.2, Section 1.2.3), in analyzing constructions of high-dimensional expanders (Section 1.2.4), and in analyzing longer random walks (Section 1.2.5).

1.2.1 Sampling Independent Sets of Fixed Size

One of the most natural simplicial complexes to consider is the independent set complex of a graph [Mes01, AB06]. Let $G = (V, E)$ be a graph. The independent set complex $I_{G,k}$ has the vertex set V as the ground set, and a subset $S \subset V$ is a face in X if and only if S is an independent set in G with $|S| \le k$.

We are interested in bounding $\lambda_2(\mathsf{P}_{k-1}^{\triangledown})$ for this simplicial complex X. The $(k-1)$-th down-up walk corresponds to a natural Markov chain on sampling independent sets of size k. Initially, the random walk starts from an arbitrary independent set S_1 of size k. In each step $t \geq 1$, we choose a uniform random vertex $u \in S_t$ and delete it from S_t, and we choose a uniform random vertex v so that $S_t - u + v$ is still an independent set of size k and set $S_{t+1} := S_t - u + v$. This Markov chain is known to mix in polynomial time for $k \leq \frac{|V|}{2\Delta+1}$ where Δ is the maximum degree of G, by using the path coupling technique [BD97, MU05]. We prove a more refined result using the simplicial complex approach.

Theorem 1.7. *Let $G = (V, E)$ be a graph with maximum degree Δ. Let $\mathsf{P}_{k-1}^{\triangledown}$ be the $(k-1)$-th down-up walk on the simplicial complex $I_{G,k}$. Let A_G be the adjacency matrix of G.*

$$\text{If} \quad k \leq \frac{|V|}{\Delta + |\lambda_{\min}(\mathsf{A}_G)|}, \quad \text{then} \quad \lambda_2(\mathsf{P}_{k-1}^{\triangledown}) \leq 1 - \frac{1}{k^2}.$$

It is well-known that $|\lambda_{\min}(A_G)| \leq \Delta$ for a graph with maximum degree Δ, and so Theorem 1.7 recovers the previous result that the Markov chain is fast mixing if $k \leq \frac{|V|}{2\Delta}$. There are various graph classes with $|\lambda_{\min}(A_G)|$ smaller than Δ, and Theorem 1.7 allows us to sample larger independent sets. For example, it is known that $|\lambda_{\min}(A_G)| \leq O(\sqrt{\Delta})$ for planar graphs and more generally for graphs with bounded arboricity [Hay06], and also for random graphs and more generally for two-sided expander graphs [HLW06].

1.2.2 Sampling Common Independent Sets in Two Partition Matroids

A matroid $M = (E, \mathcal{I})$ on the ground set E with the set of independent sets $\mathcal{I} \subset 2^E$ is a combinatorial object satisfying the following properties:

- (containment property) if $S \in \mathcal{I}$ and $T \subset S$, then $T \in S$,
- (extension property) if $S, T \in \mathcal{I}$ such that $|S| > |T|$ then there is some $x \in S \backslash T$ such that $\{x\} \cup T \in \mathcal{I}$.

A partition matroid is the special case where the ground set E is partitioned into disjoint blocks $B_1, \ldots, B_l \subseteq E$ with parameters $0 \leq d_i \leq |B_i|$ for $1 \leq i \leq l$, and a subset S is in $\mathcal{I}$ if and only if $|S \cap B_i| \leq d_i$ for $1 \leq i \leq l$.

The intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I})$ over the same ground set E can be used to formulate various interesting combinatorial optimization problems [Sch03]. We are interested in the problem of sampling a uniform random common independent set of size k, i.e. a random subset $F \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|F| = k$.

Matroids naturally correspond to simplicial complexes. Let $C_{M_1,M_2,k}$ be the matroid intersection complex with ground set E, where a subset $F \subset E$ is a face in $C_{M_1,M_2,k}$ if and only if $F \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $|F| \leq k$. The $(k-1)$-th down-up walk of this complex corresponds to a natural Markov chain on sampling common independent sets of M_1 and M_2 of size k. We show that this Markov chain is fast mixing for k up to one third the size of a maximum common independent set, when M_1 and M_2 are partition matroids and there are no two elements belonging to the same block in both matroids (i.e. there are no two elements x, y such that x and y are in the same block in M_1 and also in the same block in M_2).

Theorem 1.8. *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two given partition matroids with a common independent set of size r and no two elements belonging to the same block in both matroids. If $k \leq r/3$, then*

$$\lambda_2(\mathsf{P}_{k-1}^{\triangledown}) \leq 1 - \frac{1}{k^2},$$

where $\mathsf{P}_{k-1}^{\triangledown}$ is the $(k-1)$-th down-up walk on the matroid intersection complex $C_{M_1,M_2,k}$.

The proof of Theorem 1.8 reveals some interesting spectral property of the links of the simplicial complex $C_{M_1,M_2,k}$. For any face β of dimension $k-3$, we show that the graph G_β is the complement of the line graph of a bipartite graph. We note that this holds for any two matroids, not just for partition matroids. By the additional assumptions that the two matroids are partition matroids and there are no two elements in the same block in both matroids, the graph G_β is the line graph of a *simple* bipartite graph. Using the fact that the adjacency matrix of the line graph of a simple graph has minimum eigenvalue at least -2, we prove that $\lambda_2(G_\beta) \leq \frac{1}{k}$ as long as $k \leq \frac{r}{3}$. We can then use Corollary 1.6 to conclude Theorem 1.8.

1.2.3 Sampling Independent Sets from Hardcore Distributions

Very recently, Anari, Liu, and Oveis Gharan [ALO] use Theorem 1.5 to prove a strong result about sampling independent sets from the hardcore distribution. Given a graph $G = (V, E)$ and a parameter $\lambda > 0$, the problem is to sample an independent set S with probability $\frac{\lambda^{|S|}}{Z_G(\lambda)}$ where $Z_G(\lambda) := \sum_{S \subset V : S \text{ independent}} \lambda^{|S|}$ is the partition function. An important work of Weitz [Wei06] gave a deterministic fully polynomial time approximation scheme to estimate $Z_G(\lambda)$ for λ up to the "uniqueness threshold", but the exponent of the runtime depends on the maximum degree Δ of G. It is conjectured that the natural Markov chain for sampling independent sets mixes in polynomial time up to the uniqueness threshold. Anari, Liu, and Oveis Gharan prove this conjecture and obtain a polynomial time algorithm to estimate $Z_G(\lambda)$ up to the uniqueness threshold for any graph (even with unbounded maximum degree). They consider a pure n-dimensional simplicial complex for sampling independent sets, and prove that $\gamma_j = \Theta(\frac{1}{n-j})$ for $0 \leq j \leq n-2$ by using the techniques from correlation decay. Then it follows from Theorem 1.5 that the Markov chain is fast mixing. Note that it is crucial to have a different bound γ_j for links of different dimension in Theorem 1.5, so even when $\gamma_{n-2} = \Theta(1)$ it is still possible to conclude fast mixing.

1.2.4 Combinatorial Constructions of High Dimensional Expanders

Recently, Liu, Mohanty, and Yang [LMY19] presented an interesting combinatorial construction of a sparse simplicial complex where all higher order random walks have a constant spectral gap. Their construction is by taking a certain tensor product of a graph G on n vertices and a small H-dimensional complete simplicial complex $\mathcal{B}$ on s vertices.

Theorem 1.9 ([LMY19]). *Let G be a T-regular triangle free graph on n vertices. There is an explicit family $(X^{(s,H,G)})_{H \geq 1, s \geq H+1}$ of simplicial complexes, satisfying the following properties:*

1. *$X^{(s,H,G)}$ is a pure H-dimensional simplicial complex with $\Theta(n)$ maximal faces.*

2. *The spectral gap of the graphs of j-th dimensional links of the complex $X^{(s,H,G)}$ satisfies*

$$1 - \gamma_j \geq \begin{cases} \frac{1}{2} & \text{if } j \in [0, H-2], \\ \left(\frac{1}{2} - \frac{1}{2(T2^H+1)}\right)(1 - \sigma_2(G)) & \text{if } j = -1, \end{cases}$$

where $\sigma_2(G)$ is the second largest eigenvalue of the normalized adjacency matrix of G.

3. *For any $-1 \leq j \leq H-2$,*

$$\lambda_2(\mathsf{P}_{j+1}^{\triangledown}) = \lambda_2(\mathsf{P}_j^{\triangle}) \leq 1 - \Omega\left(\frac{1 - \sigma_2(G)}{T^2 \cdot j^2 \cdot (s-j) \cdot 2^j}\right),$$

The main technical part of their proof is in establishing Item (3) in Theorem 1.9. They use the special structures of their construction and the decomposition technique from [JSV04] to bound the spectral gap of the higher order random walks. The authors ask the question whether the spectral property in Item (2) alone is enough to prove the fast mixing result in Item (3). Note that Kaufman-Oppenheim's Theorem 1.2 does not apply in this regime.

Using Theorem 1.5, we answer their question affirmatively, by deriving Item (3) from Item (2) in a black box fashion. This slightly improves their bound and considerably simplifies their analysis.

Corollary 1.10. *Let $X := X^{(s,H,G)}$ be a complex from Theorem 1.9 satisfying Item (2). For any $-1 \le j \le H-2$,*

$$\lambda_2(\mathsf{P}^{\triangledown}_{j+1}) = \lambda_2(\mathsf{P}^{\triangle}_{j}) \le 1 - \Omega\left(\frac{1-\sigma_2(G)}{j \cdot 2^j}\right).$$

1.2.5 Longer Random Walks and Other Applications

Consider the following generalization of the up-down walk where we take "longer" steps. Initially, the random walk starts from an arbitrary α_1 face of dimension a in X. In each step $t \ge 1$, we sample a uniformly random face β of dimension $b > a$ that contains α_t, and set α_{t+1} to be a uniformly random subset of β of dimension a. We call this the a-th up-down walk through the b-th dimension, and denote its transition matrix by $\mathsf{P}^{\triangle}_{a,b}$. The k-th up-down walk defined before is the special case $\mathsf{P}^{\triangle}_{k,k+1}$. Dinur and Kaufman [DK17] derived the following result about $\mathsf{P}^{\triangle}_{a,b}$ from the result about the ordinary up-down walks.

Corollary 1.11 ([DK17])**.** *Let X be a d-dimensional pure simplicial complex. If X is a γ-local-spectral expander, then for any $0 \le a < b \le d-1$,*

$$\lambda_2(\mathsf{P}^{\triangle}_{a,b}) \le \frac{a+1}{b+1} + O(a(b-a)\gamma).$$

Using Theorem 1.5, we obtain the following improved bound. See Section 3 for the proof.

Corollary 1.12. *Let X be a d-dimensional pure simplicial complex. If X is a γ-local-spectral expander, then for any $0 \le a < b \le d-1$,*

$$\lambda_2(\mathsf{P}^{\triangle}_{a,b}) \le (1+\gamma)^{b-a} \cdot \frac{a+1}{b+1}.$$

In particular, if $\gamma \le \frac{\varepsilon}{b-a}$ for some $0 \le \varepsilon \le 1$, then $\lambda_2(\mathsf{P}^{\triangle}_{a,b}) \le e^{\varepsilon} \cdot \frac{a+1}{b+1}$.

Whereas the bound from Corollary 1.11 requires $\gamma = O(\frac{1}{b \cdot (b-a)})$ to give a nontrivial upper bound on the second eigenvalue of $\mathsf{P}^{\triangle}_{a,b}$, Corollary 1.12 only requires $\gamma \le O(\frac{1}{b-a})$ to give a comparable bound.

Corollary 1.11 has found applications in agreement testing and coding theory [DK17, DHK+19, AJQ+20]. We believe that Corollary 1.12 can be of independent interest because of those applications. One potential application would be in constructing double samplers from Ramanujan complexes under a weaker expansion assumption [DK17].

1.3 Related Work

Higher Order Random Walks and Applications

Our work follows a sequence of works [KM17, DK17, Opp18, KO18, DDFH18] which use the spectral properties of the links of simplical complexes to analyze higher order random walks. Higher order random walks on simplicial complexes were first introduced by Kaufman and Mass [KM17]. They formulated related but more combinatorial notions of skeleton expansion and colorful expansion to establish fast mixing of higher order random walks. Dinur and Kaufman [DK17] introduced the definition of two-sided γ-local-spectral expanders, which is similar to Definition 1.1 but requires all but the first eigenvalue to have absolute value at most γ (i.e. it also controls the negative eigenvalues). They used this stronger assumption to prove a similar theorem as in Theorem 1.2, and applied it to construct efficient agreement tester with applications to PCP constructions. The one-sided γ-local-expander in Definition 1.1 was first studied by Oppenheim [Opp18], where he proved Theorem 1.3. Then, Kaufman and Oppenheim [KO18] strengthened the result in [DK17] and prove Theorem 1.2.

Dikstein, Dinur, Filmus and Harsha [DDFH18] studied an alternative definition of high dimensional expanders, based on the operator norm of the difference between the (non-lazy) up-down and down-up operators. Using this definition, they show that it is possible to approximately characterize all the eigenvalues and eigenvectors of higher order random walks. Their techniques were used in [AJT19] to analyze the "swap walks" on high dimensional expanders, with applications in designing good approximation algorithms for solving constraint satisfaction problems on high-dimensional expanders. Recently, Dikstein and Dinur [DD19] study the same "swap walks" under the name "complement walks" and show applications in agreement testing.

The results in higher order random walks have also found applications in coding theory. The double samplers in [DK17] are used in [DHK+19] to design an efficient algorithm to decode direct product codes over high dimensional expanders. The swap walks in [AJT19] are used in [AJQ+20] to recover the same result and also to design an efficient algorithm to decode direct sum codes over high dimensional expanders.

Analyzing Mixing Times of Markov Chains

Mixing time of Markov chains is an extensively studied topic with various applications (see e.g. [WLP09, MT05]). There are several well-developed approaches to bound the mixing time of a Markov chain. Perhaps the most widely used approach is the coupling method (e.g. [Ald83, BD97]), which has applications in sampling graph colorings (e.g. [Jer95, Vig00]) and many other problems (see [WLP09]). The canonical path (or more generally multicommodity flow) method developed in [JS89, Sin92, Sin93] was used in the important problem of sampling perfect matchings in bipartite graphs [JS89, JSV04] and other problems including sampling matroid bases [FM92]. Geometric methods are used in the important problem of sampling a random point in a convex body [DFK91, LV06]. Analytical methods such as the (modified) log-Sobolev inequalities and Nash inequalities [DSC+96, BT06] are used in proving sharp bounds on mixing time, e.g. a recent paper [CGM19] used a modified log-Sobolev inequality to prove optimal mixing time of the natural Markov chain on sampling matroid bases.

The simplicial complex approach studied in this paper is quite different from the above approaches. It is linear algebraic and designed to bound the second eigenvalue directly using ideas from simplicial complexes. On the other hand, the coupling method is probabilistic and designed to compare two random processes, while the canonical path method and the geometric method are designed to bound the underlying expansion of the graph or the geometric object. The analytical methods are more diffcult to apply and are not as widely applicable, but when they work they could be used to prove very sharp results.

2 Preliminaries

2.1 Linear Algebra

Vectors and Inner Products

Bold faces will be used for scalar functions, i.e. $\boldsymbol{f} \in \mathbb{R}^V$. The notation $\mathbf{1}_V \in \mathbb{R}^V$ will be reserved for the all-one vector in $\mathbb{R}^V$; the subscript may be omitted when the vector space $\mathbb{R}^V$ is clear from the context.

Throughout this text, we use $\Pi \in \mathbb{R}^V$ to denote various probability distributions, i.e. $\sum_{x \in V} \Pi(x) = 1$ and $\Pi(x) \geq 0$ for $x \in V$. Given $\boldsymbol{f}, \boldsymbol{g} \in \mathbb{R}^V$, we use the notations $\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi$ and $\|\boldsymbol{f}\|_\Pi$ to denote the inner-product and the norm with respect to the distribution Π, i.e.

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi = \sum_x \Pi(x) \boldsymbol{f}(x) \boldsymbol{g}(x) \quad \text{and} \quad \|\boldsymbol{f}\|_\Pi^2 = \langle \boldsymbol{f}, \boldsymbol{f} \rangle_\Pi.$$

We reserve $\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \sum_x \boldsymbol{f}(x) \boldsymbol{g}(x)$ for the standard inner product. Given $\boldsymbol{f} \in \mathbb{R}^V$, we write $\|\boldsymbol{f}\|_{\ell_1} = \sum_{x \in V} |\boldsymbol{f}(x)|$ for its ℓ_1-norm, and $\|\boldsymbol{f}\|_{\ell_2} = (\sum_{x \in V} \boldsymbol{f}(x)^2)^{\frac{1}{2}}$ for its ℓ_2-norm.

Matrices and Eigenvalues

Serif faces will be used for matrices, i.e. $\mathsf{A} \in \mathbb{R}^{V \times V}$. Let $G = (V, E)$ be an edge-weighted undirected graph with a weight $w_e > 0$ on each edge $e \in E$. The adjacency matrix of G is denoted by $\mathsf{A}_G \in \mathbb{R}^{V \times V}$ with $\mathsf{A}_G(u, v) = w_{uv}$ for $uv \in E$ and $\mathsf{A}_G(u, v) = 0$ for $uv \notin E$. The diagonal degree matrix of G is denoted by D_G with $\mathsf{D}_G(v, v) = \deg(v) = \sum_{u: uv \in E} w_{uv}$ for $v \in V$. The random walk matrix of G is denoted by $\mathsf{M}_G := \mathsf{D}_G^{-1} \mathsf{A}_G$. Note that M_G is a row-stochastic matrix where every row sums to one. Throughout this text, we will use $\mathsf{M} \in \mathbb{R}^{U \times V}$ to denote row-stochastic operators, where $\mathsf{M}\mathbf{1}_V = \mathbf{1}_U$.

The adjoint of the operator $\mathsf{B} \in \mathbb{R}^{V \times U}$, with respect to the inner-products defined by Π_U and Π_V on U and V, is the unique operator $\mathsf{B}^* \in \mathbb{R}^{U \times V}$ such that

$$\langle \boldsymbol{f}, \mathsf{B}\boldsymbol{g} \rangle_{\Pi_U} = \langle \mathsf{B}^* \boldsymbol{f}, \boldsymbol{g} \rangle_{\Pi_V} \quad \text{for all } \boldsymbol{f} \in \mathbb{R}^U, \boldsymbol{g} \in \mathbb{R}^V.$$

If $U = V$ and $\Pi_U = \Pi_V$, the operator B is called self-adjoint if $\mathsf{B}^* = \mathsf{B}$. Note that a real symmetric matrix is self-adjoint with respect to the standard inner product.

If M is a row-stochastic self-adjoint operator (with respect to the stationary distribution Π), then the Markov chain described by M is called reversible. The random walk operator of an edge-weighted undirected graph is described by the self-adjoint row-stochastic operator M_G (with respect to the stationary distribution $\Pi = \mathsf{D}_G \mathbf{1} / \sum_v \deg(v)$) and is a reversible Markov chain.

Let $\mathsf{W} \in \mathbb{R}^{V \times V}$ be a self-adjoint operator with respect to an inner product Π. It is a fundamental result in linear algebra that W has only real eigenvalues, and an orthonormal set of eigenvectors with respect to the inner product Π, i.e. $\langle \boldsymbol{f}, \boldsymbol{g} \rangle_\Pi = 0$ for eigenvectors $\boldsymbol{f} \neq \boldsymbol{g}$. We write $\lambda_i(\mathsf{W})$ for the i-th largest eigenvalue of W so that $\lambda_1(\mathsf{W}) \geq \ldots \geq \lambda_{|V|}(\mathsf{W})$, and write $\lambda_{\min}(\mathsf{W})$ for the smallest eigenvalue of W, i.e. $\lambda_{\min}(\mathsf{W}) = \lambda_{|V|}(\mathsf{W})$.

The largest eigenvalue $\lambda_1(\mathsf{W})$ of a self-adjoint matrix W with respect to some measure Π obeys the variational formula

$$\lambda_1(\mathsf{W}) = \max\{\langle \boldsymbol{f}, \mathsf{W}\boldsymbol{f}\rangle_\Pi : \boldsymbol{f} \in \mathbb{R}^V, \|\boldsymbol{f}\|_\Pi = 1\}. \quad \text{(variational formula)}$$

It is well-known that the maximizers of the variational formula are precisely the eigenvectors of W corresponding to $\lambda_1(\mathsf{W})$, i.e. $\mathsf{W}\boldsymbol{f} = \lambda_1(\mathsf{W}) \cdot \boldsymbol{f}$ if and only if $\boldsymbol{f}$ maximizes the RHS in the variational formula.

Given an arbitrary operator $\mathsf{B} \in \mathbb{R}^{V\times U}$ we will write $\sigma_i(\mathsf{B})$ for the i-th largest singular value of B so that $\sigma_1(\mathsf{B}) \geq \ldots \geq \sigma_{\min\{|U|,|V|\}}(\mathsf{B})$. It is well known that the singular values of a real operator B coincide with the eigenvalues of the self-adjoint operator $\mathsf{B}\mathsf{B}^*$.

A self-adjoint operator $\mathsf{A} \in \mathbb{R}^{V\times V}$ with respect to inner product Π is called positive semi-definite, denoted by $\mathsf{A} \succeq_\Pi 0$, if it satisfies $\langle \boldsymbol{f}, \mathsf{A}\boldsymbol{f}\rangle_\Pi \geq 0$ for all $\boldsymbol{f} \in \mathbb{R}^V$. The condition is equivalent to the condition that $\lambda_{\min}(\mathsf{A}) \geq 0$. For self-adjoint operators $\mathsf{A} \in \mathbb{R}^{V\times V}$ and $\mathsf{B} \in \mathbb{R}^{V\times V}$ with respect to the same inner product Π, we will write $\mathsf{A} \preceq_\Pi \mathsf{B}$ if

$$\langle \boldsymbol{f}, \mathsf{A}\boldsymbol{f}\rangle_\Pi \leq \langle \boldsymbol{f}, \mathsf{B}\boldsymbol{f}\rangle_\Pi \quad \text{for all} \quad \boldsymbol{f} \in \mathbb{R}^V.$$

This is equivalent to $\mathsf{A} - \mathsf{B}$ being positive-semidefinite, i.e. $\mathsf{A} - \mathsf{B} \succeq_\Pi 0$. If Π is just the standard inner product, we will drop the subscript Π.

We will use the following results about eigenvalues in Section 3 and Section 4; see e.g. [Bha13].

Fact 2.1. *Let $\mathsf{A} \in \mathbb{R}^{U\times V}$ and $\mathsf{B} \in \mathbb{R}^{V\times U}$. Then, the non-zero spectrum of AB coincides with that of BA with the same multiplicity.*

Fact 2.2. *Let $\mathsf{A}, \mathsf{B} \in \mathbb{R}^{V\times V}$ be two self-adjoint matrices with respect to the inner product Π satisfying $\mathsf{A} \preceq_\Pi \mathsf{B}$. Then, $\lambda_i(\mathsf{A}) \leq \lambda_i(\mathsf{B})$ for all $1 \leq i \leq |V|$.*

Theorem 2.3 (Cauchy Interlacing Theorem)**.** *Let $\mathsf{A} \in \mathbb{R}^{V\times V}$ be a symmetric matrix and $\mathsf{B} \in \mathbb{R}^{U\times U}$ be a principal submatrix of A. Let $n = |V|$ and $m = |U|$. For any $0 \leq j \leq m$,*

$$\lambda_j(\mathsf{A}) \geq \lambda_j(\mathsf{B}) \geq \lambda_{n-m+j}(\mathsf{A}).$$

Theorem 2.4 (Weyl Interlacing Theorem)**.** *Let $\mathsf{A}, \mathsf{B} \in \mathbb{R}^{V\times V}$ be two symmetric matrices. For any i, j,*

$$\lambda_{i+j-1}(\mathsf{A} + \mathsf{B}) \leq \lambda_i(\mathsf{A}) + \lambda_j(\mathsf{B}).$$

2.2 Simplicial Complexes

A simplicial complex X is a collection of subsets that is downward closed, i.e. if $\beta \in X$ and $\alpha \subset \beta$ then $\alpha \in X$. The elements α, β in X are called faces/simplices of X. The dimension of a face α is defined as $|\alpha| - 1$, e.g. an edge is of dimension 1, a vertex/singleton is of dimension 0, the empty set is of dimension -1. The set of faces of dimension j is denoted by $X(j)$. The dimension of a simplicial complex is defined as the maximum dimension of its faces. A d-dimensional simplicial complex is called pure if every maximal face is of dimension d. All simplicial complexes considered in this paper are pure.

Weighted Simplicial Complexes

A simplicial complex X can be equipped with a weighted function which assigns a positive weight to each face of X. We follow the formalism in [DDFH18] where the weight function is a probability distribution Π on the faces of the same dimension. Let X be a d-dimensional simplicial complex. Given a probability distribution $\Pi := \Pi_d$ on $X(d)$, we can inductively obtain probability distributions Π_j on all $X(j)$ by considering the marginal distributions, i.e.

$$\Pi_j(\alpha) = \frac{1}{j+2} \sum_{\substack{\beta \in X(j+1), \\ \beta \supset \alpha}} \Pi_{j+1}(\beta). \tag{2.1}$$

Equivalently, we can understand Π_j as the probability distribution of the following random process: Sample a random face $\beta \in X(d)$ using the probability distribution Π_d, and then sample a uniform random subset of β in $X(j)$. The pair (X, Π) will be referred as a weighted simplicial complex. We write (X, Π) simply as X if Π is the uniform distribution.

Links and Graphs

Let (X, Π) be a pure d-dimensional weighted simplicial complex. The link X_α of a face α is the simplicial complex defined as

$$X_\alpha := \{\beta \setminus \alpha \mid \beta \in X, \beta \supset \alpha\}.$$

The probability distributions $\Pi_0, \dots, \Pi_d$ on X can be naturally used to define the probability distributions $\Pi_0^\alpha, \dots, \Pi_{d-|\alpha|}^\alpha$ on X_α using conditional probability. Suppose $\alpha \in X(j)$. The probability distribution Π_l^α for $X_\alpha(l)$ is defined as

$$\Pi_l^\alpha(\tau) = \mathbb{P}_{\beta \sim \Pi_{j+1+l}}[\beta = \alpha \cup \tau \mid \beta \supset \alpha] = \frac{\Pi_{j+l+1}(\alpha \cup \tau)}{\binom{|\alpha \cup \tau|}{|\alpha|} \cdot \Pi_j(\alpha)} \qquad \text{for } \tau \in X_\alpha(l), \tag{2.2}$$

where the latter equality is obtained by applying Eq. (2.1) repeatedly.

Given a link X_α, the graph $G_\alpha = (X_\alpha(0), X_\alpha(1), \Pi_1^\alpha)$ is defined as the 1-skeleton of X_α. More explicitly, each singleton $\{v\}$ in X_α is a vertex v in G_α, each pair $\{u, v\}$ in X_α is an edge uv in G_α, and the weight of uv in G_α is equal to $\Pi_1^\alpha(\{u, v\})$. A simple observation is that if X is a pure d-dimensional simplicial complex and Π is the uniform distribution on $X(d)$, then for any $\alpha \in X(d-2)$ the weighting Π_1^α on the edges of G_α is uniform. We will use this simple observation in Section 4.

2.3 Local Spectral Expanders

Random Walk Matrices

The definition of local spectral expanders will be based on the random walk matrix of G_α. Let A_α be the adjacency matrix of G_α. Let D_α be the diagonal degree matrix where $\mathsf{D}_\alpha(x,x) = \sum_y \mathsf{A}_\alpha(x,y) = 2\Pi_0^\alpha(x)$ where the last equality is by Eq. (2.1). The random walk matrix M_α of G_α is defined as $\mathsf{M}_\alpha := \mathsf{D}_\alpha^{-1}\mathsf{A}_\alpha$, with

$$\mathsf{M}_\alpha(x,y) = \frac{\Pi_1^\alpha(x,y)}{2\Pi_0^\alpha(x)} \qquad \text{for all } \{x,y\} \in X_\alpha(1).$$

The distribution Π_0^α is the stationary distribution of M_α, as

$$(\Pi_0^\alpha)^\top \mathsf{M}_\alpha = (\Pi_0^\alpha)^\top \mathsf{D}_\alpha^{-1}\mathsf{A}_\alpha = \mathbf{1}^\top \mathsf{A}_\alpha = (\Pi_0^\alpha)^\top.$$

The matrix M_α is self-adjoint with respect to the inner product defined by Π_0^α, as

$$\langle \boldsymbol{f}, \mathsf{M}_\alpha \boldsymbol{g} \rangle_{\Pi_0^\alpha} = \langle \boldsymbol{f}, \mathsf{D}_\alpha^{-1}\mathsf{A}_\alpha \boldsymbol{g} \rangle_{\Pi_0^\alpha} = \langle \boldsymbol{f}, \mathsf{A}_\alpha \boldsymbol{g} \rangle = \langle \mathsf{A}_\alpha \boldsymbol{f}, \boldsymbol{g} \rangle = \langle \mathsf{D}_\alpha^{-1}\mathsf{A}_\alpha \boldsymbol{f}, \boldsymbol{g} \rangle_{\Pi_0^\alpha} = \langle \mathsf{M}_\alpha \boldsymbol{f}, \boldsymbol{g} \rangle_{\Pi_0^\alpha}.$$

So, M_α have only real eigenvalues, and an orthonormal basis of eigenvectors with respect to the inner product Π_0^α. The largest eigenvalue of M_α is 1, as $\mathsf{M}_\alpha \mathbf{1} = \mathbf{1}$ and M_α is row-stochastic.

Given a vector $\boldsymbol{f}$, we will be interested in writing it as $\boldsymbol{f} = \boldsymbol{f}^{\mathbf{1}} + \boldsymbol{f}^{\perp \mathbf{1}}$, so that $\boldsymbol{f}^{\mathbf{1}} = c\mathbf{1}$ for some scalar c and $\langle \boldsymbol{f}^{\mathbf{1}}, \boldsymbol{f}^{\perp \mathbf{1}} \rangle_{\Pi_0^\alpha} = 0$. It follows that $c = \frac{\langle \boldsymbol{f}, \mathbf{1} \rangle_{\Pi_0^\alpha}}{\langle \mathbf{1}, \mathbf{1} \rangle_{\Pi_0^\alpha}} = \langle \boldsymbol{f}, \mathbf{1} \rangle_{\Pi_0^\alpha} = \mathbb{E}_{x \sim \Pi_0^\alpha}[\boldsymbol{f}(x)]$. We write $\mathsf{J}_\alpha = \mathbf{1}(\Pi_0^\alpha)^\top$ as the operator to map $\boldsymbol{f}$ to $\boldsymbol{f}^{\mathbf{1}}$, so that

$$\mathsf{J}_\alpha \boldsymbol{f} = (\mathbf{1}(\Pi_0^\alpha)^\top)\boldsymbol{f} = \langle \boldsymbol{f}, \Pi_0^\alpha \rangle \cdot \mathbf{1} = \mathop{\mathbb{E}}_{x \sim \Pi_0^\alpha}[\boldsymbol{f}(x)] \cdot \mathbf{1} = \boldsymbol{f}^{\mathbf{1}}. \qquad \text{(projector to constant functions)}$$

Local Spectral Expanders and Oppenheim's theorem

Let (X, Π) be a pure d-dimensional weighted simplicial complex. Define

$$\gamma_j := \gamma_j(X, \Pi) = \max_{\alpha \in X(j)} \lambda_2(\mathsf{M}_\alpha) \quad \text{for all } j = -1, \dots, d-2,$$

where $\lambda_2(\mathsf{M}_\alpha)$ is the second largest eigenvalue of the operator M_α. We say X is a γ-local-spectral expander if $\gamma_i \le \gamma$ for $-1 \le i \le d-2$.

Oppenheim's theorem relates the second eigenvalue of the graph of a lower-dimensional link to that of a higher-dimensional link. It works for any weighted simplicial complex with a "balanced" weight function w, where for any $\alpha \in X(k)$ and any $-1 \le k \le d-1$ it holds that

$$w(\alpha) = c_k \sum_{\substack{\beta \in X(k+1), \\ \beta \supset \alpha}} w(\beta)$$

for some constant c_k that only depends on k. Note that the weight function in Eq. (2.1) satisfies this condition with $c_k = 1/(k+2)$.

Theorem 2.5 (Oppenheim's Theorem)**.** *Let (X, Π) be a pure d-dimenisonal weighted simplicial complex where Π satisfies Eq. (2.1). For any $0 \le j \le d-2$, if G_α is connected for every $\alpha \in X(j-1)$, then*

$$\gamma_{j-1} \le \frac{\gamma_j}{1 - \gamma_j}.$$

An inductive argument proves the following corollary.

Corollary 2.6 (Oppenheim's Corollary)**.** *Let (X, Π) be a pure d-dimenisonal weighted simplicial complex where Π satisfies Eq. (2.1). If G_α is connected for every $\alpha \in X(k)$ and every $k \le d-2$, then*

$$\gamma_j \le \frac{\gamma_{d-2}}{1 - (d-2-j) \cdot \gamma_{d-2}}.$$

2.4 Higher Order Random Walks

Up and Down Operators

Let (X, Π) be a pure d-dimensional weighted simplicial complex. In the following definitions, $\alpha \in X(k)$, $\beta \in X(k+1)$, $\boldsymbol{f} \in \mathbb{R}^{X(j)}$, $\boldsymbol{g} \in \mathbb{R}^{X(k+1)}$, and $j \in \{-1, 0, \dots, d-1\}$.

The j-th up operator $\mathsf{U}_j : \mathbb{R}^{X(j)} \to \mathbb{R}^{X(j+1)}$ is defined as

$$[\mathsf{U}_j \boldsymbol{f}](\beta) = \frac{1}{j+2} \sum_{x \in \beta} \boldsymbol{f}(\beta \backslash x) = \sum_{\substack{\alpha \subset \beta, \\ \alpha \in X(j)}} \frac{\boldsymbol{f}(\alpha)}{j+2}. \qquad \text{(up operator)}$$

The $(j+1)$-st down operator $\mathsf{D}_{j+1} : \mathbb{R}^{X(j+1)} \to \mathbb{R}^{X(j)}$ is defined as

$$[\mathsf{D}_{j+1} \boldsymbol{g}](\alpha) = \sum_{x \in X_\alpha(0)} \frac{\Pi_{j+1}(\alpha \cup x)}{(j+2)\Pi_j(\alpha)} \cdot \boldsymbol{g}(\alpha \cup x) = \sum_{\substack{\beta \supset \alpha, \\ \beta \in X(j+1)}} \frac{\Pi_{j+1}(\beta) \cdot \boldsymbol{g}(\beta)}{(j+2)\Pi_j(\alpha)} \qquad \text{(down operator)}$$

It can be checked [KO18, DDFH18] that the adjoint of $\mathsf{U}_j : \mathbb{R}^{X(j)} \to \mathbb{R}^{X(j+1)}$ with respect to the inner products $\Pi_{j+1} \in \mathbb{R}^{X(j+1)}$ and $\Pi_j \in \mathbb{R}^{X(j)}$ is D_{j+1}, i.e.

$$\langle \boldsymbol{g}, \mathsf{U}_j \boldsymbol{f} \rangle_{\Pi_{j+1}} = \langle \mathsf{D}_{j+1} \boldsymbol{g}, \boldsymbol{f} \rangle_{\Pi_j} \quad \text{for all } \boldsymbol{g} \in \mathbb{R}^{X(j+1)}, \boldsymbol{f} \in \mathbb{R}^{X(j)}. \qquad \text{(adjointness)}$$

And it follows that the adjoint of D_{j+1} with respect to the inner products Π_j and Π_{j+1} is U_j, i.e. $\langle \boldsymbol{f}, \mathsf{D}_{j+1} \boldsymbol{g} \rangle_{\Pi_j} = \langle \mathsf{U}_j \boldsymbol{f}, \boldsymbol{g} \rangle_{\Pi_{j+1}}$ for all $\boldsymbol{g} \in \mathbb{R}^{X(j+1)}, \boldsymbol{f} \in \mathbb{R}^{X(j)}$.

Remark 2.7. We have stayed consistent with the notations in [DDFH18], and named U_j and D_{j+1} up and down operators with their right-action on functions (or vectors) in mind. However, in terms of random walks, U_j describes a random down-movement from $X(j+1)$ to $X(j)$, whereas D_{j+1} describes a random up-movement from $X(j)$ to $X(j+1)$, since the action of the probability distribution is from the left.

Down-Up Walk, Up-Down Walk, and Non-Lazy Up-Down Walk

We use the up and down operators to define three random walk operators on $X(j)$. The j-th down-up walk $\mathsf{P}_j^{\triangledown}$ and the j-th up-down walk $\mathsf{P}_j^{\triangle}$ are defined as

$$\mathsf{P}_j^{\triangledown} = \mathsf{U}_{j-1}\mathsf{D}_j \quad \text{and} \quad \mathsf{P}_j^{\triangle} = \mathsf{D}_{j+1}\mathsf{U}_j. \qquad \text{(down-up walk, up-down walk)}$$

One useful property of $\mathsf{P}_j^{\triangle}$ and $\mathsf{P}_j^{\triangledown}$ is that they have the same non-zero spectrum with the same multiplicity by Fact 2.1, and in particular $\lambda_2(\mathsf{P}_j^{\triangle}) = \lambda_2(\mathsf{P}_j^{\triangledown})$. Also define the j-th non-lazy up-down walk

$$\mathsf{P}_j^{\wedge} = \frac{j+2}{j+1}\left(\mathsf{P}_j^{\triangle} - \frac{1}{j+2}\mathsf{I}\right), \qquad \text{(non-lazy up-down walk)}$$

which is the up-down walk conditioned on not looping. It follows from the adjointness of U_j and D_{j+1} that all $\mathsf{P}_j^{\triangledown}$, $\mathsf{P}_j^{\triangle}$, and $\mathsf{P}_j^{\wedge}$ are self-adjoint with respect to the inner product Π_j, e.g. given any $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \mathbb{R}^{X(j)}$,

$$\langle \boldsymbol{f}_1, \mathsf{P}_j^{\triangle} \boldsymbol{f}_2 \rangle_{\Pi_j} = \langle \boldsymbol{f}_1, \mathsf{D}_{j+1}\mathsf{U}_j \boldsymbol{f}_2 \rangle_{\Pi_j} = \langle \mathsf{U}_j \boldsymbol{f}_1, \mathsf{U}_j \boldsymbol{f}_2 \rangle_{\Pi_{j+1}} = \langle \mathsf{D}_{j+1}\mathsf{U}_j \boldsymbol{f}_1, \boldsymbol{f}_2 \rangle_{\Pi_j} = \langle \mathsf{P}_j^{\triangle} \boldsymbol{f}_1, \boldsymbol{f}_2 \rangle_{\Pi_j}. \qquad (2.3)$$

These imply that Π_j is the stationary distribution for all these random walks $\mathsf{P}_j^{\triangledown}$, $\mathsf{P}_j^{\triangle}$, and $\mathsf{P}_j^{\wedge}$, e.g. putting $\boldsymbol{f}_1 = \mathbf{1}$ and $\boldsymbol{f}_2 = \chi_i$ into $\Pi_j^\top$, then

$$\Pi_j^\top \mathsf{P}_j^{\triangle}(i) = \langle \mathbf{1}, \mathsf{P}_j^{\triangle} \chi_i \rangle_{\Pi_j} = \langle \mathsf{P}_j^{\triangle} \mathbf{1}, \chi_i \rangle_{\Pi_j} = \langle \mathbf{1}, \chi_i \rangle_{\Pi_j} = \Pi_j^\top(i) \quad \Longrightarrow \quad \Pi_j^\top \mathsf{P}_j^{\triangle} = \Pi_j^\top. \qquad (2.4)$$

Combinatorial Interpretation: We can understand the higher order random walks as a random walk on a bipartite graph between $X(j)$ and $X(j+1)$ as explained in [ALOV19, DK17]. Consider the bipartite graph

$H = (X(j), X(j+1), E)$ in which a face $\alpha \in X(j)$ and a face $\beta \in X(j+1)$ are connected if and only if $\alpha \subset \beta$. The edge $\{\alpha, \beta\} \in H$ is assigned the weight $\frac{1}{j+2} \cdot \Pi_{j+1}(\beta)$. Using Eq. (2.1), it can be seen that the weighted degree of any $\alpha \in X(j)$ is $\Pi_j(\alpha)$. And the weighted degree of any $\beta \in X(k+1)$ is exactly $\Pi_{j+1}(\beta)$. Thus, the graph H has the (weighted) random walk matrix

$$\mathsf{M}_H = \begin{pmatrix} 0 & \mathsf{U}_j \\ \mathsf{D}_{j+1} & 0 \end{pmatrix}.$$

One step of the down-up walk $\mathsf{P}_{j+1}^{\bigtriangledown}$ can be thought as a two step random walk in M_H: starting from some $\beta \in X(j+1)$, the random walk will go down from $\beta \in X(j+1)$ to $\alpha \in X(j)$ by dropping an element of β, which is chosen uniformly at random as prescribed by U_j, and then the random walk will go up from $\alpha \in X(j)$ to a random face $\beta' \in X(j+1)$ which contains α as prescribed by D_{j+1}. Similarly, one step of the up-down walk $\mathsf{P}_j^{\triangle}$ can be thought as a two step random walk in M_H. More precisely,

$$\mathsf{M}_H^2 = \begin{pmatrix} 0 & \mathsf{U}_j\mathsf{D}_{j+1} \\ \mathsf{D}_{j+1}\mathsf{U}_j & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathsf{P}_{j+1}^{\bigtriangledown} \\ \mathsf{P}_j^{\triangle} & 0 \end{pmatrix}.$$

It is instructive to check that when the distribution Π of the simplicial complex is the uniform distribution, then the down-up walks and the up-down walks are as described as in the introduction.

Longer Random Walks

Suppose now $-1 \le a < b \le d$. We define the up-down walk on $X(a)$ through $X(b)$ to be

$$\mathsf{P}_{a,b}^{\triangle} = \mathsf{D}_{a+1} \cdots \mathsf{D}_b \cdot \mathsf{U}_{b-1} \cdots \mathsf{U}_a.$$

Similar to the intuition that was presented about the up-down and the down-up walks, we can think of $\mathsf{P}_{a,b}^{\triangle}$ as simulating two-steps of the random walk starting from some face $\alpha \in X(a)$ on the weighted bipartite graph $H = (X(a), X(b), E)$ where $\{\alpha, \beta\}$ is an edge of this graph with weight proportional to $\Pi_b(\beta)$ whenever $\alpha \subset \beta$.

2.5 Mixing Times of Markov Chains

Recall that two distributions Π and Π' are said to be ε-close if

$$\|\Pi - \Pi'\|_{\ell_1} = \sum_{x \in V} |\Pi(x) - \Pi'(x)| \le \varepsilon. \qquad (\varepsilon\text{-close})$$

The mixing time $T(\varepsilon, \mathsf{P})$ of the random walk operator P is defined to be the least time step where the distribution of the random walk is ε-close to the stationary distribution Π of P in the ℓ_1 distance, i.e.

$$T(\varepsilon, \mathsf{P}) = \min\{t \in \mathbb{N}_{\ge 0} : \|\mathsf{P}^t(x, \bullet) - \Pi\|_{\ell_1} \le \varepsilon\}. \qquad (\text{mixing time})$$

For our applications in sampling in Section 4, we will use the following well known relation between the mixing time of the random walk and the spectral gap of its transition matrix (see e.g. [MT05, Proposition 1.12]).

Theorem 2.8 (Spectral Mixing Time Bound)**.** *Let* $\mathsf{P} \in \mathbb{R}^{V \times V}$ *be a random walk matrix with stationary distribution* Π*. One has,*

$$T(\varepsilon, \mathsf{P}) \leq \frac{1}{1 - \sigma_2(\mathsf{P})} \cdot \log \frac{1}{\varepsilon \cdot \min_{x \in V} \Pi(x)},$$

where $\sigma_2(\mathsf{P})$ *is the second largest singular value of* P*.*

The operator of importance for us will be $\mathsf{P} = \mathsf{P}_j^{\triangledown}$. As this operator is self-adjoint as explained in Section 2.4, we have $\sigma_2(\mathsf{P}_j^{\triangledown}) = \lambda_2(\mathsf{P}_j^{\triangledown})$. Also, recall from Section 2.4 that the stationary distribution of $\mathsf{P}_j^{\triangle}$ is Π_j, we obtain

$$T(\varepsilon, \mathsf{P}_j^{\triangledown}) \leq \frac{1}{1 - \lambda_2(\mathsf{P}_j^{\triangledown})} \cdot \log \frac{1}{\varepsilon \cdot \min_{\alpha \in X(j)} \Pi_j(\alpha)}.$$

Approximate Sampling and Approximate Counting

There is a well-known equivalence between approximate sampling and approximate counting for self-reducible problems. Let $\Omega := \{\Omega_s\}_s$ be a collection of sets parametrized by some strings s, e.g. s can be describing a graph and Ω_s the set of perfect matchings in G. Suppose a randomized algorithm $\mathcal{A}$ is given whose output distribution is described by $\mu_{\mathcal{A}(s)}$. Then $\mathcal{A}$ is called a fully polynomial time randomized approximate uniform sampler (FPRAUS) for Ω_s, if for every input string s we have

$$\|\mu_{\mathcal{A}(s)} - \Pi_{\Omega_s}\|_{\ell_1} \leq \delta,$$

where Π_{Ω_s} describes the uniform distribution over Ω_s and the algorithm $\mathcal{A}$ runs in time $\text{poly}(\langle s \rangle, \log(1/\delta))$, where $\langle s \rangle$ denotes the size of the input.

Similarly, an algorithm $\mathcal{A}'$ is called a fully polynomial time randomized approximation scheme (FPRAS) for Ω, if we for every input s we have

$$\Pr[(1-\delta) \cdot |\Omega_s| \leq \mathcal{A}'(s) \leq (1+\delta) \cdot |\Omega_s|] \geq 1 - \delta,$$

and the algorithm $\mathcal{A}'$ runs in time $\text{poly}(\langle s \rangle \log(1/\delta))$.

A well-known result proven in [JVV86] asserts that approximate counting and approximate sampling are equivalent for self-reducible problems.

Theorem 2.9 (Informal)**.** *For self-reducible sets* Ω *in* NP*, the existence of an FPRAS for* Ω *is equivalent to the existence of an FPRAUS for* Ω*.*

In Section 4, we will give approximate samplers for independent sets a graph, and it follows from this equivalence we can also approximately count the number of independent sets in the graph.

3 Eigenvalue Bounds for Higher Order Random Walks

Our main result is a quantitative generalization of the basic fact that a pure d-dimensional simplicial complex X is gallery connected (i.e. $\lambda_2(\mathsf{P}_d^{\triangledown}) < 1$) if and only if the graph G_α is connected for every $\alpha \in X$ up to dimension $d-2$ (i.e. $\gamma_j < 1$ for $-1 \leq j \leq d-2$). The statement is essentially the same as in Theorem 1.5 but for more general weighted simplicial complexes.

Theorem 3.1. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. For any $0 \le k \le d$,*

$$\lambda_2(\mathsf{P}_k^{\bigtriangledown}) = \lambda_2(\mathsf{P}_{k-1}^{\triangle}) \le 1 - \frac{1}{k+1} \prod_{j=-1}^{k-2} (1 - \gamma_j).$$

Using an inductive argument as in [ALOV19, Theorem 3.3], we can prove a more general statement about the entire range of eigenvalues.

Theorem 3.2. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. Then, for any $0 \le k \le d-1$ and for any $-1 \le r \le k$, the matrix $\mathsf{P}_k^{\triangle}$ has at most $|X(r)|$ eigenvalues with value strictly greater than*

$$1 - \frac{1}{k+2} \prod_{j=r}^{k-1} (1 - \gamma_j).$$

Note that Theorem 3.1 is a special case of Theorem 3.2 where $r = -1$ (recall that $X(-1) = \{\emptyset\}$ and so $|X(-1)| = 1$). Further, Theorem 3.1 can only prove that $\lambda_2(\mathsf{P}_d^{\bigtriangledown}) \le 1 - \frac{1}{d+1}$. We observe that this bound is almost tight.

Proposition 3.3. *Let X be a d-dimensional simplicial complex. Let $n = |X(0)|$. Suppose $2(d+1) \le n$. Then $\lambda_2(\mathsf{P}_d^{\bigtriangledown}) = \lambda_2(\mathsf{P}_{d-1}^{\triangle}) \ge 1 - \frac{2}{d+1}$.*

Before we prove Theorem 3.1 and Theorem 3.2, we present two corollaries of Theorem 3.1.

Combining with Oppenheim's Corollary 2.6, Theorem 3.1 provides a bound on the second eigenvalue of the d-th down-up walk based only on the maximum second eigenvalue of the graphs in dimension $d-2$. This will be useful in Section 4.

Corollary 3.4. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. For any $0 \le k \le d$, suppose $\gamma_k \le \frac{1}{k+1}$ and $\gamma_j < 1$ for $-1 \le j \le k-2$, then*

$$\lambda_2(\mathsf{P}_k^{\bigtriangledown}) = \lambda_2(\mathsf{P}_{k-1}^{\triangle}) \le 1 - \frac{1}{(k+1)^2}.$$

Proof. Since $\gamma_{k-2} \le \frac{1}{k+1}$ and $\gamma_j < 1$ for $-1 \le j \le k-2$, it follows from Oppenheim's Corollary 2.6 that for any $-1 \le j \le k-3$,

$$\gamma_j \le \frac{\gamma_{k-2}}{1 - (k-2-j) \cdot \gamma_{k-2}} \le \frac{\frac{1}{k+1}}{1 - \frac{k-2-j}{k+1}} = \frac{1}{j+3}.$$

Therefore, by Theorem 3.1,

$$\lambda_2(\mathsf{P}_k^{\bigtriangledown}) \le 1 - \frac{1}{k+1} \prod_{j=-1}^{k-2} (1 - \gamma_j) \le 1 - \frac{1}{k+1} \prod_{j=-1}^{k-2} \frac{j+2}{j+3} = 1 - \frac{1}{(k+1)^2}.$$

□

Theorem 3.1 implies the following result for longer random walks on local-spectral expanders.

Corollary 3.5. *Let (X,Π) be a pure d-dimensional weighted simplicial complex. Let $0 \le a < b \le d-1$. If X is a γ-local-spectral expander, then*

$$\lambda_2(\mathsf{P}^{\triangle}_{a,b}) \le (1+\gamma)^{b-a} \cdot \frac{a+1}{b+1}.$$

The rest of this section is organized as follows. We will first prove Theorem 3.1 in Section 3.1, then Theorem 3.2 in Section 3.2, then Corollary 3.5 in Section 3.3, and finally Proposition 3.3 in Section 3.4.

3.1 Proof of Theorem 3.1

The key lemma in proving Theorem 3.1 is the following result that quantifies a spectral bound on the difference of the k-th non-lazy up-down walk and the k-th down-up walk in terms of the second eigenvalue of the links at dimension $k-1$.

Lemma 3.6. *Let (X,Π) be a pure d-dimensional weighted simplicial complex. For any $0 \le k \le d-1$,*

$$\mathsf{P}^{\wedge}_{k} - \mathsf{P}^{\triangledown}_{k} \preceq_{\Pi_k} \gamma_{k-1} \cdot \left(\mathsf{I} - \mathsf{P}^{\triangledown}_{k}\right).$$

The proof of Lemma 3.6, will closely follow the proof of [DDFH18, Theorem 5.5], where they prove the weaker inequality

$$\mathsf{P}^{\wedge}_{k} - \mathsf{P}^{\triangledown}_{k} \preceq_{\Pi_k} \gamma_{k-1} \cdot \mathsf{I}. \tag{3.1}$$

We remark that a similar statement was also used in [KO18] for proving Theorem 1.2.

We will first show how Lemma 3.6 implies Theorem 3.1 by an inductive argument.

Proof of Theorem 3.1 from Lemma 3.6. We prove Theorem 3.1 by induction on k. The base case is when $k=0$, where $\mathsf{P}^{\triangledown}_{0} = \mathbf{1}\Pi_0^{\top}$ is a rank one matrix and so $\lambda_2(\mathsf{P}^{\triangledown}_{0}) \le 0$, and hence Theorem 3.1 trivially holds.

For the induction step, suppose we have

$$\lambda_2(\mathsf{P}^{\triangledown}_{j+1}) = \lambda_2(\mathsf{P}^{\triangle}_{j}) \le 1 - \frac{1}{j+2}\prod_{i=-1}^{j-1}(1-\gamma_i). \tag{induction hypothesis}$$

Since $\mathsf{P}^{\triangledown}_{j+1} = \mathsf{U}_j\mathsf{D}_{j+1}$ and $\mathsf{P}^{\triangle}_{j} = \mathsf{D}_{j+1}\mathsf{U}_j$ have the same non-zero eigenvalues with the same multiplicity by Fact 2.1, we only need to prove the statement for $\mathsf{P}^{\triangle}_{j+1}$. By Lemma 3.6,

$$\mathsf{P}^{\wedge}_{j+1} \preceq_{\Pi_{j+1}} \gamma_j \cdot \mathsf{I} + (1-\gamma_j)\mathsf{P}^{\triangledown}_{j+1}$$

It follows from Fact 2.2 that

$$\lambda_2(\mathsf{P}^{\wedge}_{j+1}) \le \gamma_j + (1-\gamma_j)\cdot\lambda_2(\mathsf{P}^{\triangledown}_{j+1}) \le 1 - \frac{1}{j+2}\prod_{i=-1}^{j}(1-\gamma_i),$$

where the last equality is by plugging in the induction hypothesis. The theorem now follows from the definition of non-lazy up-down walk that

$$\mathsf{P}^{\wedge}_{j+1} = \frac{j+3}{j+2}\left(\mathsf{P}^{\triangle}_{j+1} - \frac{1}{j+3}\mathsf{I}\right) \quad\Longleftrightarrow\quad \mathsf{P}^{\triangle}_{j+1} = \frac{j+2}{j+3}\cdot\mathsf{P}^{\wedge}_{j+1} + \frac{1}{j+3}\mathsf{I}.$$

Therefore,

$$\lambda_2(\mathsf{P}^{\triangle}_{j+1}) = \frac{j+2}{j+3}\cdot\lambda_2(\mathsf{P}^{\wedge}_{j+1}) + \frac{1}{j+3} \le 1 - \frac{1}{j+3}\prod_{i=-1}^{j}(1-\gamma_i),$$

and this proves the induction step. □

3.1.1 Proof of Lemma 3.6

The proof of Lemma 3.6 will rest on few useful identities established in [KO18, DDFH18], which can be obtained through the "Garland Method", which decomposes the higher order random walk matrices into the random walk matrices of the links.

In the following, given $\boldsymbol{f} \in \mathbb{R}^{X(k)}$ and $\alpha \in X(k-1)$, we use $\boldsymbol{f}_\alpha$ to denote the restriction of $\boldsymbol{f}$ to the entries in $\{\alpha \cup \{x\} \mid x \in X_\alpha(0)\}$. And recall that J_α is the projector to constant functions defined in Section 2.3

Lemma 3.7. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. For all $\boldsymbol{f} \in \mathbb{R}^{X(j)}$ the following hold,*

1. $\langle \boldsymbol{f}, \mathsf{I}\boldsymbol{f}\rangle_{\Pi_j} = \mathbb{E}_{\alpha\sim\Pi_{j-1}}\|\boldsymbol{f}_\alpha\|^2_{\Pi_0^\alpha} = \mathbb{E}_{\alpha\sim\Pi_{j-1}}\langle \boldsymbol{f}_\alpha, \boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}$,
2. $\langle \boldsymbol{f}, \mathsf{P}^{\bigtriangledown}_j \boldsymbol{f}\rangle_{\Pi_j} = \mathbb{E}_{\alpha\sim\Pi_{j-1}}\|\mathsf{J}_\alpha\boldsymbol{f}_\alpha\|^2_{\Pi_0^\alpha} = \mathbb{E}_{\alpha\sim\Pi_{j-1}}\langle \boldsymbol{f}_\alpha, \mathsf{J}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}$,
3. $\langle \boldsymbol{f}, \mathsf{P}^{\wedge}_j \boldsymbol{f}\rangle_{\Pi_j} = \mathbb{E}_{\alpha\sim\Pi_{j-1}}\langle \boldsymbol{f}_\alpha, \mathsf{M}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}$.

We will provide a proof of Lemma 3.7 in Section 3.1.2 for completeness. We are ready to prove Lemma 3.6.

Proof of Lemma 3.6. Let $\boldsymbol{f} \in \mathbb{R}^{X(j)}$ be arbitrary. By Items (2) and (3) in Lemma 3.7, we write

$$\langle \boldsymbol{f}, (\mathsf{P}^{\wedge}_j - \mathsf{P}^{\bigtriangledown}_j)\boldsymbol{f}\rangle_{\Pi_j} = \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\left[\langle \boldsymbol{f}_\alpha, (\mathsf{M}_\alpha - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}\right].$$

Notice that since M_α is a row-stochastic matrix (with first eigenvector $\mathbf{1}$), the matrix J_α is the projector to its top eigenspace. Since both M_α and J_α are both self-adjoint with respect to the inner product Π_0^α (see Section 2.3), it follows that

$$\mathsf{M}_\alpha - \mathsf{J}_\alpha \preceq_{\Pi_0^\alpha} \lambda_2(\mathsf{M}_\alpha)\cdot\mathsf{I}.$$

Moreover, since the matrix $\mathsf{M}_\alpha - \mathsf{J}_\alpha$ is only supported on the subspace perpendicular to $\mathbf{1}$, writing $\boldsymbol{f}_\alpha^{\perp\mathbf{1}}$ for the component of $\boldsymbol{f}_\alpha$ that is perpendicular to $\mathbf{1}$, we have

$$\langle \boldsymbol{f}_\alpha, (\mathsf{M}_\alpha - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha} = \langle \boldsymbol{f}_\alpha^{\perp\mathbf{1}}, (\mathsf{M}_\alpha - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha^{\perp\mathbf{1}}\rangle_{\Pi_0^\alpha}.$$

As J_α is the projector to constant functions we have, $\boldsymbol{f}_\alpha^{\perp\mathbf{1}} = (\mathsf{I} - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha$ and thus

$$\langle \boldsymbol{f}_\alpha, (\mathsf{M}_\alpha - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\rangle \;\le\; \lambda_1(\mathsf{M}_\alpha - \mathsf{J}_\alpha)\cdot\|\boldsymbol{f}_\alpha^{\perp\mathbf{1}}\|^2_{\Pi_0^\alpha} \;\le\; \lambda_2(\mathsf{M}_\alpha)\cdot\|(\mathsf{I} - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\|^2_{\Pi_0^\alpha}, \tag{3.2}$$

where the first inequality is by the variational formula. Therefore,

$$\begin{aligned}\langle \boldsymbol{f}, (\mathsf{P}_j^{\wedge} - \mathsf{P}_j^{\triangledown})\boldsymbol{f}\rangle_{\Pi_j} &= \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\left[\langle \boldsymbol{f}_\alpha, (\mathsf{M}_\alpha - \mathsf{J}_\alpha)\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}\right], && \text{(by Items (2) and (3) in Lemma 3.7)}\\ &\le \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\left[\lambda_2(\mathsf{M}_\alpha)\cdot\|(\mathsf{I}-\mathsf{J}_\alpha)\boldsymbol{f}_\alpha\|^2_{\Pi_0^\alpha}\right], && \text{(by Eq. (3.2))}\\ &\le \gamma_{j-1}\cdot \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\left[\|(\mathsf{I}-\mathsf{J}_\alpha)\boldsymbol{f}_\alpha\|^2_{\Pi_0^\alpha}\right],\\ &= \gamma_{j-1}\cdot \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\left[\langle \boldsymbol{f}_\alpha, (\mathsf{I}-\mathsf{J}_\alpha)\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}\right], && \text{(by } \langle \mathsf{J}_\alpha\boldsymbol{f}_\alpha, \mathsf{J}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha} = \langle \boldsymbol{f}_\alpha, \mathsf{J}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha})\\ &= \gamma_{j-1}\cdot\langle \boldsymbol{f}, (\mathsf{I}-\mathsf{P}_j^{\triangledown})\boldsymbol{f}\rangle_{\Pi_j} && \text{(by Items (1) and (2) in Lemma 3.7).}\end{aligned}$$

This proves $\mathsf{P}_j^{\wedge} - \mathsf{P}_j^{\triangledown} \preceq_{\Pi_j} \gamma_{j-1}(\mathsf{I}-\mathsf{P}_j^{\triangledown})$. □

3.1.2 Proof of Lemma 3.7

Here we provide a proof of Section 3.1.2 for completeness. These arguments are from [KO18, DDFH18].

Proof. Item (1) can be proven from the identity

$$\Pi_j(\beta) = \sum_{\substack{\alpha\in X(j-1), x\in X(0),\\ \alpha\cup x=\beta}} \frac{\Pi_j(\alpha\cup\{x\})}{k+1} = \sum_{\substack{\alpha\in X(j-1), x\in X(0),\\ \alpha\cup x=\beta}} \Pi_{j-1}(\alpha)\cdot\Pi_0^\alpha(x),$$

where the last equality is by Eq. (2.2) that $\Pi_0^\alpha(x) = \frac{\Pi_j(\alpha\cup\{x\})}{(j+1)\cdot\Pi_{j-1}(\alpha)}$. Then,

$$\begin{aligned}\langle \boldsymbol{f}, \mathsf{I}\boldsymbol{f}\rangle_{\Pi_j} &= \sum_{\beta\in X(j)} \Pi_j(\beta)\cdot\boldsymbol{f}(\beta)^2\\ &= \sum_{\beta\in X(j)} \sum_{\substack{\alpha\in X(j-1), x\in X(0),\\ \alpha\cup x=\beta}} \Pi_{j-1}(\alpha)\cdot\Pi_0^\alpha(x)\cdot\boldsymbol{f}_\alpha(x)^2\\ &= \sum_{\alpha\in X(j-1)} \Pi_{j-1}(\alpha)\cdot \sum_{x\in X_\alpha(0)} \Pi_0^\alpha(x)\cdot\boldsymbol{f}_\alpha(x)^2\\ &= \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}} \langle \boldsymbol{f}_\alpha, \boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha}.\end{aligned}$$

Item (2) follows by appealing to the definition of the down-up walk that $\mathsf{P}_j^{\triangledown} = \mathsf{U}_{j-1}\mathsf{D}_j$, and so

$$\langle \boldsymbol{f}, \mathsf{P}_j^{\triangledown}\boldsymbol{f}\rangle_{\Pi_j} = \langle \boldsymbol{f}, \mathsf{U}_{j-1}\mathsf{D}_j\boldsymbol{f}\rangle_{\Pi_j} = \langle \mathsf{D}_j\boldsymbol{f}, \mathsf{D}_j\boldsymbol{f}\rangle_{\Pi_{j-1}}.$$

By the definition of the down operator and $\Pi_0^\alpha(x) = \frac{\Pi_j(\alpha\cup\{x\})}{(j+1)\cdot\Pi_{j-1}(\alpha)}$ from Eq. (2.2), it follows that $[\mathsf{D}_j\boldsymbol{f}](\alpha) = \sum_{x\in X_\alpha(0)} \Pi_0^\alpha(x)\cdot\boldsymbol{f}(\alpha\cup\{x\}) = \mathbb{E}_{x\sim\Pi_0^\alpha}\boldsymbol{f}_\alpha(x)$ and thus

$$\langle \boldsymbol{f}, \mathsf{P}_j^{\triangledown}\boldsymbol{f}\rangle_{\Pi_j} = \sum_{\alpha\in X(j-1)} \Pi_{j-1}(\alpha)\left(\mathop{\mathbb{E}}_{x\sim\Pi_0^\alpha}\boldsymbol{f}_\alpha(x)\right)^2 = \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\left[\left(\mathop{\mathbb{E}}_{x\sim\Pi_0^\alpha}\boldsymbol{f}_\alpha(x)\right)^2\right].$$

Observing that $\mathsf{J}_\alpha\boldsymbol{f}_\alpha = \mathbf{1}\cdot\mathbb{E}_{x\sim\Pi_0^\alpha}\boldsymbol{f}_\alpha(x)$ by the definition of the projector to constant functions and therefore $\|\mathsf{J}_\alpha\boldsymbol{f}_\alpha\|^2_{\Pi_0^\alpha} = \left(\mathbb{E}_{x\sim\Pi_0^\alpha}\boldsymbol{f}_\alpha(x)\right)^2$. Hence, Item (2) follows as

$$\langle \boldsymbol{f}, \mathsf{P}_j^{\triangledown}\boldsymbol{f}\rangle_{\Pi_j} = \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\|\mathsf{J}_\alpha\boldsymbol{f}_\alpha\|^2_{\Pi_0^\alpha} = \mathop{\mathbb{E}}_{\alpha\sim\Pi_{j-1}}\langle \boldsymbol{f}_\alpha, \mathsf{J}_\alpha\boldsymbol{f}_\alpha\rangle_{\Pi_0^\alpha},$$

where we used that J_α is an orthogonal projection and so $\langle \mathsf{J}_\alpha \boldsymbol{f}_\alpha, \mathsf{J}_\alpha \boldsymbol{f}_\alpha \rangle_{\Pi_0^\alpha} = \langle \boldsymbol{f}_\alpha, \mathsf{J}_\alpha \boldsymbol{f}_\alpha \rangle_{\Pi_0^\alpha}$.

For Item (3), by the definition of $\mathsf{P}_j^{\triangle} = \mathsf{D}_{j+1}\mathsf{U}_j$ and the definition of up operator,

$$\langle \boldsymbol{f}, \mathsf{P}_j^{\triangle} \boldsymbol{f} \rangle_{\Pi_j} = \langle \mathsf{U}_j \boldsymbol{f}, \mathsf{U}_j \boldsymbol{f} \rangle_{\Pi_{j+1}} = \sum_{\beta \in X(j+1)} \Pi_{j+1}(\beta) \cdot \sum_{x,y \in \beta} \frac{1}{|\beta|^2} \boldsymbol{f}(\beta \backslash x) \boldsymbol{f}(\beta \backslash y).$$

Now, by the definition of non-lazy up-down walk, we see that

$$\begin{aligned} \langle \boldsymbol{f}, \mathsf{P}_j^{\wedge} \boldsymbol{f} \rangle_{\Pi_j} &= \frac{j+2}{j+1} \cdot \langle \boldsymbol{f}, \mathsf{P}_j^{\triangle} \boldsymbol{f} \rangle_{\Pi_j} - \frac{1}{j+1} \langle \boldsymbol{f}, \boldsymbol{f} \rangle_{\Pi_j}, \\ &= \sum_{\beta \in X(j+1)} \frac{\Pi_{j+1}(\beta)}{|\beta| \cdot (|\beta| - 1)} \sum_{x,y \in \beta} \boldsymbol{f}(\beta \backslash x) \boldsymbol{f}(\beta \backslash y) - \frac{1}{j+1} \sum_{\alpha \in X(j)} \Pi_j(\alpha) \boldsymbol{f}(\alpha) \boldsymbol{f}(\alpha), \end{aligned}$$

where we got the second inequality using $|\beta| = j+2$. Now, notice that sampling $\alpha \sim \Pi_j$ is the same as first sampling $\beta \sim \Pi_{j+1}$ and then sampling $x \sim \beta$ uniformly and considering $\beta \backslash x$, so we can get by Eq. (2.1) that

$$\begin{aligned} \langle \boldsymbol{f}, \mathsf{P}_j^{\wedge} \boldsymbol{f} \rangle_{\Pi_j} &= \sum_{\beta \in X(j+1)} \sum_{x,y \in \beta} \frac{\Pi_{j+1}(\beta)}{|\beta| \cdot (|\beta| - 1)} \boldsymbol{f}(\beta \backslash x) \boldsymbol{f}(\beta \backslash y) - \frac{1}{(j+1)} \sum_{\beta \in X(j+1)} \Pi_{j+1}(\beta) \cdot \sum_{x \in \beta} \frac{\boldsymbol{f}(\beta \backslash x) \boldsymbol{f}(\beta \backslash x)}{j+2}, \\ &= \sum_{\beta \in X(j+1)} \Pi_{j+1}(\beta) \sum_{\{x,y\} \in \beta} \frac{1}{\binom{|\beta|}{2}} \boldsymbol{f}(\beta \backslash x) \boldsymbol{f}(\beta \backslash y) \end{aligned}$$

where we have obtained the last inequality by using $|\beta| = j+2$ and noticing that the sum kills the diagonal terms. Using $\tau = \beta \backslash \{x, y\}$ and the identity $\frac{\Pi_{j+1}(\beta)}{\binom{|\beta|}{2}} = \Pi_{j-1}(\tau) \cdot \Pi_1^\tau(\{x, y\})$ from Eq. (2.2), we can rewrite it as

$$\begin{aligned} \langle \boldsymbol{f}, \mathsf{P}_j^{\wedge} \boldsymbol{f} \rangle_{\Pi_j} &= \sum_{\beta \in X(j+1)} \sum_{\{x,y\} \in \beta} \Pi_1^\tau(\{x, y\}) \cdot \Pi_{j-1}(\tau) \cdot \boldsymbol{f}(\tau \cup x) \boldsymbol{f}(\tau \cup y) \\ &= \sum_{\tau \in X(j-1)} \Pi_{j-1}(\tau) \sum_{\{x,y\} \in X_\tau(1)} \Pi_1^\tau(\{x, y\}) \boldsymbol{f}(\tau \cup x) \boldsymbol{f}(\tau \cup y). \end{aligned}$$

On the other hand, using the equation

$$\langle \boldsymbol{f}, \mathsf{M}_\tau \boldsymbol{f} \rangle_{\Pi_0^\tau} = \sum_{x \in X_\tau(0)} \Pi_0^\tau(x) \cdot \boldsymbol{f}(x) \cdot [\mathsf{M}_\tau \boldsymbol{f}](x) = \sum_{\{x,y\} \in X_\tau(1)} \boldsymbol{f}(x) \boldsymbol{f}(y) \cdot \Pi_1^\tau(x, y).$$

where we use $\mathsf{M}_\tau(x, y) = \frac{\Pi_1^\tau(x,y)}{2\Pi_0^\tau(x)}$ from Section 2.3, we can also write

$$\mathop{\mathbb{E}}_{\tau \sim \Pi_{j-1}} \left[\langle \boldsymbol{f}_\tau, \mathsf{M}_\tau, \boldsymbol{f}_\tau \rangle_{\Pi_0^\tau} \right] = \sum_{\tau \in X(j-1)} \Pi_{j-1}(\tau) \cdot \sum_{\{x,y\} \in X_\tau(1)} \Pi_1^\tau(\{x, y\}) \cdot \boldsymbol{f}_\tau(x) \cdot \boldsymbol{f}_\tau(y),$$

and this proves $\langle \boldsymbol{f}, \mathsf{P}_j^{\wedge} \boldsymbol{f} \rangle_{\Pi_j} = \mathbb{E}_{\tau \sim \Pi_{j-1}} \left[\langle \boldsymbol{f}_\tau, \mathsf{M}_\tau, \boldsymbol{f}_\tau \rangle_{\Pi_0^\tau} \right]$. □

3.2 Proof of Theorem 3.2

We will prove Theorem 3.2 about the entire spectrum of the higher order random walks.

Theorem 3.2. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. Then, for any $0 \le k \le d-1$ and for any $-1 \le r \le k$, the matrix $\mathsf{P}_k^{\triangle}$ has at most $|X(r)|$ eigenvalues with value strictly greater than*

$$1 - \frac{1}{k+2}\prod_{j=r}^{k-1}(1-\gamma_j).$$

Proof. We prove by induction on k. The base case is when $k = 0$, where $\mathsf{P}_0^{\triangle} = \frac{1}{2}\mathsf{M}_{\varnothing} + \frac{1}{2}\mathsf{I}$. The claim states that we have at most $|X(-1)| = 1$ eigenvalue that is strictly greater than $\frac{1}{2} + \frac{\gamma_0}{2} = \frac{1}{2} + \frac{\lambda_2(\mathsf{M}_{\varnothing})}{2}$ (which is true by the definition of $\lambda_2(\mathsf{M}_{\varnothing})$), and there are at most $|X(0)|$ eigenvalues strictly greater than $1/2$ (which is true as the $\mathsf{M}_{\varnothing}$ is of rank at most $|X(0)|$).

For the induction step, suppose that there exists some $j \ge 1$ such that the claim of the theorem is true for all $-1 \le r \le j$. By Fact 2.1, $\mathsf{P}_{j+1}^{\triangledown}$ and $\mathsf{P}_j^{\triangle}$ have the same non-zero eigenvalues. By Lemma 3.6, $\mathsf{P}_{j+1}^{\wedge} \preceq_{\Pi_{j+1}} \gamma_j \mathsf{I} + (1-\gamma_j)\mathsf{P}_j^{\triangledown}$, and thus for $-1 \le r \le j$ the matrix $\mathsf{P}_{j+1}^{\wedge}$ has at most $|X(r)|$ eigenvalues with value greater than

$$\gamma_j + (1-\gamma_j)\cdot\left(1 - \frac{1}{j+2}\prod_{i=r}^{j-1}(1-\gamma_i)\right) = 1 - \frac{1}{j+2}\prod_{i=r}^{j}(1-\gamma_i).$$

Using the definition of the non-lazy up-down walk, we have that for $-1 \le r \le j$, $\mathsf{P}_{j+1}^{\triangle}$ has at most $|X(r)|$ eigenvalues with value greater than

$$\frac{j+2}{j+3}\left(1 - \frac{1}{j+2}\prod_{i=r}^{j}(1-\gamma_i)\right) + \frac{1}{j+3} = 1 - \frac{1}{j+3}\prod_{i=r}^{j}(1-\gamma_i).$$

For $r = j+1$, it is trivial that there at most $|X(j+1)|$ eigenvalues greater than $\frac{j+1}{j+2}$ since $\mathsf{P}_{j+1}^{\triangle}$ is an operator of rank at most $|X(j+1)|$. □

3.3 Proof of Corollary 3.5

Corollary 3.5. *Let (X, Π) be a pure d-dimensional weighted simplicial complex. Let $0 \le a < b \le d-1$. If X is a γ-local-spectral expander, then*

$$\lambda_2(\mathsf{P}_{a,b}^{\triangle}) \le (1+\gamma)^{b-a}\cdot\frac{a+1}{b+1}.$$

We will use two basic facts in the proof.

Fact 3.8. *Let $\mathsf{M}_1 \in \mathbb{R}^{V\times U}$ and $\mathsf{M}_2 \in \mathbb{R}^{U\times W}$ be two row-stochastic matrices. Then, we have $\sigma_2(\mathsf{M}_1\cdot\mathsf{M}_2) \le \sigma_2(\mathsf{M}_1)\cdot\sigma_2(\mathsf{M}_2)$.*

Fact 3.9 (Bernoulli's Inequality)**.** *Let $x \ge -1$ and $r \ge 1$ be real numbers. Then, $(1+x)^r \ge 1 + r\cdot x$.*

Proof. Recall that $\mathsf{P}_{a,b}^{\triangle} = \mathsf{D}_{a+1}\cdots\mathsf{D}_b\cdot\mathsf{U}_{b-1}\cdots\mathsf{U}_a$. As $\mathsf{P}_{a,b}^{\triangle}$ is self-adjoint, $\sigma_2(\mathsf{P}_{a,b}^{\triangle}) = \lambda_2(\mathsf{P}_{a,b}^{\triangle})$. By Fact 3.8,

$$\lambda_2(\mathsf{P}_{a,b}^{\triangle}) = \sigma_2(\mathsf{P}_{a,b}^{\triangle}) \le \sigma_2(\mathsf{D}_{a+1})\cdots\sigma_2(\mathsf{D}_b)\cdot\sigma_2(\mathsf{U}_{b-1})\cdots\sigma_2(\mathsf{U}_a).$$

Notice that as $\mathsf{P}_j^{\triangle} = \mathsf{D}_{j+1}\mathsf{U}_j$ and $\mathsf{D}_{j+1}^* = \mathsf{U}_j$, we have $\lambda_2(\mathsf{P}_j^{\triangle}) = \sigma_2(\mathsf{U}_j) \cdot \sigma_2(\mathsf{D}_{j+1})$. Thus, by rearranging we obtain,

$$\begin{aligned}
\lambda_2(\mathsf{P}_{a,b}^{\triangle}) &\leq \prod_{j=0}^{b-a-1} \lambda_2(\mathsf{P}_{a+j}^{\triangle}), \\
&\leq \prod_{j=0}^{b-a-1} \left(1 - \frac{(1-\gamma)^{a+j+1}}{a+j+2}\right), && \text{(by Theorem 3.1)} \\
&\leq \prod_{j=0}^{b-a-1} \left(1 - \frac{1-(a+j+1)\cdot\gamma}{a+j+2}\right), && \text{(by Fact 3.9)} \\
&= \prod_{j=0}^{b-a-1} \left((1+\gamma)\frac{a+j+1}{a+j+2}\right).
\end{aligned}$$

By cancellations in the telescoping product, we have

$$\lambda_2(\mathsf{P}_{b,a}^{\triangle}) \leq (1+\gamma)^{b-a} \cdot \frac{a+1}{b+1}.$$

□

3.4 Proof of Proposition 3.3

We first recall some basic definitions and results from spectral graph theory. Let $\mathsf{M} \in \mathbb{R}^{V \times V}$ be a reversible Markov chain with stationary distribution Π. We will write $(X_t)_{t\geq 0}$ for the random variable describing the state of this Markov chain. The conductance $\Phi(S)$ of a set $S \subset V$ is the probability that a random step of the Markov chain leaving the set S conditioned on having started from a random state in S, i.e.

$$\Phi(S) = \Pr[X_1 \notin S \mid X_0 \in S] = \sum_{x \in S} \frac{\Pi(x)}{\Pi(S)} \Pr[X_1 \notin S \mid X_0 = x].$$

We recall that the conductance of the chain M is defined to be,

$$\Phi(\mathsf{M}) = \min_{\substack{S \subset V, \\ \Pi(S) \leq 1/2}} \Phi(S).$$

The Alon-Milman-Cheeger inequality tells that the parameter $\Phi(M)$ is closely related to the parameter $\lambda_2(\mathsf{M})$.

Theorem 3.10 ([AM85, Che70])**.** *Let* M *be a row-stochastic matrix describing a reversible Markov chain. Then,*

$$\frac{1-\lambda_2(\mathsf{M})}{2} \leq \Phi(\mathsf{M}) \leq \sqrt{\frac{1-\lambda_2(\mathsf{M})}{2}}.$$

Proof of Proposition 3.3. It is clear that there should exist a vertex $v \in X(0)$ such that $\Pi_0(v) \leq \frac{1}{|X(0)|} = \frac{1}{n}$.

We consider the set $A_v \subset X(d)$ consisting of all faces in $X(d)$ containing the vertex v, i.e. $A_v = \{\beta \in X(d) : v \in \beta\}$. Note that,

$$\begin{aligned} \Pi_d(A_v) &= (d+1) \cdot \Pi_0(v), && \text{(by using Eq. (2.1) repeatedly)}, \\ &\leq \frac{d+1}{n}, && \text{(by using } \Pi_0(v) \leq \tfrac{1}{n}), \\ &\leq \frac{1}{2}. && \text{(by using } 2(d+1) \leq n) \end{aligned}$$

By Theorem 3.10,

$$\frac{1-\lambda_2(\mathsf{P}_d^{\nabla})}{2} \leq \min_{S:\Pi_d(S)\leq 1/2} \Phi(S) \leq \Phi(A_v).$$

We recall that the random-walk P_d^{∇} starting from a face $\beta \in X(d)$ first picks an index $i \sim \beta$ uniformly at random, and then picks some face $\beta' \supset (\beta \backslash i)$ with probability proportional to $\Pi_d(\beta')$. If $\beta \in A_v$ the only way we leave A_v in a single step is when the index i we pick from β is v, which happens with probability $1/|\beta| = 1/(d+1)$.

Writing $(X_t)_{t\geq 0}$ for the state of the random walk , this means for any $\beta \in A_v$ we have

$$\Pr[X_1 \notin A_v \mid X_0 = \beta] \leq \frac{1}{d+1}.$$

It follows that

$$\frac{1-\lambda_2(\mathsf{P}_d^{\nabla})}{2} \leq \Phi(A_v) = \sum_{\beta\in A_v} \frac{\Pi_d(\beta)}{\Pi_d(A_v)} \cdot \Pr[X_1 \notin A_v \mid X_0 = \beta] \leq \max_{\beta\in A_v} \Pr[X_1 \notin A_v \mid X_0 = \beta] \leq \frac{1}{d+1}.$$

Solving the expression for $\lambda_2(\mathsf{P}_d^{\nabla})$ proves the proposition. □

4 Analyzing Mixing Times of Markov Chains

In this section, we will use Corollary 3.4 to analyze Markov chains for sampling independent sets of a graph of fixed size and sampling common independent sets of two partition matroids.

4.1 Independent Sets

Let $G = (V, E)$ be a graph. A subset of vertices $S \subset V$ is called an independent set if $uv \notin E$ for every pair $u, v \in S$. We are interested in the problem of sampling a uniformly random independent set of size k. We will analyze a natural Markov chain for the problem by analyzing the down-up walk of a corresponding simplicial complex.

Define the $(k-1)$-dimensional simplicial complex $I_{G,k}$ of $G = (V, E)$ as

$$I_{G,k} = \{S \subset V : |S| \leq k \text{ and } S \text{ is independent}\},$$

the complex consisting of all independent sets in G of cardinality at most k. We endow $I_{G,k}$ with the uniform distribution Π_{k-1} on $I_{G,k}(k-1)$, i.e. the set of independent sets of size k. We simply write $I_{G,k}$ for the weighted simplicial complex $(I_{G,k}, \Pi_{k-1})$.

The $(k-1)$-th down-up walk $\mathsf{P}_{k-1}^{\triangledown}$ on $I_{G,k}$ corresponds to a natural Markov chain to sample independent sets of size k. It is known that this Markov chain is fast mixing when $k \le \frac{|V|}{2\Delta+1}$ using coupling techniques [BD97, MU05]. The main result in this subsection is the following improved bound using higher order random walks on simplicial complexes.

Theorem 1.7. *Let $G = (V, E)$ be a graph with maximum degree Δ. Let $\mathsf{P}_{k-1}^{\triangledown}$ be the $(k-1)$-th down-up walk on the simplicial complex $I_{G,k}$. Let A_G be the adjacency matrix of G.*

$$\text{If} \quad k \le \frac{|V|}{\Delta + |\lambda_{\min}(\mathsf{A}_G)|}, \quad \text{then} \quad \lambda_2(\mathsf{P}_{k-1}^{\triangledown}) \le 1 - \frac{1}{k^2}.$$

It is well-known that $|\lambda_{\min}(\mathsf{A}_G)| \le \Delta$ for a graph with maximum degree Δ, and so Theorem 1.7 recovers the previous result that the Markov chain is fast mixing if $k \le \frac{|V|}{2\Delta}$. There are various graph classes with $|\lambda_{\min}(\mathsf{A}_G)|$ smaller than Δ, and Theorem 1.7 allows us to sample larger independent sets. For example, it is known that $|\lambda_{\min}(\mathsf{A}_G)| \le O(\sqrt{\Delta})$ for planar graphs and more generally for graphs with bounded arboricity [Hay06], and also for random graphs and more generally for two-sided expander graphs [HLW06].

Using the simple bound $\min_{S \in I_{G,k}(k-1)} \Pi_{k-1}(S) \ge n^{-k}$ as Π_{k-1} is the uniform distribution, the following mixing time result follows from Theorem 2.8.

Corollary 4.1. *Let $G = (V, E)$ be a graph with maximum degree Δ and let A_G be the adjacency matrix of G. For any $k \le n/(\Delta + |\lambda_{\min}(\mathsf{A}_G)|)$, the down-up walk $\mathsf{P}_{k-1}^{\triangledown}$ on the simplicial complex $I_{G,k}$ samples a random independent set of G of size k whose distribution is ε-close to the uniform distribution on all independent sets of size k in*

$$T(\varepsilon, \mathsf{P}_{k-1}^{\triangledown}) \le k^2 \cdot \left(\log\left(\frac{1}{\varepsilon}\right) + k \cdot \log n \right)$$

many time steps.

This implies a polynomial time algorithm to approximately sample a uniform random independent set and also a FPRAS for approximately counting the number of independent set of size k for $k \le \frac{n}{\Delta + |\lambda_{\min}(\mathsf{A}_G)|}$.

4.1.1 Proof of Theorem 1.7

The plan is to use Corollary 3.4 to prove Theorem 1.7. To apply Corollary 3.4, we need to prove that:

1. $I_{G,k}$ is a pure simplicial complex. It is a simple exercise that this complex is pure when $k \le \frac{n}{\Delta+1}$.
2. For each $S \in I_{G,k}$ with $|S| \le k-2$, the random walk matrix M_S of the graph G_S of the link $(I_{G,k})_S$ satisfies $\lambda_2(\mathsf{M}_S) < 1$. This is proved in Lemma 4.2.
3. For each $S \in I_{G,k}$ with $|S| = k-2$, the random walk matrix M_S of the graph G_S satisfies $\lambda_2(\mathsf{M}_S) \le 1/k$. This is proved in Lemma 4.3.

Assuming the three items are proven, Theorem 1.7 follows immediately from Corollary 3.4. We will prove the second item in Section 4.1.2 and the third item in Section 4.1.3.

4.1.2 Proof of Lemma 4.2

Let $H_S = (V_S, E_S)$ be the underlying support graph of G_S of the link $(I_{G,k})_S$, i.e. G_S without edge weights. Let M_S be the random walk matrix of G_S as defined in Section 2.3. Note that $\lambda_2(\mathsf{M}_S) < 1$ if and only if H_S is connected.

We introduce some notation to describe H_S. We write $N_G[S]$ as the union of S and the set of vertices which are connected to a vertex in S in G, i.e.

$$N_G[S] = S \cup \{v : \text{there exists some } uv \in E(G) \text{ such that } u \in S\}.$$

For a subset of vertices $S \subset V(G)$, we write $\overline{S} = V(G) \setminus S$ for the complement of S in G, and $G[S]$ for the induced subgraph of G on S. For a graph H, we write $\overline{H}$ for the complement graph of H.

Recall that a vertex v is in V_S if and only if $S \cup \{v\}$ is an independent set in G of size $|S| + 1$, and so V_S is exactly $V - N_G[S] = \overline{N_G[S]}$. Two vertices $u, v \in V_S$ have an edge in H_S if and only if $S \cup \{u, v\}$ is an independent set in G of size $|S| + 2$, and so $uv \in E_S$ if and only if $uv \notin E(G)$. Therefore, we see that

$$H_S = \overline{G[V_S]} = \overline{G[\overline{N[S]}]}.$$

With the description of H_S, we are ready to prove the second item in Section 4.1.1.

Lemma 4.2. *Let $G = (V, E)$ be a graph with maximum degree Δ. Suppose $k \le \frac{|V|}{\Delta+1}$. For any $S \in I_{G,k}$ with $|S| \le k - 2$, the random walk matrix M_S of the graph G_S of the link $(I_{G,k})_S$ satisfies $\lambda_2(\mathsf{M}_S) < 1$.*

Proof. Note that $\lambda_2(\mathsf{M}_S) < 1$ if and only if the underlying support graph H_S of G_S is connected, so we focus on proving the latter. To prove that H_S is connected, we prove the stronger claim that every two vertices $u, v \in H_S$ has a path of length at most two. If uv is an edge in H_S, then there is a path of length one. Suppose uv is not an edge in H_S. Then uv is an edge in G. Since G is of maximum degree Δ, it implies that $|N_G[\{u, v\}]| \le (\deg_G(u) + 1) + (\deg_G(v) + 1) - 2 \le 2\Delta$, and also

$$|V_S| = |V| - |N_G[S]| \ge |V| - |S| \cdot (\Delta + 1) \ge 2\Delta + 2,$$

where we use the assumptions that $|S| \le k - 2 \le \frac{|V|}{\Delta+1} - 2$ in the last inequality. So, there must be some vertex w such that $w \in V_S \setminus N_G[\{u, v\}]$. This implies that $wu \notin E(G)$ and $wv \notin E(G)$, and thus $wu \in E(H_S)$ and $wv \in E(H_S)$ and so there is a path of length two connecting u and v in H_S. ☐

4.1.3 Proof of Lemma 4.3

We observe that G_S is an unweighted graph for S with $|S| = k - 2$ when the distribution on $I_{G,k}(k-1)$ is the uniform distribution. Therefore, G_S is simply a scaled version of H_S, and the random walk matrix M_S of G_S is the same as the random walk matrix of H_S. To bound the second eigenvalue, we will use some simple interlacing arguments. We need the stronger assumption that $k \le \frac{|V(G)|}{\Delta + |\lambda_{\min}(\mathsf{A}_G)|}$ in the proof of the following lemma. (Note that for any unweighted graph G, we have $|\lambda_{\min}(\mathsf{A}_G)| \ge 1$.)

Lemma 4.3. *Let $G = (V, E)$ be a graph with maximum degree Δ. Suppose $k \le |V|/(\Delta + |\lambda_{\min}(\mathsf{A}_G)|)$. For any $S \in I_{G,k}$ with $|S| = k - 2$, the random walk matrix M_S of the graph G_S of the link $(I_{G,k})_S$ satisfies $\lambda_2(\mathsf{M}_S) \le \frac{1}{k}$.*

Proof of Lemma 4.3. Recall that for S with $|S| = k-2$, the random walk matrix M_S of G_S is the same as the random walk matrix of H_S, and so we will focus on the latter. Let D_H be diagonal degree matrix of H_S. As argued above, the random walk matrix M_S of G_S is equal to $\mathsf{M}_S = \mathsf{D}_H^{-1}\mathsf{A}_H$. We can write the adjacency matrix A_H of H_S as

$$\mathsf{A}_H = \mathbf{1}\mathbf{1}^\top - \mathsf{I} - \mathsf{A}_{G[\overline{N[S]}]},$$

where $\mathsf{A}_{G[\overline{N[S]}]}$ is the adjacency matrix of $G[\overline{N[S]}]$. By Weyl's interlacing theorem,

$$\begin{aligned}
\lambda_2(\mathsf{M}_S) &\leq \lambda_2(\mathsf{D}_H^{-1/2}\mathbf{1}\mathbf{1}^\top\mathsf{D}_H^{-1/2}) + \lambda_1(\mathsf{D}_H^{-1/2}(-\mathsf{A}_{G[\overline{N[S]}]} - \mathsf{I})\mathsf{D}_H^{-1/2}), && \text{(by Weyl Interlacing Theorem 2.4)}\\
&= \lambda_1(\mathsf{D}_H^{-1/2}(-\mathsf{A}_{G[\overline{N[S]}]} - \mathsf{I})\mathsf{D}_H^{-1/2}), && \text{(since } \mathsf{D}_H^{-1/2}\mathbf{1}\mathbf{1}^\top\mathsf{D}_H^{-1/2} \text{ is of rank 1)}\\
&\leq \left\|\mathsf{D}_H^{-1}\right\| \cdot \lambda_1(-\mathsf{A}_{G[\overline{N[S]}]} - \mathsf{I}), && \text{(by the variational formula)} \qquad (4.1)\\
&\leq \left\|\mathsf{D}_H^{-1}\right\| \cdot (|\lambda_{\min}(\mathsf{A}_{G[\overline{N[S]}]})| - 1), && \text{(by using } \lambda_1(-\mathsf{A}_{G[\overline{N[S]}]}) = -\lambda_{\min}(\mathsf{A}_{G[\overline{N[S]}]}))\\
&\leq \left\|\mathsf{D}_H^{-1}\right\| \cdot (|\lambda_{\min}(\mathsf{A}_G)| - 1), && \text{(by using Cauchy-Interlacing Theorem 2.3)}
\end{aligned}$$

For Eq. (4.1), we have used the variational formula $\lambda_1(\mathsf{W}) = \max\{\langle \boldsymbol{f}, \mathsf{W}\boldsymbol{f}\rangle : \boldsymbol{f} \in \mathbb{R}^V, \|\boldsymbol{f}\| = 1\}$ in the following way: Let $\mathsf{W} = -\mathsf{A}_L - \mathsf{I}$ and $\boldsymbol{g}$ be an unit top-eigenvecor of $\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2}$, i.e. $\|\boldsymbol{g}\| = 1$ and $\langle \boldsymbol{g}, \mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2}\boldsymbol{g}\rangle = \lambda_1(\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2})$. Then,

$$\lambda_1(\mathsf{W}) \geq \left\langle \frac{\mathsf{D}_H^{-1/2}\boldsymbol{g}}{\|\mathsf{D}_H^{-1/2}\boldsymbol{g}\|}, \mathsf{W}\frac{\mathsf{D}_H^{-1/2}\boldsymbol{g}}{\|\mathsf{D}_H^{-1/2}\boldsymbol{g}\|} \right\rangle = \frac{\lambda_1(\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2})}{\|\mathsf{D}_H^{-1/2}\|^2} \geq \frac{\lambda_1(\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2})}{\|\mathsf{D}_H^{-1}\|}.$$

It remains to bound $\|\mathsf{D}_H^{-1}\| = (\min_v \deg_{H_S}(v))^{-1}$. As $H_S = \overline{G[\overline{N[S]}]} = \overline{G[V - N[S]]}$,

$$\deg_{H_S}(v) = |V| - |N[S]| - (\deg_{G[\overline{N[S]}]}(v) + 1) \geq |V| - (\Delta+1)(|S|+1),$$

where the last inequality uses that $|N[S]| \leq |S| \cdot (\Delta+1)$ and $\deg_{G[\overline{N[S]}]}(v) \leq \deg_G(v) \leq \Delta$. Therefore, using our bound $\lambda_2(\mathsf{M}_S) \leq \left\|\mathsf{D}_H^{-1}\right\| \cdot (|\lambda_{\min}(\mathsf{A}_G)| - 1)$, we obtain

$$\lambda_2(\mathsf{M}_S) \leq \frac{|\lambda_{\min}(\mathsf{A}_G)| - 1}{|V| - (\Delta+1)\cdot(|S|+1)} = \frac{|\lambda_{\min}(\mathsf{A}_G)| - 1}{|V| - (\Delta+1)\cdot(k-1)},$$

where we use $|S| = k-2$. Finally, plugging in the assumption

$$k \leq \frac{|V|}{\Delta + |\lambda_{\min}(\mathsf{A}_G)|} \quad \Longrightarrow \quad \lambda_2(\mathsf{M}_S) \leq \frac{1}{k}.$$

□

4.2 Matroid Intersection

A matroid $M = (E, \mathcal{I})$ on the ground set E with the set of independent sets $\mathcal{I} \subset 2^E$ is a combinatorial object satisfying the following properties:

- (containment property) if $S \in \mathcal{I}$ and $T \subset S$, then $T \in S$,
- (extension property) if $S, T \in \mathcal{I}$ such that $|S| > |T|$ then there is some $x \in S \backslash T$ such that $\{x\} \cup T \in \mathcal{I}$.

A partition matroid is the special case where the ground set E is partitioned into disjoint blocks $B_1, \ldots, B_l \subseteq E$ with parameters $0 \leq d_i \leq |B_i|$ for $1 \leq i \leq l$, and a subset S is in $\mathcal{I}$ if and only if $|S \cap B_i| \leq d_i$ for $1 \leq i \leq l$.

The intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I})$ over the same ground set E can be used to formulate various interesting combinatorial optimization problems [Sch03]. We are interested in the problem of sampling a uniform random common independent set of size k, i.e. a random subset $F \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|F| = k$. We will analyze a natural Markov chain for the problem by analying the down-up walk of a corresponding simplicial complex.

Define the $(k-1)$-dimensional matroid intersection complex $C_{M_1,M_2,k}$ of $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ as

$$C_{M_1,M_2,k} = \{S \in \mathcal{I}_1 \cap \mathcal{I}_2 : |S| \leq k\},$$

the complex consisting of all common independent sets of both matroids containing at most k elements. We endow $C_{M_1,M_2,k}(k-1)$ with the uniform distribution Π_{k-1} on the common independent sets $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|S| = k$. We write $C_{M_1,M_2,k}$ for the weighted simplicial complex $(C_{M_1,M_2,k}, \Pi_{k-1})$.

The $(k-1)$-th down-up walk $\mathsf{P}_{k-1}^{\bigtriangledown}$ on $C_{M_1,M_2,k}$ corresponds to the following natural Markov chain to sample common independent sets of size k. Initially, the random walk starts from a common independent set S_1 of size k. In each step $t \geq 1$, we choose a uniform random element $i \in S_t$ and delete i from S_t, and set S_{t+1} to be a uniform random common independent set of size k that contains $S_t \setminus \{i\}$. The stationary distribution of $\mathsf{P}_{k-1}^{\bigtriangledown}$ is the uniform distribution Π_{k-1}; see Eq. (2.4).

The main result in this subsection is the following upper bound on the second eigenvalue of $\mathsf{P}_{k-1}^{\bigtriangledown}$.

Theorem 1.8. *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two given partition matroids with a common independent set of size r and no two elements belonging to the same block in both matroids. If $k \leq r/3$, then*

$$\lambda_2(\mathsf{P}_{k-1}^{\bigtriangledown}) \leq 1 - \frac{1}{k^2},$$

where $\mathsf{P}_{k-1}^{\bigtriangledown}$ is the $(k-1)$-th down-up walk on the matroid intersection complex $C_{M_1,M_2,k}$.

4.2.1 Proof of Theorem 1.8

The plan is to use Corollary 3.4 to prove Theorem 1.8. To apply Corollary 3.4, we need to prove that:

1. $C_{M_1,M_2,k}$ is a pure simplicial complex. This is a simple proof in Claim 4.4.
2. For each $S \in C_{M_1,M_2,k}$ with $|S| \leq k-2$, the random walk matrix M_S of the graph G_S of the link $(C_{M_1,M_2,k})_S$ satisfies $\lambda_2(\mathsf{M}_S) < 1$. This is proved in Lemma 4.5, showing that the underlying graph of G_S is the complement of the line graph of a bipartite graph.
3. For each $S \in C_{M_1,M_2,k}$ with $|S| = k-2$, the random walk matrix M_S of the graph G_S satisfies $\lambda_2(\mathsf{M}_S) \leq 1/k$. This is proved in Lemma 4.8, using the fact that the minimum eigenvalue of the adjacency matrix of the line graph of a simple graph is at least -2.

Assuming the three items are proven, Theorem 1.8 follows immediately from Corollary 3.4.

It remains to prove the three items. We will prove the second item in Section 4.2.2 and the third item in Section 4.2.3. We note that the first two items hold for any two matroids, and we only use the additional assumptions for the third item. The following is a simple proof for the first item.

Claim 4.4. *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two matroids with a common independent set $T \in \mathcal{I}_1 \cap \mathcal{I}_2$ of size $|T| = r$. Any common independent set $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ with $|S| < r/2$ is contained in a larger common independent set. In particular, this implies that the simplicial complex $C_{M_1,M_2,k}$ is a pure simplicial complex as long as $k \le r/2$.*

Proof. By the extension property of matroids, there is a subset $T_1 \subset T$ with $|T_1| \ge r - |S|$ such that $S \cup \{x\}$ is an independent set in $\mathcal{I}_1$ for any $x \in T_1$. Similarly, there is a subset $T_2 \subset T$ with $|T_2| \ge r - |S|$ such that $S \cup \{y\}$ is an independent set in $\mathcal{I}_2$ for any $y \in T_2$. As $|S| < r/2$, this implies that $T_1 \cap T_2 \ne \emptyset$, and $S \cup \{z\}$ is a larger independent set that contains S for any $z \in T_1 \cap T_2$. ☐

4.2.2 Proof of Lemma 4.5

Let $H_S = (E_S, F_S)$ be the underlying support graph of G_S of the link $(C_{M_1,M_2,k})_S$, that is, H_S is G_S without edge weights. The vertex set of H_S is $E_S = \{x \in E \mid S \cup \{x\} \in \mathcal{I}_1 \cap \mathcal{I}_2\}$ and the edge set of H_S is $F_S = \{\{x, y\} \mid x, y \in E \text{ and } S \cup \{x, y\} \in \mathcal{I}_1 \cap \mathcal{I}_2\}$. Let M_S be the random walk matrix of G_S as defined in Section 2.3. It is a basic fact in spectral graph theory that $\lambda_2(\mathsf{M}_S) < 1$ if and only if H_S is connected.

We will see that H_S is the complement of the line graph of a bipartite graph B. To define the bipartite graph B, we first introduce the matroid partition property (see e.g. [ALOV19]). The matroid partition property says that there is a partition $\mathcal{P} := \{P_1, \dots, P_p\}$ of the vertex set E_S (i.e. $\bigcup_{i=1}^p P_i = E_S$ and $P_i \cap P_j = \emptyset$ for $i \ne j$) with the property that for any $x, y \in E_S$,

$$S \cup \{x, y\} \notin \mathcal{I}_1 \quad \Longleftrightarrow \quad x, y \in P_i \text{ for some } 1 \le i \le p.$$

In words, there is a partition $\mathcal{P}$ of the vertex set E_S such that two elements x, y in E_S can be added to S to form an independent set in the first matroid M_1 if and only if x, y do not belong to the same class of the partition $\mathcal{P}$. Similarly, there is a partition $\mathcal{Q} := \{Q_1, \dots, Q_q\}$ of the vertex set E_S such that for any two elements $x, y \in E_S$, we have $S \cup \{x, y\} \notin \mathcal{I}_2$ if and only if $x, y \in Q_i$ for some $1 \le i \le q$.

We use the partitions $\mathcal{P}$ and $\mathcal{Q}$ to define the bipartite graph B as follows. The vertex set of B is $P \sqcup Q$, where we create a vertex $i \in P$ in B for each P_i in $\mathcal{P}$, and we create a vertex $j \in Q$ in B for each Q_j in $\mathcal{Q}$. Each edge in B corresponds to an element in E_S. For each element $x \in E_S$, we create the edge $e_x = ij$ in B if and only if $x \in P_i$ and $x \in Q_j$. Note that the edge e_x for $x \in E_S$ is well-defined by the matroid partition property. By construction, it should be clear that the biparite graph B satisfies the following important property:

$$e_x \text{ and } e_y \text{ do not share a vertex in } B \quad \Longleftrightarrow \quad S \cup \{x, y\} \in \mathcal{I}_1 \cap \mathcal{I}_2 \quad \Longleftrightarrow \quad \{x, y\} \in F_S. \tag{4.2}$$

Recall that the line graph $L(B)$ of a graph B is defined as follows: the vertex set of $L(B)$ is the edge set of B, and two vertices in $L(B)$ have an edge if and only if the corresponding edges in B share an endpoint. Let $\overline{L(B)}$ be the complement of $L(B)$ where $\overline{L(B)}$ and $L(B)$ have the same vertex set and two vertices in $\overline{L(B)}$ have an edge if and only if the corresponding vertices in $L(B)$ do not have an edge. Then, we see from Eq. (4.2) that

$$H_S = \overline{L(B)}. \tag{4.3}$$

Using the bipartite graph B, it is easy to show the second item in Section 4.2.1.

Lemma 4.5. *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two matroids with a common independent set $T \in \mathcal{I}_1 \cap \mathcal{I}_2$ of size $|T| = r$. Suppose $k < r/2 - 1$. For any $S \in C_{M_1,M_2,k}$ with $|S| \le k - 2$, the random walk matrix M_S of the graph G_S of the link $(C_{M_1,M_2,k})_S$ satisfies $\lambda_2(\mathsf{M}_S) < 1$.*

Proof. It is well known that $\lambda_2(\mathsf{M}_S) < 1$ if and only if the underlying support graph H_S of G_S is connected, so we focus on proving the latter. Since $|S| \leq k-2 < r/2-3$, it follows from Claim 4.4 that there are four elements $a,b,c,d \in E$ such that $S \cup \{a,b,c,d\} \in \mathcal{I}_1 \cap \mathcal{I}_2$. In the bipartite graph B in Eq. (4.3), the four elements a,b,c,d correspond to four vertex-disjoint edges e_a, e_b, e_c, e_d in B by Eq. (4.2). To prove that H_S is connected, we prove the stronger claim that every two vertices $u,v \in H_S$ has a path of length at most two. If uv is an edge in H_S, then there is a path of length one. Suppose uv is not an edge in H_S. Then e_u and e_v shares a vertex in B and so they span at most three vertices in B. This implies that $e_u \cup e_v$ cannot intersect all four (vertex-disjoint) edges e_a, e_b, e_c, e_d. So there must be an edge, say e_a, which is vertex-disjoint from both e_u and e_v. Then u-a-v is path of length two in H_S by Eq. (4.2), which completes the proof. □

4.2.3 Proof of Lemma 4.8

For the third term, we need to prove that for each $S \in C_{M_1,M_2,k}$ with $|S| = k-2$, the random walk matrix M_S of the graph G_S satisfies $\lambda_2(\mathsf{M}_S) \leq \frac{1}{k}$. We use the additional assumptions for the following property.

Claim 4.6. *If M_1 and M_2 are two partition matroids and there are no two elements x,y such that x,y belongs to the same block in M_1 and also the same block in M_2, then Eq. (4.3) holds with the property that the bipartite graph B is a simple graph.*

Observe that G_S is an unweighted graph for S with $|S| = k-2$ when the distribution on $C_{M_1,M_2,k}(k-1)$ is the uniform distribution (i.e. the distribution on the common independent sets of size k is the uniform distribution). This is because when $|S| = k-2$, for any $x,y \in E$, either $S \cup \{x,y\}$ is contained in exactly one or zero set of size k in $C_{M_1,M_2,k}$, and each set of size k is assigned the same weight in the uniform distribution (more formally see Eq. (2.2) for the definition of the weight). Therefore, G_S is simply a scaled version of H_S, and the random walk matrix M_S of G_S is the same as the random walk matrix of H_S.

Fact 4.7. *Let $G = (V,E)$ be any simple graph and $\mathsf{A}_{L(G)}$ be the adjacency matrix of the line graph of G. It holds that $\lambda_{\min}(\mathsf{A}_{L(G)}) \geq -2$.*

Proof. Define $\mathsf{B} \in \mathbb{R}^{E \times V}$ to be the edge-vertex incidence matrix of $G = (V,E)$, i.e. $\mathsf{B}(e,v) = 1[v \in e]$. Observe that

$$2\mathsf{I} + \mathsf{A}_{L(G)} = \mathsf{B}\mathsf{B}^\top \quad \Longrightarrow \quad \lambda_{\min}\left(2\mathsf{I} + \mathsf{A}_{L(G)}\right) = \lambda_{\min}\left(\mathsf{B}\mathsf{B}^\top\right) \geq 0,$$

as $\mathsf{B}\mathsf{B}^\top$ is a positive semidefinite matrix. This implies that $\lambda_{\min}(\mathsf{A}_{L(G)}) \geq -2$. □

We are ready to bound the second eigenvalue of M_S. We need the stronger assumption that $k \leq \frac{r}{3}$ in the proof of the following lemma.

Lemma 4.8. *Let $M_1 = (E,\mathcal{I}_1)$ and $M_2 = (E,\mathcal{I}_2)$ be two partition matroids with a common independent set $T \in \mathcal{I}_1 \cap \mathcal{I}_2$ of size $|T| = r$ and there are no two elements belonging to the same block in both matroids. Suppose $k \leq r/3$. For any $S \in C_{M_1,M_2,k}$ with $|S| = k-2$, the random walk matrix M_S of the graph G_S of the link $(C_{M_1,M_2,k})_S$ satisfies $\lambda_2(\mathsf{M}_S) \leq 1/k$.*

Proof. Recall that for S with $|S| = k-2$, the random walk matrix M_S of G_S is the same as the random walk matrix of H_S, and so we will focus on the latter. By Claim 4.6, H_S is the complement of the line graph of a simple graph, and so we can write the adjacency matrix A_H of H_S as

$$\mathsf{A}_H = \mathbf{1}\mathbf{1}^\top - \mathsf{I} - \mathsf{A}_L,$$

where A_L is the adjacency matrix of the line graph $L(B)$ in Eq. (4.3). Let D_H be diagonal degree matrix of H_S. As argued above, the random walk matrix M_S of G_S is equal to $\mathsf{M}_S = \mathsf{D}_H^{-1}\mathsf{A}_H$. Using that D_H^{-1} and $\mathsf{D}_H^{-1/2}\mathsf{A}_H\mathsf{D}_H^{-1/2}$ are similar matrices and have the same spectrum, we have

$$\begin{aligned}\lambda_2(\mathsf{M}_S) &= \lambda_2(\mathsf{D}_H^{-1/2}\mathsf{A}_H\mathsf{D}_H^{-1/2}),\\ &= \lambda_2(\mathsf{D}_H^{-1/2}(\mathbf{1}\mathbf{1}^\top - \mathsf{A}_L - \mathsf{I})\mathsf{D}_H^{-1/2}),\\ &= \lambda_2\Big(\mathsf{D}_H^{-1/2}\mathbf{1}\mathbf{1}^\top\mathsf{D}_H^{-1/2} + \mathsf{D}_H^{-1/2}(-\mathsf{A}_L - \mathsf{I})\mathsf{D}_H^{-1/2}\Big).\end{aligned}$$

Using the Weyl Interlacing Theorem 2.4,

$$\begin{aligned}\lambda_2(\mathsf{M}_S) &\le \lambda_2(\mathsf{D}_H^{-1/2}\mathbf{1}\mathbf{1}^\top\mathsf{D}_H^{-1/2}) + \lambda_1(\mathsf{D}_H^{-1/2}(-\mathsf{A}_L - \mathsf{I})\mathsf{D}_H^{-1/2}), && \text{(by Weyl Interlacing Theorem 2.4)}\\ &= \lambda_1(\mathsf{D}_H^{-1/2}(-\mathsf{A}_L - \mathsf{I})\mathsf{D}_H^{-1/2}), && \text{(since } \mathsf{D}_H^{-1/2}\mathbf{1}\mathbf{1}^\top\mathsf{D}_H^{-1/2} \text{ is of rank 1)}\\ &\le \left\|\mathsf{D}_H^{-1}\right\| \cdot \lambda_1(-\mathsf{A}_L - \mathsf{I}), && \text{(by the variational formula)} \quad (4.4)\\ &\le \left\|\mathsf{D}_H^{-1}\right\| \cdot (|\lambda_{\min}(\mathsf{A}_L)| - 1), && \text{(by using } \lambda_1(-\mathsf{A}_L) = -\lambda_{\min}(\mathsf{A}_L))\\ &= \left\|\mathsf{D}_H^{-1}\right\|. && \text{(by using Fact 4.7)} \quad (4.5)\end{aligned}$$

For Eq. (4.4), we have used the variational formula $\lambda_1(\mathsf{W}) = \max\big\{\langle \boldsymbol{f}, \mathsf{W}\boldsymbol{f}\rangle : \boldsymbol{f} \in \mathbb{R}^V, \|\boldsymbol{f}\| = 1\big\}$ in the following way: Let $\mathsf{W} = -\mathsf{A}_L - \mathsf{I}$ and $\boldsymbol{g}$ be an unit top-eigenvecor of $\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2}$, i.e. $\|\boldsymbol{g}\| = 1$ and $\langle \boldsymbol{g}, \mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2}\boldsymbol{g}\rangle = \lambda_1(\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2})$. Then,

$$\lambda_1(\mathsf{W}) \ge \left\langle \frac{\mathsf{D}_H^{-1/2}\boldsymbol{g}}{\|\mathsf{D}_H^{-1/2}\boldsymbol{g}\|}, \mathsf{W}\frac{\mathsf{D}_H^{-1/2}\boldsymbol{g}}{\|\mathsf{D}_H^{-1/2}\boldsymbol{g}\|} \right\rangle = \frac{\lambda_1(\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2})}{\|\mathsf{D}_H^{-1/2}g\|^2} \ge \frac{\lambda_1(\mathsf{D}_H^{-1/2}\mathsf{W}\mathsf{D}_H^{-1/2})}{\|\mathsf{D}_H^{-1}\|}.$$

It remains to bound $\|\mathsf{D}_H^{-1}\| = (\min_x \deg_{H_S}(x))^{-1}$. By the definition of H_S, the degree $\deg_{H_S}(x)$ of $x \in E$ is equal to the number of elements $y \in E \setminus (S \cup \{x\})$ such that $S \cup \{x, y\} \in \mathcal{I}_1 \cap \mathcal{I}_2$. By our assumption, there is a common independent set $T \in \mathcal{I}_1 \cap \mathcal{I}_2$ of size r. Since $|S \cup \{x\}| = k-1$, by the extension property of the first matroid M_1, there are at least $r - k + 1$ elements $y \in T$ such that $S \cup \{x, y\} \in \mathcal{I}_1$. Similarly, there are at least $r - k + 1$ elements $y \in T$ such that $S \cup \{x, y\} \in \mathcal{I}_2$. Therefore, there are at least $r - 2k + 2$ elements $y \in T$ such that $S \cup \{x, y\} \in \mathcal{I}_1 \cap \mathcal{I}_2$. This implies that for any $x \in V(H_S)$,

$$\deg_{H_S}(x) \ge r - 2k + 2 \ge k \quad \Longrightarrow \quad \lambda_2(\mathsf{M}_S) \le \|\mathsf{D}_H^{-1}\| \le \frac{1}{k},$$

where we use the assumption that $k \le \frac{r}{3}$. □

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