Approximate Min-Max Theorems of Steiner Rooted-Orientations of Hypergraphs

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Abstract

Given an undirected hypergraph and a subset of vertices $S \subseteq V$ with a specified root vertex $r \in S$, the STEINER ROOTED-ORIENTATION *problem is to find an orientation of all the hyperedges so that in the resulting directed hypergraph the "connectivity" from the root r to the vertices in S is maximized. This is motivated by a multicasting problem in undirected networks as well as a generalization of some classical problems in graph theory. The main results of this paper are the following approximate min-max relations:*

- *Given an undirected hypergraph H, if S is $2k$-hyperedge-connected in H, then H has a Steiner rooted k-hyperarc-connected orientation.*
- *Given an undirected graph G, if S is $2k$-element-connected in G, then G has a Steiner rooted k-element-connected orientation.*

Both results are tight in terms of the connectivity bounds. These also give polynomial time constant factor approximation algorithms for both problems. The proofs are based on submodular techniques, and a graph decomposition technique used in the STEINER TREE PACKING *problem. Some complementary hardness results are presented at the end.*

1 Introduction

Let $H = (V, \mathcal{E})$ be an undirected hypergraph. An *orientation* of H is obtained by assigning a direction to each hyperedge in H. In our setting, a *hyperarc* (a directed hyperedge) is a hyperedge with a designated *tail vertex* and other vertices as *head vertices*. Given a set $S \subseteq V$ of *terminal vertices* (the vertices in $V - S$ are called the *Steiner vertices*) and a *root vertex* $r \in S$, we say a directed hypergraph is *Steiner rooted k-hyperarc-connected* if there are k hyperarc-disjoint paths from the root vertex r to each terminal vertex in S. Here, a *path* in a directed hypergraph is an alternating sequence of distinct vertices and hyperarcs $\{v_0, a_0, v_1, a_1, \ldots, a_{k-1}, v_k\}$ so that v_i is the tail of a_i and v_{i+1} is a head of a_i for all $0 \leq i < k$. The STEINER ROOTED-ORIENTATION problem is to find an orientation of H so that the resulting directed hypergraph is Steiner rooted k-hyperarc-connected, and our objective is to maximize k.

When the STEINER ROOTED-ORIENTATION problem specializes to graphs, it is a common generalization of some classical problems in graph theory. When there are only two terminals ($S = \{r, v\}$), it is the edge-disjoint paths problem solved by Menger. When all vertices in the graph are terminals ($S = V$), it can be shown to be equivalent to the edge-disjoint spanning trees problem solved by Tutte [32] and Nash-Williams [31]. An alternative common generalization of the above problems is the STEINER TREE PACKING problem studied in [21, 18, 22]. Notice that if a graph G has k edge-disjoint *Steiner trees* (i.e. trees that connect the terminal vertices S), then G has a Steiner rooted k arc-connected orientation. The converse, however, is not true. As we shall see, significantly sharper approximate min-max relations and also approximation ratio can be achieved for the STEINER ROOTED-ORIENTATION problem, especially when we consider hyperarc-connectivity and element-connectivity. This has implications in the network multicasting problem, which will be discussed later.

Given a hypergraph H, we say S *is k-hyperedge-connected in H* if there are k hyperedge-disjoint paths between every pair of vertices in S. It is not difficult to see that for a hypergraph H to have a Steiner rooted k-hyperarc-connected orientation, S must be at least k-hyperedge-connected in H. The main focus of this paper is to determine the smallest constant c so that the following holds: If S is ck-hyperedge-connected in H, then H has a Steiner rooted k-hyperarc-connected orientation.

Previous Work: Graph orientations is a well-studied subject in the literature, and there are many ways to look at such questions (see [2]). Here we focus on graph orientations achieving high connectivity. A directed graph is

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strongly k-arc-connected if there are k arc-disjoint paths between every ordered pair of vertices. The starting point of this line of research is a theorem by Robbins which says that an undirected graph G has a strongly 1-arc-connected orientation if and only if G is 2-edge-connected. In the following $\lambda(x, y)$ denotes the maximum number of edge-disjoint paths from x to y, which is called the *local-edge-connectivity* from x to y. Nash-Williams [30] proved the following deep generalization of Robbins' theorem which achieves optimal local-arc-connectivity for all pairs of vertices: "Every undirected graph G has an orientation D so that $\lambda_D(x, y) \geq \lfloor \lambda_G(x, y)/2 \rfloor$ for all $x, y \in V$".

Nash-Williams' original proof is quite complicated, and until now this is the only known orientation result achieving high *local*-arc-connectivity. Subsequently, Frank, in a series of works [9, 10, 12, 14], developed a general framework to solve graph orientation problems achieving high *global*-arc-connectivity by using the *submodular flow* problem introduced by Edmonds and Giles [6]. With this powerful tool, Frank greatly extended the range of orientation problems that can be solved concerning global-arc-connectivity. Some examples include finding a strongly k-arc-connected orientation with minimum weight [10], with in-degree constraints [9] and in mixed graphs [12]. Recently, this framework has been generalized to solve hypergraph orientation problems achieving high global-hyperarc-connectivity [15].

Extending graph orientation results to local hyperarc-connectivity or to vertex-connectivity is more challenging. For the STEINER ROOTED-ORIENTATION problem, the only known result follows from Nash-Williams' orientation theorem: if S is $2k$-edge-connected in an undirected graph G, then G has a Steiner rooted k-arc-connected orientation. For hypergraphs, there is no known orientation result concerning Steiner rooted-hyperarc-connectivity. A closely related problem of characterizing hypergraphs that have a Steiner strongly k-hyperarc-connected orientation is posted as an open problem in [8] (and more generally an analog of Nash-Williams' orientation theorem in hypergraphs). For orientation results concerning vertex-connectivity, very little is known even for global rooted-vertex-connectivity (when there are no Steiner vertices). Frank [13] made a conjecture on a necessary and sufficient condition for the existence of a strongly k-vertex-connected orientation, which in particular would imply that a $2k$-vertex-connected graph has a strongly k-vertex-connected orientation (and hence a rooted k-vertex-connected orientation). The only positive result along this line is a sufficient condition due to Jordán [20] for the case $k = 2$: Every 18-vertex-connected graph has a strongly 2-vertex-connected orientation.

Results: The main result of this paper is the following approximate min-max theorem on hypergraphs, which is tight in terms of the connectivity bound. This gives a positive answer to the rooted version of the question in [8].

Theorem 1.1 *Suppose H is an undirected hypergraph, S is a subset of terminal vertices with a specified root vertex $r \in S$. Then H has a Steiner rooted k-hyperarc-connected orientation if S is $2k$-hyperedge-connected in H.*

The proof is constructive, and also implies a polynomial time constant factor approximation algorithm for the problem. When the above theorem specializes to graphs, this gives a new and simpler algorithm (without using Nash-Williams' orientation theorem) to find a Steiner rooted k-arc-connected orientation in a graph when S is $2k$-edge-connected in G. On the other hand, we prove that finding an orientation which maximizes the Steiner rooted-arc-connectivity in a graph is NP-complete (Theorem 6.1).

Following the notation on approximation algorithms on graph connectivity problems, by an *element* we mean either an edge or a Steiner vertex. For graph connectivity problems, element-connectivity is regarded as of intermediate difficulty between vertex-connectivity and edge-connectivity (see [19, 7]). A directed graph is *Steiner rooted k-element-connected* if there are k element-disjoint directed paths from r to each terminal vertex in S. We prove the following approximate min-max theorem on element-connectivity, which is tight in terms of the connectivity bound. We also prove the NP-completeness of this problem (Theorem 6.2).

Theorem 1.2 *Suppose G is an undirected graph, S is a subset of terminal vertices with a specified root vertex $r \in S$. Then G has a Steiner rooted k-element-connected orientation if S is $2k$-element-connected in G.*

Techniques: Since Nash-Williams' orientation theorem, little progress has been made on orientation problems concerning local-arc-connectivity, local-hyperarc-connectivity or vertex-connectivity. The difficulty is largely due to a lack of techniques to work with these more sophisticated connectivity notions. The main technical contribution of this paper is a new method to use the submodular flow problem. A key ingredient in the proof of Theorem 1.1 is the use of an "extension property" (see [22, 23]) to help decompose a general hypergraph into hypergraphs with substantially simpler structures. Then, in those simpler hypergraphs, we apply the submodular flows technique in a very effective way to solve the problem (and also prove the extension property). An important building block of our approach is the following class of polynomial time solvable graph orientation problems, which we call the DEGREE-SPECIFIED STEINER ROOTED-ORIENTATION problem.

Theorem 1.3 *Suppose G is an undirected graph, S is a subset of terminal vertices with a specified root vertex $r \in S$, and m is an in-degree specification on the Steiner vertices (i.e. $m : (V(G) - S) \rightarrow \mathbb{Z}^+$). Then deciding whether G has a Steiner rooted k-arc-connected orientation with the specified in-degrees can be solved in polynomial time.*

Perhaps Theorem 1.3 does not seem to be very useful at first sight, but it turns out to be surprisingly powerful in some situations when we have a rough idea on what the indegrees of Steiner vertices should be like. To prove Theorem 1.3, we shall reduce this problem to a submodular flow problem from which we can also derive a sufficient and necessary condition for the existence of a Steiner rooted k-arc-connected orientation. This provides us with a crucial tool in establishing the approximate min-max relations.

Interestingly, the proof of Theorem 1.2 is also based on the DEGREE-SPECIFIED STEINER ROOTED-ORIENTATION problem (Theorem 1.3) which is designed for edge-connectivity problems. For a similar step in the hypergraph orientation problem, we shall use a technique in [4] to obtain a graph with simpler structures.

The Network Multicasting Problem: The STEINER ROOTED-ORIENTATION problem is motivated by the *multicasting* problem in computer networks, where the root vertex (the sender) must transmit all its data to the terminal vertices (the receivers) and the goal is to maximize the transmission rate that can be achieved simultaneously for all receivers. The connection is through a beautiful min-max theorem by Ahlswede et. al. [1]: "Given a directed multigraph with unit capacity on each arc, if there are k arc-disjoint paths from the root vertex to each terminal vertex, then the root vertex can transmit k units of data to all terminal vertices simultaneously". They prove the theorem by introducing the innovative idea of *network coding* [1], which has generated much interest from information theory to computer science. These studies focus on directed networks, for example the Internet, where the direction of data movement on each link is fixed a priori. On the other hand, there are practical networks which are undirected, i.e. data can be sent in either direction along a link. By using the theorem by Ahlswede et. al., computing the maximum multicasting rate in undirected networks (with network coding supported) reduces to the STEINER ROOTED-ORIENTATION problem. This has been studied in the graph model [25, 26] and efficient (approximation) algorithms have been proposed. An important example of undirected networks is wireless networks (equipped with omni-directional antennas), for which many papers have studied the advantages of incorporating network coding (see [28] and the references therein). However, there are some aspects of wireless communications that are not captured by a graph model. One distinction is that wireless communications in such networks are inherently one-to-many instead of one-to-one. This motivates researchers to use the directed hypergraph model (see [5, 28]) to study the multicasting problem in wireless networks. A simple reduction shows that the above theorem by Ahlswede el. al. applies to directed hypergraphs as well. Therefore, computing the maximum multicasting rate in an undirected hypergraph (with network coding supported) reduces to the STEINER ROOTED-ORIENTATION problem of hypergraphs.

In the multicasting problem, the STEINER TREE PACKING problem is used to transmit data when network coding is not supported. However, one cannot hope for analogous results of Theorem 1.1 or Theorem 1.2 for the corresponding STEINER TREE PACKING problems. In fact, both the hyperedge-disjoint Steiner tree packing problem and the element-disjoint Steiner tree packing problem are shown to be NP-hard to approximate within a factor of $\Omega(\log n)$ [4]. (It was also shown in [3] that no constant connectivity bound implies the existence of two hyperedge-disjoint spanning sub-hypergraphs.) As a consequence, Theorem 1.1 indicates that multicasting with network coding in the hypergraph model could be much more efficient in terms of the throughput achieved (an $\Omega(\log n)$ gap in the worst case).

2 The Basics

Let $H = (V, \mathcal{E})$ be an undirected hypergraph. Given $X \subseteq V$, we say a hyperedge e *enters* X if $0 < |e \cap X| < |e|$. We define $\delta_H(X)$ to be the set of hyperedges that enter X, and $d_H(X) := |\delta_H(X)|$. We also define $E(X)$ to be the number of induced hyperedges in X. In a directed hypergraph $\vec{H} = (V, \overrightarrow{\mathcal{E}})$, a hyperarc a *enters* a set X if the tail of a is not in X and some head of a is in X. We define $\delta^{in}_{\vec{H}}(X)$ to be the set of hyperarcs that enter X, and $d^{in}_{\vec{H}}(X) := |\delta^{in}_{\vec{H}}(X)|$. Similarly, a hyperarc a *leaves* a set X if a enters $V - X$. We define $\delta^{out}_{\vec{H}}(X)$ to be the set of hyperarcs that leave X, and $d^{out}_{\vec{H}}(X) := |\delta^{out}_{\vec{H}}(X)|$.

Let V be a finite ground set. Two subsets X and Y are *intersecting* if $X - Y$, $Y - X$, $X \cap Y$ are all non-empty. X and Y are *crossing* if they are intersecting and $X \cup Y \neq V$. For a function $m : V \to \mathbb{R}$ we use the notation $m(X) := \sum(m(x) : x \in X)$. Let $f : 2^V \to \mathbb{R}$ be a function defined on the subsets of V. The set-function f is called (intersecting, crossing) *submodular* if the following inequality holds for any two (intersecting, crossing) subsets X and Y of V:

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y). \tag{1}$$

The set function f is called (intersecting, crossing) *supermodular* if the reverse inequality of (1) holds for any two (intersecting, crossing) subsets X and Y of V.

Submodular Flows and Graph Orientations: Now we introduce the submodular flow problem. Let $D = (V, A)$ be a digraph, $\mathcal{F}$ be a crossing family of subsets of V (if X, Y are two crossing sets in $\mathcal{F}$, then $X \cup Y, X \cap Y \in \mathcal{F}$), and $b : \mathcal{F} \to \mathbb{Z}$ be a crossing submodular function. Given such $D, \mathcal{F}, b$, a *submodular flow* is a function $x : A \to \mathbb{R}$ satisfying:

$$x^{in}(U) - x^{out}(U) \leq b(U) \text{ for each } U \in \mathcal{F}.$$

Given two functions $f : A \rightarrow \mathbb{Z} \cup \{-\infty\}$ and $g : A \rightarrow \mathbb{Z} \cup \{\infty\}$, a submodular flow is *feasible* with respect to f, g if $f(a) \leq x(a) \leq g(a)$ holds for all $a \in A$. The Edmonds-Giles theorem [6] (roughly) says that the set of feasible submodular flows (with respect to given $D, \mathcal{F}, b, f, g$) has an integer optimal solution for any objective function $\min\{\sum_{a \in A(D)} c(a) \cdot x(a)\}$. From the Edmonds-Giles theorem, Frank [11] derived a necessary and sufficient condition to have a feasible submodular flow if b is intersecting submodular. From this characterization, using the same approach as in [12, 14], we can derive the following theorem for finding an orientation covering an intersecting supermodular function. Let $h : 2^V \rightarrow \mathbb{Z}$ be an integer valued set-function with $h(\emptyset) = h(V) = 0$. We say an orientation $\vec{H}$ *covers* h if $d^{in}_{\vec{H}}(X) \geq h(X)$ for all $X \subseteq V$.

Theorem 2.1 *Let $G = (V, E)$ be an undirected graph. Let $h : 2^V \rightarrow \mathbb{Z} \cup \{-\infty\}$ be an intersecting supermodular function with $h(\emptyset) = h(V) = 0$. Then there exists an orientation D of G satisfying*

$$d^{in}_D(X) \geq h(X) \text{ for all } X \subset V \text{ iff } e_{\mathcal{P}} \geq \sum h(X_i)$$

holds for every subpartition $\mathcal{P} = \{X_1, X_2, \ldots, X_t\}$ of V. Here $e_{\mathcal{P}}$ counts the number of edges which enter some member of $\mathcal{P}$.

Our original approach used Theorem 2.1 as the basis for the results of Section 3 (see [24]), which works for arbitrary intersecting supermodular functions. For non-negative intersecting supermodular functions (which include the DEGREE-SPECIFIED STEINER ROOTED-ORIENTATION problem), we can simplify the proofs by using the following results.

Lemma 2.2 *([16]) Let $G = (V, E)$ be an undirected graph, $x : V \rightarrow \mathbb{Z}^+$ an indegree specification, and $h : 2^V \rightarrow \mathbb{Z}^+$ a non-negative function. Then G has an orientation D that covers h and $d^{in}_D(v) = x(v)$ for every $v \in V$ if and only if $x(X) \geq E(X) + h(X)$ for every $X \subseteq V$.*

Theorem 2.3 *(see [27]) Let $h : 2^V \rightarrow \mathbb{Z}^+$ be a non-negative intersecting supermodular set function, and let l be a non-negative integer. The polyhedron*

$$\mathcal{B} := \{x \in \mathbb{R}^V : x(X) \geq h(X) \text{ for } X \subseteq V, x(V) = l\}$$

is non-empty if and only if the following conditions hold:

1. *$h(\emptyset) = 0$,*
2. *$\sum_{X \in \mathcal{F}} h(X) \leq l$ for every partition $\mathcal{F}$ of V.*

If $\mathcal{B}$ is non-empty, then it is a base polyhedron, so its vertices are integral.

Mader's Splitting-Off Theorem: Let G be an undirected graph. *Splitting-off* a pair of edges $e = uv, f = vw$ means that we replace e and f by a new edge uw (parallel edges may arise). The resulting graph will be denoted by G^{ef}. The following theorem by Mader [29] proves to be very useful in attacking edge-connectivity problems.

Theorem 2.4 *Let $G = (V, E)$ be a connected undirected graph in which $0 < d_G(s) \neq 3$ and there is no cut-edge incident with s. Then there exists a pair of edges $e = su, f = st$ so that $\lambda_G(x, y) = \lambda_{G^{ef}}(x, y)$ holds for every $x, y \in V - s$.*

3 Degree-Specified Steiner Orientations

In this section we consider the DEGREE-SPECIFIED STEINER ORIENTATION problem, which will be the basic tool for proving the main theorems. Note that we shall only consider this problem in graphs. Given a graph $G = (V, E)$, a terminal set $S \subseteq V(G)$ and a connectivity requirement function $h : 2^S \rightarrow \mathbb{Z}$, we say the connectivity requirement function $h^* : 2^V \rightarrow \mathbb{Z}$ is the *Steiner extension* of h if $h^*(X) = h(X \cap S)$ for every $X \subseteq V$. Suppose G, S, h are given as above, and an indegree specification $m(v)$ for each Steiner vertex is given. The goal of the DEGREE-SPECIFIED STEINER ORIENTATION problem is to find an orientation D of G that covers the Steiner extension h^* of h, with an additional requirement that $d^{in}_D(v) = m(v)$ for every $v \in V(G) - S$.

We show that the DEGREE-SPECIFIED STEINER ORIENTATION problem can be solved in polynomial time if h is a non-negative intersecting supermodular set function. Notice that h^* is not an intersecting submodular function in general, and therefore Theorem 2.3 (or Theorem 2.1) cannot be directly applied. Nonetheless, we can reformulate the problem so that we can use Theorem 2.3.

Since the indegrees of the vertices in $V - S$ are fixed, we have to determine the indegrees of the vertices in S. By Lemma 2.2, a vector $x : S \rightarrow \mathbb{Z}^+$ with $x(S) = |E| - m(V - S)$ is the vector of indegrees of a degree-specified Steiner orientation if and only if $x(X) + m(Z) \geq h(X) + E(X \cup Z)$ for every $X \subseteq S$ and $Z \subseteq V - S$. Let us define the following set function on S:

$$h'(X) := h(X) + \max_{Z \subseteq V - S} (E(X \cup Z) - m(Z)) \text{ for } X \subseteq S.$$

It follows that there is a degree-specified Steiner orientation such that x is the vector of indegrees of the vertices of S if and only if $x(X) \geq h'(X)$ for every $X \subseteq S$ and $x(S) = |E| - m(V - S)$.

Lemma 3.1 *The set function h' is intersecting supermodular if h is intersecting supermodular.*

Proof. Let $X_1 \subseteq S$ and $X_2 \subseteq S$ be two intersecting sets. There are sets $Z_1 \subseteq V - S$ and $Z_2 \subseteq V - S$ such that

$h'(X_1) = h(X_1) + E(X_1 \cup Z_1) - m(Z_1)$ and $h'(X_2) = h(X_2) + E(X_2 \cup Z_2) - m(Z_2)$. By the properties of the set functions involved, we have the following inequalities:

- $h(X_1) + h(X_2) \le h(X_1 \cap X_2) + h(X_1 \cup X_2)$.
- $E(X_1 \cup Z_1) + E(X_2 \cup Z_2) \le E((X_1 \cap X_2) \cup (Z_1 \cap Z_2)) + E((X_1 \cup X_2) \cup (Z_1 \cup Z_2))$.
- $m(Z_1) + m(Z_2) = m(Z_1 \cap Z_2) + m(Z_1 \cup Z_2)$.

Thus

$$\begin{aligned}
& h'(X_1) + h'(X_2) \\
= \ & h(X_1) + h(X_2) + E(X_1 \cup Z_1) + E(X_2 \cup Z_2) \\
& -m(Z_1) - m(Z_2) \\
\le \ & h(X_1 \cap X_2) + E((X_1 \cap X_2) \cup (Z_1 \cap Z_2)) \\
& -m(Z_1 \cap Z_2) + h(X_1 \cup X_2) \\
& +E((X_1 \cup X_2) \cup (Z_1 \cup Z_2)) - m(Z_1 \cup Z_2) \\
\le \ & h'(X_1 \cap X_2) + h'(X_1 \cup X_2).
\end{aligned}$$

∎

Let us consider the following polyhedron:

$$\mathcal{B} := \{x \in \mathbb{R}^S : \ x(X) \ge h'(X) \text{ for every } X \subseteq S, \\ x(S) = |E| - m(V - S).\}$$

The integer vectors of this polyhedron correspond to indegree vectors of degree-specified Steiner orientations. By Theorem 2.3, $\mathcal{B}$ is non-empty if and only if the following two conditions hold:

1. $h'(\emptyset) = 0$,
2. $\sum_{X \in \mathcal{F}} h'(X) \le |E| - m(V - S)$ for every partition $\mathcal{F}$ of S.

If $\mathcal{B}$ is non-empty, then it is a base polyhedron, so its vertices are integral. As we have seen, such a vertex is the indegree vector of a degree-specified Steiner orientation. Thus the non-emptiness of $\mathcal{B}$ is equivalent to the existence of a degree-specified orientation. Since a vertex of a base polyhedron given by an intersecting supermodular set function can be found in polynomial time, we obtained the following results:

Theorem 3.2 *Let $G = (V, E)$ be an undirected graph with a terminal set $S \subseteq V$. Let $h : 2^S \to \mathbb{Z}^+$ be a non-negative intersecting supermodular set function and $m : (V - S) \to \mathbb{Z}^+$ be an indegree specification. Then G has an orientation covering the Steiner extension h^* of h with the specified indegrees if and only if $E(Z) \le m(Z)$ for every $Z \subseteq V - S$ and for every partition $\mathcal{F}$ of S*

$$\sum_{X \in \mathcal{F}} (h(X) + \max_{Z \subseteq V - S} (E(X \cup Z) - m(Z))) \le |E| - m(V - S).$$

Theorem 3.3 *If h is non-negative and intersecting supermodular, then the* DEGREE-SPECIFIED STEINER ORIENTATION *problem can be solved in polynomial time.*

Steiner Rooted-Orientations of Graphs: In the following we focus on the STEINER ROOTED ORIENTATION problem. First we derive Theorem 1.3 as a corollary of Theorem 3.2. In contrast with Theorem 3.3, the STEINER ROOTED ORIENTATION problem is NP-complete (Theorem 6.1). That said, in general, finding an in-degree specification for the Steiner vertices to maximize the Steiner rooted-edge-connectivity is hard.

Proof of Theorem 1.3: Let S be the set of terminal vertices and $r \in S$ be the root vertex. Set $h(X) := k$ for every $X \subseteq S$ with $r \notin X$, and $h(X) := 0$ otherwise. Then h is an intersecting supermodular function on S. By Menger's theorem, an orientation is Steiner rooted k-arc-connected if and only if it covers the Steiner extension of h. Thus, by Theorem 3.2, the problem of finding a Steiner rooted-orientation with the specified indegrees can be solved in polynomial time. ∎

The following theorem can be derived from Theorem 3.2, which will be used to prove a special case of Theorem 1.1. This is one of the examples that the DEGREE-SPECIFIED STEINER ORIENTATION problem is useful. The key observation is that we can "hardwire" the indegrees of the Steiner vertices to be 1.

Theorem 3.4 *Let $G = (V, E)$ be an undirected graph with terminal set $S \subseteq V(G)$. If every Steiner vertex (vertices in $V(G) - S$) is of degree at most 3 and there is no edge between two Steiner vertices in G, then G has a Steiner rooted k-edge-connected orientation if and only if*

$$e_{\mathcal{P}} \ge k(t - 1)$$

holds for every partition $\mathcal{P} = (V_1, \dots, V_t)$ of $V(G)$ such that each V_i contains a terminal vertex, where $e_{\mathcal{P}}$ denotes the number of crossing edges. In fact, there exists such an orientation with every Steiner vertex of indegree 1.

4 Proof of Theorem 1.1

In this section, we present the proof of the main result of this paper (Theorem 1.1). We shall consider a minimal counterexample $\mathcal{H}$ of Theorem 4.2 with the minimum number of edges and then the minimum number of vertices. Note that Theorem 4.2 is a stronger version of Theorem 1.1 with an "extension property" introduced (Definition 4.1). The extension property allows us to apply a graph decomposition procedure to simplify the structures of $\mathcal{H}$ significantly (Corollary 4.5, Corollary 4.6). With these structures, we can construct a bipartite graph representation B of $\mathcal{H}$. Then, the DEGREE-SPECIFIED STEINER ROOTED ORIENTATION problem can be applied in the bipartite graph B to establish a tight approximate min-max relation (Theorem 4.10). To better illustrate the proof idea, we also include a proof of Theorem 4.2 in the special case of *rank* 3

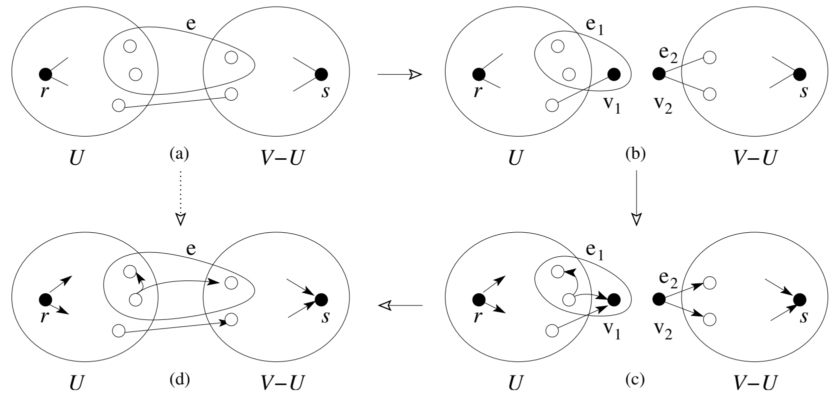

Figure 1. **An illustration of the proof of Lemma 4.4.**

hypergraphs (Lemma 4.7), where every hyperedge is of size at most 3.

We need some notation to state the extension property. A hyperarc a is in $\delta^{in}(X;\overline{Y})$ if a enters X and $a \cap Y = \emptyset$. If Y is an emptyset, then $\delta^{in}(X;\overline{Y})$ is the same as $\delta^{in}(X)$. We use $d^{in}(X;\overline{Y})$ to denote $|\delta^{in}(X;\overline{Y})|$. A hyperarc a is in $\overrightarrow{E}(X,Y)$ if a leaves X and enters Y. We use $\overrightarrow{d}(X,Y)$ to denote $|\overrightarrow{E}(X,Y)|$. The following extension property is at the heart of our approach.

Definition 4.1 *Given $H = (V,\mathcal{E})$, $S \subseteq V$ and a vertex $s \in S$, a Steiner rooted-orientation D of H* extends *s if:*

1. *$d^{in}_D(s) = d_H(s)$;*
2. *$d^{in}_D(Y;\overline{s}) \geq \overrightarrow{d}_D(Y,s)$ for every $Y \subseteq V$ for which $Y \cap S = \emptyset$.*

As mentioned previously, we shall prove the following stronger theorem which immediately implies Theorem 1.1.

Theorem 4.2 *Suppose H is an undirected hypergraph, S is a subset of terminal vertices with a specified root vertex $r \in S$. Then H has a Steiner rooted k-hyperarc-connected orientation if S is $2k$-hyperedge-connected in H. In fact, given any vertex $s \in S$ of degree $2k$, H has a Steiner rooted k-hyperarc-connected orientation that extends s. We call the special vertex s the* sink *of H.*

The next lemma shows that the choice of the root vertex does not matter. The proof idea is that we can reverse the directions of the arcs in the r,v-paths.

Lemma 4.3 *Suppose there exists a Steiner rooted k-hyperarc-connected orientation that extends s with r as the root. Then there exists a Steiner rooted k-hyperarc-connected orientation that extends s with v as the root for every $v \in S - s$.*

In the following we say a set X is *tight* if $d_{\mathcal{H}}(X) = 2k$; X is *nontrivial* if $|X| \geq 2$ and $|V(\mathcal{H}) - X| \geq 2$. The following is the key lemma where we use the graph decomposition technique (see Figure 1 for an illustration).

Lemma 4.4 *There is no nontrivial tight set in $\mathcal{H}$.*

Proof. Suppose there exists a nontrivial tight set U, i.e. $d_{\mathcal{H}}(U) = 2k$, $|U| \geq 2$ and $|V(\mathcal{H}) - U| \geq 2$. Contract $V(\mathcal{H}) - U$ of $\mathcal{H}$ to a single vertex v_1 and call the resulting hypergraph H_1; similarly, contract U of $\mathcal{H}$ to a single vertex v_2 and call the resulting hypergraph H_2. We assume $s \in H_2$. See Figure 1 (b) for an illustration. So, $V(H_1) = U \cup \{v_1\}$, $V(H_2) = (V(\mathcal{H}) - U) \cup \{v_2\}$ and there is an one-to-one correspondence between the hyperedges in $\delta_{H_1}(v_1)$ and the hyperedges in $\delta_{H_2}(v_2)$. To be more precise, for a hyperedge $e \in \mathcal{E}(\mathcal{H})$, it decomposes into $e_1 = (e \cap V(H_1)) \cup \{v_1\}$ in H_1 and $e_2 = (e \cap V(H_2)) \cup \{v_2\}$ in H_2 and we refer them as the corresponding hyperedges of e in H_1 and H_2 respectively.

Since U is non-trivial, both H_1 and H_2 are smaller than $\mathcal{H}$. We set $S_1 := (S \cap V(H_1)) \cup v_1$ and $S_2 = (S \cap V(H_2)) \cup v_2$, and set the sink of H_1 to be v_1 and the sink of H_2 to be s. Clearly, S_1 is $2k$-hyperedge-connected in H_1 and S_2 is $2k$-hyperedge-connected in H_2. By the minimality of $\mathcal{H}$, H_2 has a Steiner rooted k-hyperarc-connected orientation D_2 that extends s. By Lemma 4.3, we can choose the root of D_2 to be v_2. Similarly, by the minimality of $\mathcal{H}$, H_1 has a Steiner rooted k-hyperarc-connected orientation D_1 that extends v_1. Let the root of D_1 be r. See Figure 1 (c) for an illustration.

We shall prove that the concatenation D of the two orientations D_1, D_2 gives a Steiner rooted k-hyperarc-connected orientation of $\mathcal{H}$ that extends s. Notice for a hyperedge e in $\delta_{\mathcal{H}}(U)$, its corresponding hyperedge e_1 in H_1 is oriented with v_1 as a head (by the extension property of D_1), and its corresponding hyperedge e_2 in H_2 is oriented so that v_2 is the tail (as v_2 is the root of D_2). So, in D, the orientation of e is well-defined and has its tail in H_1. See Figure 1 (d) for an illustration. Now we show that D is a Steiner rooted k-hyperarc-connected orientation. By Menger's theorem, it suffices to show that $d^{in}_D(X) \geq k$ for any $X \subseteq V(\mathcal{H})$ for which $r \notin X$ and $X \cap S \neq \emptyset$.

Suppose $X \cap S_1 = \emptyset$ (the case that $X \cap S_1 \neq \emptyset$ is easy). Let $X_1 = X \cap H_1$ and $X_2 = X \cap H_2$. The case that $X_1 = \emptyset$ follows from the properties of D_2. So we assume both X_1 and X_2 are non-empty. We have the following inequality:

$$d^{in}_D(X) \geq d^{in}_{D_1}(X_1; \overline{v_1}) + d^{in}_{D_2}(X_2) - \overrightarrow{d}_D(X_1, X_2). \quad (2)$$

Note that $\overrightarrow{d}_{D_1}(X_1, v_1) \geq \overrightarrow{d}_D(X_1, X_2)$. So, by property (ii) of Definition 4.1, $d^{in}_{D_1}(X_1; \overline{v_1}) \geq \overrightarrow{d}_{D_1}(X_1, v_1) \geq \overrightarrow{d}_D(X_1, X_2)$. Hence $d^{in}_D(X) \geq d^{in}_{D_2}(X_2) \geq k$, where the second inequality is by the properties of D_2.

This implies that D is a Steiner rooted k-hyperarc-connected orientation of $\mathcal{H}$. To finish the proof, we need to check that D extends s. The first property of Definition 4.1 follows immediately from our construction. The second property of Definition 4.1 can be shown by a similar argument as above. This shows that D extends s, which contradicts that $\mathcal{H}$ is a counterexample. ∎

The following are two important properties obtained from Lemma 4.4.

Corollary 4.5 *Each hyperedge of $\mathcal{H}$ of size at least 3 contains only terminal vertices.*

Proof. Suppose e is a hyperedge of $\mathcal{H}$ of size at least 3 and $t \in e$ is a Steiner vertex. Let H' be a hypergraph with the same vertex and edge set as $\mathcal{H}$ except we replace e by $e' := e - t$. By minimality of $\mathcal{H}$, there exists a set X which separates two terminals with $d_{\mathcal{H}}(X) = 2k$ and $d_{H'}(X) < 2k$. So $e \in \delta_{\mathcal{H}}(X)$. Suppose $t \in X$. Since X contains a terminal, $|X| \geq 2$. Also, $e - t$ must be contained in $V(\mathcal{H}) - X$; otherwise $d_{\mathcal{H}}(X) = d_{H'}(X)$. Hence $|V(\mathcal{H}) - X| \geq |e - t| \geq 2$. Therefore, X is a nontrivial tight set, which contradicts Lemma 4.4. ∎

The proof of the following corollary is similar.

Corollary 4.6 *There is no edge between two Steiner vertices in $\mathcal{H}$.*

4.1 The Bipartite Representation of $\mathcal{H}$

Using Corollary 4.5 and Corollary 4.6, we shall construct a bipartite graph from $\mathcal{H}$, which allows us to apply the results on the DEGREE-SPECIFIED STEINER ROOTED-ORIENTATION problem to $\mathcal{H}$. Let S be the set of terminal vertices in $\mathcal{H}$. Let $\mathcal{E}'$ be the set of hyperedges in $\mathcal{H}$ which do not contain a Steiner vertex, i.e. a hyperedge e is in $\mathcal{E}'$ if $e \cap (V(\mathcal{H}) - S) = \emptyset$. We construct a bipartite graph

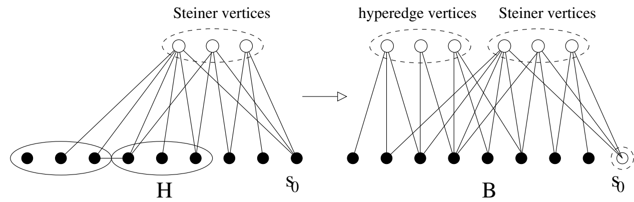

Figure 2. **The bipartite representation B of $\mathcal{H}$.**

$B = (S, (V(\mathcal{H}) - S) \cup \mathcal{E}'; E)$ from the hypergraph $\mathcal{H}$ as follows. Every vertex v in $\mathcal{H}$ corresponds to a vertex v in B, and also every hyperedge $e \in \mathcal{E}'$ corresponds to a vertex v_e in B. By Corollary 4.5, hyperedges which intersect $V(\mathcal{H}) - S$ are graph edges (i.e. hyperedges of size 2); we add these edges to $E(B)$. For every hyperedge $e \in \mathcal{E}'$, we add $v_e w$ to $E(B)$ if and only if $w \in e$ in $\mathcal{H}$. Let the set of terminal vertices in B be S; all other vertices are non-terminal vertices in B. By Corollary 4.5 and Corollary 4.6, there is no edge between two non-terminal vertices in B. Hence B is a bipartite graph. To distinguish the non-terminal vertices corresponding to Steiner vertices in $\mathcal{H}$ and the non-terminal vertices corresponding to hyperedges in $\mathcal{E}'$, we call the former the Steiner vertices and the latter the *hyperedge vertices*. See Figure 2 for an illustration.

4.2 Rank 3 Hypergraphs

To better illustrate the idea of the proof, we first prove Theorem 4.2 for the case of rank 3 hypergraphs. This motivates the proof for general hypergraphs, which is considerably more complicated.

Lemma 4.7 *$\mathcal{H}$ is not a rank 3 hypergraph.*

Proof. Since $\mathcal{H}$ is of rank 3, all hyperedge vertices in B are of degree at most 3. The crucial use of the rank 3 assumption is the following simple observation.

Proposition 4.8 *S is $2k$-hyperedge-connected in $\mathcal{H}$ if and only if S is $2k$-edge-connected in B.*

We remark that Proposition 4.8 does not hold for hypergraphs of rank greater than 3. With Proposition 4.8, we can apply Mader's splitting off theorem to prove the following.

Lemma 4.9 *Steiner vertices of $\mathcal{H}$ are of degree at most 3.*

Proof. If a Steiner vertex v is not of degree 3 in $\mathcal{H}$, then it is not of degree 3 in B. So we can apply Mader's splitting-off theorem (Theorem 2.4) to find a suitable splitting at v in B. Let $e_1 = s_1 v$ and $e_2 = v s_2$ be the pair of edges that we split-off, and $e = s_1 s_2$ be the new edge. By Corollary 4.6, s_1 and s_2 are terminal vertices. We add a new

Steiner vertex v_e to $V(B)$ and replace the edge s_1s_2 by two new edges v_es_1 and v_es_2. Since B is bipartite, the resulting graph, denoted by B', is bipartite. Notice that B' corresponds to a hypergraph H' with $V(H') = V(\mathcal{H})$ and $E(H') = E(\mathcal{H}) - \{e_1, e_2\} + \{e\}$. By Proposition 4.8, S is k-hyperedge-connected in H'. By the minimality of $\mathcal{H}$, there is a Steiner rooted k-hyperarc-connected orientation of H'. Suppose s_1s_2 in H' is oriented as $\overrightarrow{s_1s_2}$ in H', then we orient vs_1 and vs_2 as $\overrightarrow{s_1v}$ and $\overrightarrow{vs_2}$ in $\mathcal{H}$. All other hyperedges in $\mathcal{H}$ have the same orientations as the corresponding hyperedges in H'. It is easy to see that this orientation is a Steiner rooted k-hyperarc-connected orientation of $\mathcal{H}$, a contradiction. ∎

Now we are ready to finish the proof of Lemma 4.7. Construct $B' = B - s$, where we remove all edges in B which are incident with s. We shall use Theorem 3.4 to prove that there is a Steiner rooted k-arc-connected orientation of B'. Since S is $2k$-edge-connected in B, for any partition $\mathcal{P} = \{P_1, \ldots, P_t\}$ of $V(B')$ such that each P_i contains a terminal vertex, we have $\sum_{i=1}^{t} d_{B'}(P_i) = \sum_{i=1}^{t} d_B(P_i) - d_B(s) \geq 2kt - 2k = 2k(t-1)$. So there are at least $k(t-1)$ edges crossing $\mathcal{P}$ in B'.

By Theorem 3.4, there is a Steiner rooted k-edge-connected orientation D' of B' with the additional property that each Steiner vertex has indegree exactly 1. By orienting the edges in $\delta_B(s)$ to have s as the head, we obtain an orientation D of B. Note that D corresponds to a hypergraph orientation of $\mathcal{H}$. Also, by this construction, property (i) of Definition 4.1 is satisfied.

Consider an arbitrary Y for which $Y \cap S = \emptyset$. Since every vertex y in Y is of degree at most 3 by Lemma 4.9, y can have at most one outgoing arc to s; otherwise $d_{\mathcal{H}}(\{s, y\}) < 2k$ which contradicts our connectivity assumption since $d_{\mathcal{H}}(s) = 2k$. Since Y induces an independent set by Corollary 4.6 and each vertex in Y has indegree exactly 1, each $y \in Y$ has an incoming arc from outside Y. So we have $d_D^{in}(Y; \overline{s}) \geq \overrightarrow{d}(Y, s)$. This implies that D satisfies property (ii) of Definition 4.1 as well.

Finally we verify that D is a Steiner rooted k-hyperedge-connected orientation. Consider a subset $X \subseteq V(\mathcal{H})$ which contains a terminal but not the root. If X contains a terminal other than s, then clearly $d_D^{in}(X) \geq k$. So suppose $X \cap S = s$. As argued above, v has at most one outgoing arc to s. As each Steiner vertex is of indegree 1 and there is no edge between two Steiner vertices, we have $d_D^{in}(X) \geq d_D^{in}(s) = 2k$. This shows that D is a Steiner rooted k-hyperarc-connected orientation that extends s, a contradiction. ∎

4.3 General Hypergraphs

For the proof of Theorem 4.2 for the case of rank 3 hypergraphs, a crucial step is to apply Mader's splitting-off lemma to the bipartite representation B of $\mathcal{H}$ to obtain Lemma 4.9. In general hypergraphs, however, a suitable splitting at a Steiner vertex which preserves the edge-connectivity of S in B might not preserve the hyperedge-connectivity of S in $\mathcal{H}$. And there is no analogous edge splitting-off result which preserves hyperedge-connectivity.

Our key observation is that, if we were able to apply Mader's lemma as in the proof of Lemma 4.7, then every Steiner vertex would end up with indegree $\lfloor d(v)/2 \rfloor$ in the resulting orientation of B. This motivates us to apply the DEGREE-SPECIFIED STEINER ROOTED-ORIENTATION problem by "hardwiring" $m(v) = \lfloor d(v)/2 \rfloor$ to "simulate" the splitting-off process. Also, we "hardwire" the indegree of the sink to be $2k$ for the extension property. (In the example of Figure 2, the indegrees of the Steiner vertices are specified to be 3,2,1 from left to right; the sink becomes a non-terminal vertex with specified indegree $2k$.) Quite surprisingly, such an orientation always exists when S is $2k$-hyperedge connected in $\mathcal{H}$. The following theorem is the final (and most technical) step to the proof of Theorem 4.2, which shows that a minimal counterexample of Theorem 4.2 does not exist.

Theorem 4.10 *Suppose that S is $2k$-hyperedge-connected in H, there is no edge between two Steiner vertices, and no hyperedge of size at least 3 contains a Steiner vertex. Let $s_0 \in S$ be a vertex of degree $2k$. Then H has a Steiner rooted k-hyperarc-connected orientation that extends s_0.*

Proof. We will use the theorem on the DEGREE-SPECIFIED STEINER ROOTED-ORIENTATION problem of graphs (Theorem 3.2). To get an instance of that problem, we consider the bipartite representation $(B' = (V', E'), S', m')$ of H as in the proof of Lemma 4.7. Let the set of terminals be $S' := S - s_0$. The indegree specification $m' : V' - S' \to \mathbb{Z}^+$ is defined by

$$m'(v) := \begin{cases} \lfloor d_H(v)/2 \rfloor & \text{if } v \text{ is a Steiner vertex} \\ 1 & \text{if } v \text{ is a hyperedge vertex} \\ 2k & \text{if } v = s_0 \text{ is the sink} \end{cases}$$

We shall show that if B' has a Steiner rooted k-arc-connected orientation with the specified indegrees, then H has a Steiner rooted k-hyperarc-connected orientation that extend s_0. By Theorem 3.2, this graph has a Steiner rooted k-arc-connected orientation with the specified indegrees if and only if the following conditions hold:

$$E_{B'}(Z) \leq m'(Z) \quad \text{for every } Z \subseteq V' - S', \tag{3}$$

$$\sum_{X \in \mathcal{F}} (h(X) + \max_{Y \subseteq V' - S'} (E_{B'}(X \cup Y) - m'(Y))) \leq |E'| - m'(V' - S') \tag{4}$$

for all partition $\mathcal{F}$ of S', where $h : S' \to \mathbb{Z}^+$ is defined by

$$h(X) := \begin{cases} k & \text{if } \emptyset \neq X \subseteq S' - r, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.11 *Condition* (3) *is always satisfied.*

Proposition 4.12 *Conditions* (4) *is satisfied if*

$$\sum_{e \in \mathcal{E}} (|\{X \in \mathcal{F} :\ e \cap X \neq \emptyset\}| - 1) + \sum_{v \notin \cup\mathcal{F} + s_0} \left\lceil \frac{d_H(v)}{2} \right\rceil \geq k(|\mathcal{F}| - 1) \tag{5}$$

for every subpartition $\mathcal{F}$ of V for which $S \cap X \neq \emptyset$ for every $X \in \mathcal{F}$, and $S \cap (\cup\mathcal{F}) = S - s_0$.

Notice that Proposition 4.12 is formulated in terms of the original hypergraph H. We will prove that the bipartite representation B' of H has the desired degree-specified orientation by showing that the conditions in Proposition 4.12 are satisfied if S is $2k$-hyperedge-connected in H.

Let $\mathcal{F}$ be a subpartition of V for which $S \cap X \neq \emptyset$ for every $X \in \mathcal{F}$, and $S \cap (\cup\mathcal{F}) = S - s_0$. Let $\mathcal{E}_1$ denote the set of hyperedges of H which enter exactly 1 member of $\mathcal{F}$, and let $\mathcal{E}_2$ denote the set of hyperedges of H which enter at least 2 members of $\mathcal{F}$. Let $U := V - (\cup\mathcal{F} + s_0)$. Then the only hyperedges that are disjoint from every member of $\mathcal{F}$ are the edges between U and s_0, so

$$\begin{aligned} &\sum_{e \in \mathcal{E}} \Big(|\{X \in \mathcal{F} :\ e \cap X \neq \emptyset\}| - 1\Big) + \sum_{v \notin \cup\mathcal{F} + s_0} \left\lceil \frac{d_H(v)}{2} \right\rceil \\ &\geq \sum_{X \in \mathcal{F}} \frac{d_{\mathcal{E}_2}(X)}{2} - d_H(U, s_0) + \sum_{v \in U} \frac{d_H(v)}{2} \\ &= \sum_{X \in \mathcal{F}} \frac{d_{\mathcal{E}_2}(X)}{2} + \frac{d_H(U, S - s_0)}{2} - \frac{d_H(U, s_0)}{2}. \end{aligned} \tag{6}$$

Here

$$d_H(U, S - s_0) = \sum_{X \in \mathcal{F}} d_{\mathcal{E}_1}(X) - d_H(V - U, s_0),$$

and so

$$d_H(U, S - s_0) - d_H(U, s_0) = \sum_{X \in \mathcal{F}} d_{\mathcal{E}_1}(X) - 2k.$$

Using this identity in inequality (6) we get that

$$\begin{aligned} &\sum_{e \in \mathcal{E}} \Big(|\{X \in \mathcal{F} :\ e \cap X \neq \emptyset\}| - 1\Big) + \sum_{v \notin \cup\mathcal{F} + s_0} \left\lceil \frac{d_H(v)}{2} \right\rceil \\ &\geq \sum_{X \in \mathcal{F}} \Big(\frac{d_{\mathcal{E}_2}(X) + d_{\mathcal{E}_1}(X)}{2}\Big) - k \geq k(|\mathcal{F}| - 1), \end{aligned}$$

where the last inequality holds because $d_{\mathcal{E}_2}(X) + d_{\mathcal{E}_1}(X) \geq 2k$ for every $X \in \mathcal{F}$ as S is $2k$-hyperedge-connected in H.

Therefore, we have the desired degree-specified orientation of the bipartite representation B of H. Since every hyperedge vertex has indegree 1 in B, this orientation corresponds to a Steiner rooted k-hyperarc-connected orientation of H. To check the second property of the extension property, we use a similar argument as in Lemma 4.7. The properties that we use are that there are no edges between two Steiner vertices and each Steiner vertex v has indegree $\lfloor d(v)/2 \rfloor$. This finishes the proof of Theorem 4.10. ∎

Since a minimal counterexample $\mathcal{H}$ must satisfy the condition of Theorem 4.10, Theorem 4.10 proves that $\mathcal{H}$ does not exist. So Theorem 4.2 (and hence Theorem 1.1) is proven.

5 Proof of Theorem 1.2

In this section we sketch another application of the Degree-Specified Steiner Orientation problem. The proof of Theorem 1.2 consists of two steps. The first step is to reduce the problem from general graphs to the graphs with no edges between Steiner vertices. This technique was used in [17, 4]. The second step is to reduce the problem in this special instance into the Degree-Specified Steiner Rooted Orientation problem. The idea is that if we specify the indegree of each Steiner vertex to be 1, then a Steiner rooted k-arc-connected orientation is a Steiner rooted k-element-connected orientation, since each Steiner vertex cannot be in two edge-disjoint paths. It turns out that such a degree-specified orientation always exists when S is $2k$-element-connected in G.

Lemma 5.1 *(See also [17, 4].) Given an undirected graph G and a set S of terminal vertices. Suppose S is k-element-connected in G. Then we can construct in polynomial time a graph G' with the following properties:*

1. *there is no edge between Steiner vertices in G';*
2. *S is k-element-connected in G';*
3. *if there is a Steiner rooted k'-element-connected orientation in G' with the indegrees of the Steiner vertices being 1, then there is a Steiner rooted k'-element-connected orientation in G.*

The following lemma can be shown to be a special case of Theorem 4.10. Then Theorem 1.2 follows immediately from Lemma 5.2 and Lemma 5.1.

Lemma 5.2 *Given an undirected graph $G = (V, E)$ and a set S of terminal vertices. If S is $2k$-element-connected in G and there are no edges between vertices in $V(G) - S$, then G has a Steiner rooted k-element-connected orientation with the indegrees of the Steiner vertices being 1.*

6 Hardness Results

Theorem 6.1 *Given a graph G, a set of terminals S, and a root vertex $r \in S$, it is NP-complete to determine if G has a Steiner rooted k-arc-connected orientation.*

Theorem 6.2 *Given a graph G, a set of terminals S, and a root vertex $r \in S$, it is NP-complete to determine if G has a Steiner rooted k-element-connected orientation.*

7 Concluding Remarks

The questions of generalizing Nash-Williams' theorem to hypergraphs and obtaining graph orientations achieving high vertex-connectivity remain wide open. We believe that substantially new ideas are required to solve these problems. The following problem seems to be a concrete intermediate problem which captures the main difficulty: If S is $2k$-element-connected in an undirected graph G, is it true that G has a Steiner strongly k-element-connected orientation? We believe that settling it would be a major step towards the above questions.

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