Packing Steiner Forests

(Extended Abstract)

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Abstract. Given an undirected multigraph G and a set $\mathcal{S} := \{S_1, \ldots, S_t\}$ of disjoint subsets of vertices of G, a Steiner $\mathcal{S}$-forest F is an acyclic subgraph of G such that each S_i is connected in F for $1 \leq i \leq t$. In this paper, we study the STEINER FOREST PACKING problem where we seek a largest collection of edge-disjoint $\mathcal{S}$-forests. The main result is a connectivity-type sufficient condition for the existence of k edge-disjoint $\mathcal{S}$-forest, that yields the first polynomial time approximation algorithm for the STEINER FOREST PACKING problem. We end this paper by a conjecture in a more general setting.

1 Introduction

Given an undirected multigraph G and a set $\mathcal{S} := \{S_1, \ldots, S_t\}$ of disjoint subsets of vertices of G, a Steiner $\mathcal{S}$-forest F is an acyclic subgraph of G such that each S_i is connected in F for $1 \leq i \leq t$. The STEINER FOREST PACKING problem is to find a largest collection of edge-disjoint $\mathcal{S}$-forests. This is a generalization of the STEINER TREE PACKING problem where we are given G and a subset of vertices $S \subseteq V(G)$, and the goal is to find a largest collection of edge-disjoint trees that each connects S. The STEINER TREE PACKING problem is well-studied in the literature as it generalizes the edge-disjoint s,t-paths problem [11] as well as the spanning tree packing problem [14, 12], and also it has applications in network broadcasting and VLSI circuit design. We say a set $S \subseteq V(G)$ is k-edge-connected in G if there are k edge-disjoint paths between u, v for all $u, v \in S$, and a tree is a S-tree if it connects S. A necessary condition for G to have k edge-disjoint S-trees is that S is k-edge-connected in G. The following conjecture by Kriesell [7] gives a sufficient condition for the existence of k edge-disjoint S-trees:

> **Kriesell conjecture:** If S is $2k$-edge-connected in G, then G has k edge-disjoint S-trees.

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This beautiful conjecture is the best possible and has generated much interest [13, 3, 6, 7, 2, 8, 9], recently it is shown in [9] that the conjecture is true if $2k$ is replaced by $26k$.

A related problem is the MINIMUM STEINER FOREST problem, where the goal is to find a minimum-cost $\mathcal{S}$-forest F in G. Goemans and Williamson [4] gave a primal-dual 2-approximation algorithm for the MINIMUM STEINER FOREST problem. Very recently, Chekuri and Shepherd [1] present a new 2-approximation algorithm for the MINIMUM STEINER FOREST problem via the STEINER FOREST PACKING problem. Specifically, they show that given an *Eulerian graph* G, if each S_i is $2k$-edge-connected in G, then there are k edge-disjoint $\mathcal{S}$-forests in G. They then show that this result implies the integrality gap of the standard LP formulation of the MINIMUM STEINER FOREST problem is at most 2.

Motivated by the above results, we study the STEINER FOREST PACKING problem in general graphs. The main result is the following:

Theorem 1. *Given an undirected multigraph G and a set $\mathcal{S} := \{S_1, \ldots, S_t\}$ of disjoint subsets of vertices of G. If each S_i is Qk-edge-connected in G, then there are k edge-disjoint $\mathcal{S}$-forests in G.*

Remark: The best upper bound on Q that we can achieve is 32; however, the proof is fairly involved. In this extended abstract, for the sake of clarity, we only give a proof for $Q = 56$ which contains most of the underlying ideas. We also remark that the proof of Theorem 1 is constructive, and this yields the first polynomial time constant factor approximation algorithm for the STEINER FOREST PACKING problem.

2 Preliminaries

The organization of this section is as follows. We first develop the necessary notations and concepts in Section 2.1. Then, we review the basic theorems that we will be using in Section 2.2. Finally, we give an overview of the proof of our result in Section 2.3.

2.1 Basic definitions

Given an undirected multigraph G and a set $\mathcal{S} := \{S_1, \ldots, S_t\}$ of disjoint subsets of vertices of G. We call each S_i a **group**, note that we can assume that each group is of size at least 2. A subgraph H of G is a **$\mathcal{S}$-subgraph** if each S_i is connected in H for $1 \leq i \leq t$; a subgraph H is a **double $\mathcal{S}$-subgraph** of G if H is a $\mathcal{S}$-subgraph of G and every vertex in $S_1 \cup \ldots \cup S_t$ is of degree at least 2 in H. Given $S \subseteq V(G)$, we say a subgraph H of G is a **S-subgraph** if H is *connected* and S is connected in H; a subgraph H is a **double S-subgraph** of G if H is a S-subgraph of G and every vertex in S is of degree at least 2 in H. A subgraph H **spans** a subset of vertices U if $U \subseteq V(H)$.

In the rest of this paper, we denote $S := S_1 \cup \ldots \cup S_t$. A subset of vertices X is a **$\mathcal{S}$-separating cut** if (i) $S \cap X \neq \emptyset$, $S \cap (V(G) - X) \neq \emptyset$ and (ii) for each S_i, either $S_i \subseteq X$ or $S_i \subseteq V(G) - X$. We denote by $\delta_G(X)$ the set of edges in G with one endpoint in X and the other endpoint in $V(G) - X$, the open neighbourhood set of a vertex v in G by $N_G(v)$, and the induced subgraph on a set of vertices X by $G[X]$. A **core** C is a $\mathcal{S}$-separating cut with $\delta_G(C) \leq Qk$ and $|C|$ minimal. Let R be a specified set of vertices such that $R \cap S = \emptyset$ and each vertex in R is of degree at least Qk. A subset of vertices X is a R-**isolating cut** if $R \cap X \neq \emptyset$ and $S \cap X = \emptyset$.

Given a vertex v, we denote by $E(v)$ the set of edges with an endpoint in v. $\mathcal{P}_k(v) := \{E_1(v), \ldots, E_k(v)\}$ is a **balanced edge-subpartition** of $E(v)$ if (i) $E_1(v) \cup E_2(v) \cup \ldots \cup E_k(v) \subseteq E(v)$, (ii) $|E_i(v)| \geq 2$ for $1 \leq i \leq k$ and (iii) $E_i(v) \cap E_j(v) = \emptyset$ for $i \neq j$. We denote the set of neighbours of u in $E_i(u)$ by $N_{E_i}(u)$. Given k edge-disjoint subgraphs $\{H_1, \ldots, H_k\}$ of G, a vertex v is called **balanced-extendible** with respect to $\{H_1, \ldots, H_k\}$ (or just balanced-extendible if the context is clear) if there exists a balanced edge-subpartition $\{E_1(v), \ldots, E_k(v)\}$ of $E(v)$ so that $H_i \cap E(v) \subseteq E_i(v)$. An easy fact that will be used many times later: if there are $2k$ edges in $E(v)$ not used in any of $\{H_1, \ldots, H_k\}$, then v is balanced-extendible (with respect to $\{H_1, \ldots, H_k\}$). And, of course, if v is of degree at least 2 in each H_i, then v is balanced-extendible.

Cut decomposition: Given a multigraph G a subset of vertices $Y \subset V(G)$, the cut decomposition operation constructs two multigraphs G_1 and G_2 from G as follows. G_1 is obtained from G by contracting $V(G) - Y$ to a single vertex v_1, and keeping all edges from Y to v_1 (even if this produce multiple edges). Similarly, G_2 is obtained from G by contracting Y to a single vertex v_2, and keeping all edges from $V(G) - Y$ to v_2. So, $V(G_1) = Y \cup \{v_1\}, \delta_G(Y) \subseteq E(G_1)$ and $V(G_2) = (V(G) - Y) \cup \{v_2\}, \delta_G(Y) \subseteq E(G_2)$. Notice that for each edge $e \in \delta_G(Y)$, e will appear in both G_1 and G_2 (i.e. e in G_1 is incident with v_1 where e in G_2 is incident with v_2). So, given an edge e incident with v_1 in G_1, we refer to the same edge in G_2 incident with v_2 the **corresponding edge** of e in G_2, and vice versa. The cut decomposition operation will be used several times later, the following are two basic properties of G_1 and G_2:

1. For each pair of vertices u, v in G_i, the number of edge-disjoint paths between u, v in G_i is at least the number of edge-disjoint paths between u, v in G. In particular, if a set S is Qk-edge-connected in G, then $S \cap V(G_1)$ and $S \cap V(G_2)$ are Qk-edge-connected in G_1 and G_2 respectively.
2. The degree of each vertex v in $V(G_i) - \{v_i\}$ is equal to the degree of v in G.

2.2 Basic tools

The following Mader's splitting-off lemma has been a standard tool in many edge-connectivity related problems. Let G be a graph and $e_1 = xy, e_2 = yz$ be two edges of G, the operation of obtaining $G(e_1, e_2)$ from G by replacing $\{e_1, e_2\}$ by a new edge $e' = xz$ is called **splitting off at** y. The splitting-off operation is said to be **suitable**, if the number of edge-disjoint u,v-paths in $G(e_1,e_2)$ is the same as the number of edge-disjoint u,v-paths in G for $u,v \in V(G)-y$.

Lemma 1. (MADER'S SPLITTING LEMMA) *[10] Let x be a vertex of a graph G. Suppose that x is not a cut vertex and that x is incident with at least 4 edges and adjacent to at least 2 vertices. Then there exists a suitable splitting-off operation of G at x.*

By repeatedly applying Mader's splitting-off lemma, one can make the following assumption on the degree of vertices in $V(G)-S$. The proof of Lemma 2 is now standard and appeared in [3, 6, 7, 9], we omit the details here.

Lemma 2. *Every vertex in $V(G)-S$ is incident with exactly three edges and adjacent to exactly three vertices.*

The following are two results on the STEINER TREE PACKING problem which we will be using. Somewhat surprisingly, to prove the result in this paper, we could not use Theorem 3 as a black box. In fact, we need to revise and generalize many parts of the proof of Theorem 3 in order to meet our requirements, this point will be elaborated more in the next subsection.

Theorem 2. *[3] If there is no edge between any two vertices in $V(G)-S$ and S is $3k$-edge-connected in G, then G has k edge-disjoint S-trees.*

Theorem 3. *[9] If S is $26k$-edge-connected in G, then G has k edge-disjoint S-trees.*

2.3 Overview

Here we outline the approach of Chekuri and Shepherd [1]. In [1], they consider the STEINER FOREST PACKING problem when G is Eulerian. This allows them to assume that $V(G)=S$ (recall that $S := S_1 \cup \ldots \cup S_t$) by repeatedly applying the Mader's splitting-off lemma. Then they find a core C in G and show that $G[C]$ has k edge-disjoint spanning trees $T_1,\ldots,T_k$, by using Tutte [14] and Nash-Williams [12] result on spanning tree packing. Now they contract C in G and obtain a new graph G^*. Note that C contains a group and thus has at least two vertices, so G^* has fewer vertices than G. By induction, G^* has k edge-disjoint Steiner forests $F_1,F_2,\ldots,F_k$. Now, as each tree is *spanning* in C, $F_1 \cup T_1,\ldots,F_k \cup T_k$ are the desired k edge-disjoint Steiner forests in G.

However, if G is non-Eulerian, we cannot assume that $V(G)=S$. So, even if we assume the existence of a core C of G so that $G[C]$ has k edge-disjoint Steiner trees $T_1,\ldots,T_k$, and also the existence of k edge-disjoint Steiner forests $F_1,\ldots,F_k$ of G^* as constructed above, the main difficulty is that we cannot guarantee that for all pair of vertices which are connected in F_i are still connected in $F_i \cup T_i$. Here the extension property introduced in [9] comes into the picture. Roughly, we show that there are k edge-disjoint Steiner trees in $G[C]$ that "extend" $F_1,\ldots,F_k$ so that $F_1 \cup T_1,\ldots,F_k \cup T_k$ are actually k edge-disjoint

Steiner forests. Unlike the situation in the Steiner Tree Packing problem, however, we also need to prove structural properties on $F_1, \dots, F_k$ so that they can actually be extended (not every $F_1, \dots, F_k$ can be extended). This requires us to revise and generalize the extension theorem in [9], in particular we need to add an additional constraint on the extension theorem, which causes the constant in Theorem 1 (i.e. approximation ratio) to be slightly bigger than that in Theorem 3.

The organization of the proof is as follows. In Section 3, we prove structural results on R-isolating cut, $\mathcal{S}$-separating cut and core. With these, in Section 4, we show how to reduce a stronger version (see Theorem 4) of Theorem 1 to the Steiner Tree Packing problem with the extension property (defined in subsection 4.1) and an additional requirement (see Theorem 5). Then, in Section 5, we prove the extension theorem (Theorem 5).

3 Structural results

This section contains structural results on R-isolating cut and $\mathcal{S}$-separating cut; these results will be used in Section 4 to reduce the Steiner Forest Packing problem to a modified version of the Steiner Tree Packing problem (see Theorem 5).

3.1 *R*-isolating cut

Let $\mathcal{G}$ be a minimal counterexample of Theorem 1 (or Theorem 4) in the following.

Lemma 3. *$\mathcal{G}$ has no R-isolating cut Y with $|\delta_{\mathcal{G}}(Y)| \leq (Q-2)k$.*

Proof. Suppose not. Consider a R-isolating cut Y with $|\delta_{\mathcal{G}}(Y)|$ minimum. Apply the cut decomposition operation on $\mathcal{G}$ and Y to obtain two graphs G_1 and G_2, let $R_1 := R \cap V(G_1)$ and $R_2 := R \cap V(G_2)$. Also, by definition of Y, $S \subseteq V(G_2)$. From the properties of the cut decomposition operation, S is Qk-edge-connected in G_2 and every vertex of R_2 is of degree at least Qk in G_2. Note that $R_1 \neq \emptyset$, hence $|Y| \geq 2$ as each vertex in R is of degree at least Qk while $|\delta_{\mathcal{G}}(Y)| \leq (Q-2)k < Qk$. As $|Y| \geq 2$, G_2 is smaller than $\mathcal{G}$. By the choice of $\mathcal{G}$, G_2 has k edge-disjoint $\mathcal{S}$-subgraphs $\{H'_1, \dots, H'_k\}$ such that every vertex in R_2 is balanced-extendible.

Let $l := |\delta_{\mathcal{G}}(Y)|$, then $l \leq (Q-2)k$. We claim that each vertex $r \in R_1$ has l edge-disjoint paths to v_1 in G_1. Suppose not, then there exists $Y' \subseteq V(G_1)$ so that $r \in Y'$, $v_1 \notin Y'$ and $|\delta_{G_1}(Y')| < l$. Hence, Y' is a R-isolating cut in $\mathcal{G}$ with $|\delta_{\mathcal{G}}(Y')| < l$, but this contradicts the minimality of $|\delta_{\mathcal{G}}(Y)|$. Therefore, each vertex in R_1 has l edge-disjoint paths to v_1. Choose a vertex $r^* \in R_1$ so that the total length of the l edge-disjoint paths $\{P_1, \dots, P_l\}$ from v_1 to r^* is minimized. Let $P := \{P_1, \dots, P_l\}$ and $H := P_1 \cup P_2 \cup \ldots \cup P_l$. We claim that r is of degree at most l in H for each $r \in R_1$. Suppose not, let $r \in R_1$ be of degree at least $l+1$ in H. Then $r \neq r^*$ and at least $(l+1)/2$ paths from

v_1 to r^* pass through r, say $\{P_1, \ldots, P_{(l+1)/2}\}$. For $1 \le i \le (l+1)/2$, let the subpath of P_i from v_1 to r be P'_i, and the subpath of P_i from r^* to r be P''_i. Then we claim that $H - P''_{(l+1)/2}$ contains l edge-disjoint paths from v_1 to r; indeed, $\{P'_1, \ldots, P'_{(l+1)/2}, P_{(l+1)/2+1} \cup P''_1, \ldots, P_l \cup P''_{(l-1)/2}\}$ are l edge-disjoint paths from v_1 to r in $H - P''_{(l+1)/2}$. This contradicts with the choice of r^* and thus r is of degree at most $l \le (Q-2)k$ for each $r \in R_1$ in H. In particular, each $r \in R_1$ is balanced-extendible with respect to $\{P_1, \ldots, P_l\}$.

Now, we show that the k edge-disjoint double $\mathcal{S}$-subgraphs $\{H'_1, \ldots, H'_k\}$ in G_2 can be extended to k edge-disjoint double $\mathcal{S}$-subgraphs in $\mathcal{G}$, by using the edge-disjoint paths from v_1 to r^* in G_1 constructed in the previous paragraph. For each H'_i in G_2, let $E_i(v_2) := E(H'_i) \cap E(v_2)$ and $E_i(v_1)$ be the corresponding edge set in G_1. Let P_i be the paths in P of G_1 using the corresponding edges in $E_i(v_1)$ (note that $|P_i| = |E_i(v_1)|$), and we set $H_i := H'_i \cup P_i$ to be a subgraph of $\mathcal{G}$. Now each path in H'_i using v_2 in G_2 is still connected in $\mathcal{G}$ via the paths of P_i in G_1. Since H'_i is a double $\mathcal{S}$-subgraph in G_2, H_i is a double $\mathcal{S}$-subgraph in $\mathcal{G}$. Also, since H'_i and H'_j are edge-disjoint in G_2 for $i \neq j$, and P_i and P_j are edge-disjoint in G_1 for $i \neq j$, H_i and H_j are edge-disjoint in $\mathcal{G}$ for $i \neq j$. Finally, since vertices in R_1 and R_2 are balanced-extendible in G_1 and G_2 respectively, vertices in R are balanced-extendible in $\mathcal{G}$ with respect to $\{H_1, \ldots, H_k\}$. As a result, $\{H_1, \ldots, H_k\}$ are k edge-disjoint double $\mathcal{S}$-subgraphs in $\mathcal{G}$ so that every vertex in R is balanced-extendible. This contradiction completes the proof of the lemma. □

3.2 $\mathcal{S}$-separating cut and core

Let C be a core of $\mathcal{G}$. We apply the cut decomposition operation on $\mathcal{G}$ and C to obtain two graphs G_1 and G_2, and let $C \subseteq V(G_1)$, $R_1 := R \cap V(G_1)$ and $S_1 := S \cap V(G_1)$ (recall that $S := S_1 \cup \ldots \cup S_t$). We prove some structural results of G_1 in the following lemma.

Lemma 4. *Let G_1 be defined above. Then S_1 is Qk-edge-connected in G_1 and $S_1 \cup R_1$ is $(Q-2)k$-edge-connected in G_1. Besides, either one of the following must be true:*

1. *$S_1 \cup \{v_1\}$ is Qk-edge-connected in G_1.*
2. *$N_{G_1}(v_1) \subseteq S_1 \cup R_1$.*

Proof. First we prove that S_1 is Qk-edge-connected in G_1 (note that S_1 maybe the union of several groups). Suppose not, then there exists $Y \subseteq V(G_1)$ such that $Y \cap S_1 \neq \emptyset$, $(V(G_1) - Y) \cap S_1 \neq \emptyset$, and $|\delta_{G_1}(Y)| < Qk$. Without loss of generality, we can assume that $v_1 \notin Y$ and hence Y is also a cut in $\mathcal{G}$. Since $|\delta_{G_1}(Y)| < Qk$, each group in G_1 is either contained in Y or disjoint from Y; otherwise this contradicts with the assumption that each group is Qk-edge-connected in $\mathcal{G}$. So, Y is a $\mathcal{S}$-separating cut in $\mathcal{G}$ with $|\delta_{\mathcal{G}}(Y)| < Qk$ and $|Y| < |C|$ (as $(V(G_1) - Y) \cap S_1 \neq \emptyset$). This contradicts with the fact that C is a core, and so S_1 is Qk-edge-connected in G_1.

Next we prove that $S_1 \cup R_1$ is $(Q-2)k$-edge-connected in G_1. Suppose not, then there exists $Y \subseteq V(G_1)$ such that $Y \cap (R_1 \cup S_1) \neq \emptyset$, $(V(G_1) - Y) \cap (R_1 \cup S_1) \neq \emptyset$, and $|\delta_{G_1}(Y)| < (Q-2)k$. Since S_1 is Qk-edge-connected in G_1, either $S_1 \subseteq Y$ or $S_1 \subseteq V(G_1) - Y$. Without loss of generality, we assume that $v_1 \notin Y$ and hence Y is also a cut in $\mathcal{G}$. If $S_1 \subseteq Y$, by a similar argument as in the previous paragraph, Y is a $\mathcal{S}$-separating cut in $\mathcal{G}$ with $|\delta_{\mathcal{G}}(Y)| < Qk$ and $|Y| < |C|$ (as $(V(G_1) - Y) \cap R_1 \neq \emptyset$), and this contradicts with the fact that C is a core. So $S_1 \subseteq V(G_1) - Y$. This, however, implies that Y is a R-isolating cut in $\mathcal{G}$ with $|\delta_{\mathcal{G}}(Y)| < (Q-2)k$, which contradicts Lemma 3. As a result, $S_1 \cup R_1$ is $(Q-2)k$-edge-connected in G_1.

Finally, we prove that if $N_{G_1}(v_1) \nsubseteq S_1 \cup R_1$, then $S_1 \cup \{v_1\}$ is Qk-edge-connected in G_1. First, we show that v_1 must be of degree Qk. Suppose not, then v_1 is a vertex of degree less than Qk. Let $w \in N_{G_1}(w_1)$ be a vertex in $V(G_1) - S_1 - R_1$. Since w is of degree 3 by Lemma 2, $|\delta_{\mathcal{G}}(C - w)| \leq Qk$ and hence $C - w$ is also a $\mathcal{S}$-separating cut which contradicts the fact that C is a core. So, v_1 is of degree exactly Qk. Suppose indirectly that $S_1 \cup \{v_1\}$ is not Qk-edge-connected in G_1, then there exists $Y \subseteq V(G_1)$ such that $v_1 \notin Y$, $Y \cap S_1 \neq \emptyset$ and $|\delta_{G_1}(Y)| < Qk$. Since S_1 is Qk-edge-connected in G_1, we have $S_1 \subseteq Y$. Also, since v_1 is of degree Qk but $|\delta_{G_1}(Y)| < Qk$, we have $|V(G_1) - Y| \geq 2$ and hence $|Y| < |C|$. This implies that Y is also a $\mathcal{S}$-separating cut which contradicts with the fact that C is a core. Therefore, if $N_{G_1}(v_1) \nsubseteq S_1 \cup R_1$, then $S_1 \cup \{v_1\}$ is Qk-edge-connected in G_1. This completes the proof of the lemma. □

4 Proof of the main theorem

We are going to prove the following stronger theorem, which helps us to reduce the Steiner Forest Packing problem to a modified version of the Steiner Tree Packing problem (see Theorem 5).

Theorem 4. *Given G, $\mathcal{S} := \{S_1, \ldots, S_t\}$, $R \subseteq V(G)$. If each S_i is Qk-edge-connected in G and each vertex in R is of degree at least Qk, then there are k edge-disjoint double $\mathcal{S}$-subgraphs $\{H_1, \ldots, H_k\}$ of G so that every vertex in R is balanced-extendible with respect to $\{H_1, \ldots, H_k\}$.*

Theorem 4 immediately implies Theorem 1 (just set $R := \emptyset$). In the following, we consider a minimal counterexample $\mathcal{G}$ of Theorem 4, and show that it does not exist.

4.1 The extension theorem

The extension property defined below is crucial in applying divide-and-conquer strategy to decompose the original problem instance to smaller instances with restricted structures.

Definition 1. (The Extension Property) *Given G, $S \subseteq V(G)$, and an edge-subpartition $\mathcal{P}_k(v) := \{E_1(v), \ldots, E_k(v)\}$ of a vertex v. $\{H_1, \ldots, H_k\}$ are k*

edge-disjoint double S-subgraphs that **extend** $\mathcal{P}_k(v)$ *if for* $1 \le i \le k$*:*
(1) $E_i(v) \subseteq E(H_i)$*;*
(2) $H_i - v$ *is a double* $(S - v)$*-subgraph that spans* $N_{E_i}(v)$.

The following lemma illustrates how to use the cut decomposition operation to reduce the Steiner Tree Packing problem (with the extension property and the balanced-extendible constraint) in a graph to the same problem in two smaller graphs.

Lemma 5. *Given* G*,* $S, R \subseteq V(G)$*,* $v \in S$ *and an edge-subpartition* $\mathcal{P}_k(v)$ *of* v*. Let* G_1 *and* G_2 *be two graphs obtained from the cut decomposition operation of a graph* G *and assume* $v \in G_2$*, let* $S_1 := S \cap V(G_1), S_2 := S \cap V(G_2), R_1 := R \cap V(G_1)$*, and* $R_2 := R \cap V(G_2)$*. Suppose* $\{H_1^2, \ldots, H_k^2\}$ *are* k *edge-disjoint double* S_2*-subgraphs of* G_2 *that extend* $\mathcal{P}_k(v)$ *so that every vertex in* R_2 *is balanced-extendible. Let* $\mathcal{P}_k(v_2) := \{H_1^2 \cap E(v_2), \ldots, H_k^2 \cap E(v_2)\}$ *and* $\mathcal{P}_k(v_1)$ *be the corresponding edge-subpartition of* v_1 *in* G_1*. If* $\{H_1^1, \ldots, H_k^1\}$ *are* k *edge-disjoint double* S_1*-subgraphs of* G_1 *that extend* $\mathcal{P}_k(v_1)$ *so that every vertex in* R_1 *is balanced-extendible, then* $\{H_1^1 \cup H_1^2, \ldots, H_k^1 \cup H_k^2\}$ *are* k *edge-disjoint double* S*-subgraphs of* G *that extend* $\mathcal{P}_k(v)$ *so that every vertex in* R *is balanced-extendible.*

Proof. The statement of the lemma is somewhat long, but the proof is quite straightforward and a similar proof (without the balanced-extendible constraint) is available in Lemma 3.2 of [9]. We omit the proof here for the sake of space. □

Of course, not every edge-subpartition of an arbitrary vertex can be extended. The following extension theorem, which is at the heart of the proof of Theorem 4, gives a sufficient condition for an edge-subpartition of a vertex to be extendible. The proof of Theorem 5 is in Section 5.

Theorem 5. *Given* G*,* $S, R \subseteq V(G)$*. If* S *is* Qk*-edge-connected in* G *and* $S \cup R$ *is* $(Q-2)k$*-edge-connected in* G*, then there are* k *edge-disjoint double* S*-subgraphs in* G *such that every vertex in* R *is balanced-extendible. Furthermore, given a balanced edge-subpartition* $\mathcal{P}_k(v)$ *of a vertex* $v \in S$ *of degree* Qk*, then there are* k *edge-disjoint double* S*-subgraphs that extend* $\mathcal{P}_k(v)$ *such that every vertex in* R *is balanced-extendible.*

4.2 Proof of Theorem 4

Now, with Theorem 5, we are ready to prove Theorem 4.

Proof. First suppose that S is Qk-edge-connected in $\mathcal{G}$. Then, by Lemma 3, $S \cup R$ is $(Q-2)k$-edge-connected in $\mathcal{G}$. Hence, Theorem 5 (without using the extension property) implies the theorem in this case.

So S is not Qk-edge-connected in $\mathcal{G}$, then a core C exists. We apply the cut decomposition operation on $\mathcal{G}$ and C to obtain two graphs G_1 and G_2, and assume without loss of generality that $C \subset V(G_1)$. Let $S_1 := S \cap V(G_1), S_2 := S \cap V(G_2) R_1 := R \cap V(G_1), R_2 := R \cap V(G_2)$, and $\mathcal{S}_1$ and $\mathcal{S}_2$ be the groups contained in S_1 and S_2 respectively. By the properties of the cut decomposition operation, each group in $\mathcal{S}_1$ and $\mathcal{S}_2$ is Qk-edge-connected in G_1 and G_2 respectively. Also, vertices in R_1 and R_2 are of degree at least Qk in G_1 and G_2 respectively. By the choice of $\mathcal{G}$, there are k edge-disjoint double $\mathcal{S}_2$-subgraphs $\{H_1^2, \dots, H_k^2\}$ in G_2 such that every vertex in R_2' is balanced-extendible. Based on Lemma 4, we distinguish two cases based on the possible structures of v_1.

The first case is that $S_1 \cup \{v_1\}$ is Qk-edge-connected in G_1, let $S_1' := S_1 \cup \{v_1\}$ and $R_2' := R_2 \cup \{v_2\}$. Let $\mathcal{P}_k^2(v_2) := \{H_1^2 \cap E(v_2), \dots, H_k^2 \cap E(v_2)\}$ be the edge-subpartition on v_2 induced by $\{H_1^2, \dots, H_k^2\}$ in G_2, and $\mathcal{P}_k^1(v_1)$ be the corresponding edge-subpartition defined on $E(v_1)$ in G_1. Since $v_2 \in R_2'$, v_2 is balanced-extendible with respect to $\{H_1^2, \dots, H_k^2\}$. Now, we apply Theorem 5 on G_1 with S_1', R_1 and $\mathcal{P}_k^1(v_1)$ to get k edge-disjoint double $\mathcal{S}_1'$-subgraphs $\{H_1^1, \dots, H_k^1\}$ in G_1 that extend $\mathcal{P}_k^1(v_1)$, so that every vertex in R_1 is balanced-extendible. Then, by Lemma 5, we obtain k edge-disjoint double $\mathcal{S}$-subgraphs $\{H_1, \dots, H_k\}$ in $\mathcal{G}$ such that every vertex in R is balanced-extendible.

The second case is that $N_{G_1}(v_1) \subseteq S_1 \cup R_1$. Consider $G_1 - v_1$. Since v_1 is of degree at most Qk, $S_1 \cup R_1$ is at least $(Q-2)k - Qk/2 \geq 26k$-edge-connected in G_1. By Theorem 3, G_1 has k edge-disjoint double $S_1 \cup R_1$-subgraphs $\{H_1', \dots, H_k'\}$ in $G_1 - v_1$. Setting $H_i^1 := H_i' \cup E_i$, $\{H_1^1, \dots, H_k^1\}$ are k edge-disjoint double $S_1 \cup R_1$-subgraphs in G_1 that extend $\mathcal{P}_k(v_1)$. Then, by Lemma 5, we obtain k edge-disjoint double $\mathcal{S}$-subgraphs $\{H_1, \dots, H_k\}$ in $\mathcal{G}$ such that every vertex in R is balanced-extendible.

By Lemma 4, these are the only two cases that could happen in G_1. Therefore, $\mathcal{G}$ is not a counterexample and the theorem follows. $\square$

5 Proof of the extension theorem

In this section, we will prove Theorem 5 by showing that a minimal counterexample $\mathcal{G}$ of Theorem 5 does not exist. In Section 5.1, we first prove that there is no edge between two vertices in $V(\mathcal{G}) - S - R$, which allow us to apply Theorem 2 to prove Theorem 5. Then, in Section 5.2, we construct an auxiliary graph G' from $\mathcal{G}$ and show some properties of G'. We then introduce the concept of diverging paths and common paths in Section 5.3. Finally, we distinguish two cases of the structure of G' and show that each is impossible in Section 5.4 and Section 5.5.

5.1 There is no edge between two vertices in $V(\mathcal{G}) - S - R$

Lemma 6. *There is no edge in $\mathcal{G}$ with both endpoints in $V(\mathcal{G}) - S - R$.*

Proof. Suppose indirectly that e is such an edge. If S is Qk-edge-connected and $S \cup R$ is $(Q-2)k$-edge-connected in $\mathcal{G} - e$, then by the choice of $\mathcal{G}$, $\mathcal{G} - e$ satisfies Theorem 5 and thus $\mathcal{G}$. So we may assume that either (i) there exists $Y \subseteq V(\mathcal{G})$ and $e \in \delta_{\mathcal{G}}(Y)$ so that $Y \cap S \neq \emptyset$, $(V(\mathcal{G}) - Y) \cap S \neq \emptyset$ and $|\delta_{\mathcal{G}}(Y)| = Qk$; or (ii) there exists $Y \subseteq V(\mathcal{G})$ and $e \in \delta_{\mathcal{G}}(Y)$ so that $S \subseteq Y$, $(V(\mathcal{G}) - Y) \cap R \neq \emptyset$ and

$|\delta_{\mathcal{G}}(Y)| = (Q-2)k$. In either case, we apply the cut decomposition operation on $\mathcal{G}$ and Y to obtain two graphs G_1 and G_2, and let $Y \subset G_1$, $S_1 := S \cap V(G_1)$, $R_1 := R \cap V(G_1)$, $S_2 := S \cap V(G_2)$ and $R_2 := R \cap V(G_2)$. Notice that since both G_1 and G_2 contain a vertex in $V(\mathcal{G}) - S - R$ (an endpoint of e), both G_1 and G_2 are smaller than $\mathcal{G}$ and hence they both satisfy Theorem 5.

We first consider case (i). In this case, S_1 and S_2 are non-empty. Hence, $S_1 \cup \{v_1\}$ and $S_2 \cup \{v_2\}$ are Qk-edge-connected in G_1 and G_2, respectively. We let $S_1' := S_1 \cup \{v_1\}$ and $S_2' := S_2 \cup \{v_2\}$. Without loss of generality, we assume that v is in G_2, where v is the vertex to be extended in $\mathcal{G}$. By the choice of $\mathcal{G}$, there are k edge-disjoint double S_2'-subgraphs $\{H_1^2, \ldots, H_k^2\}$ in G_2 that extend $\mathcal{P}_k(v)$ such that every vertex in R_2 is balanced-extendible. Since $v_2 \in S_2'$, $\mathcal{P}_k(v_2) := \{H_1^2 \cap E_{G_2}(v_2), \ldots, H_k^2 \cap E_{G_2}(v_2)\}$ is a balanced edge-subpartition of v_2. Let $\mathcal{P}_k(v_1)$ be the corresponding balanced edge-subpartition of v_1 in G_1. Again, by the choice of $\mathcal{G}$, there are k edge-disjoint double S_1'-subgraphs $\{H_1^1, \ldots, H_k^1\}$ in G_1 that extend $\mathcal{P}_k(v_1)$ such that every vertex in R_1 is balanced-extendible. Now, by applying Lemma 5, there are k edge-disjoint double S-subgraphs in $\mathcal{G}$ that extend $\mathcal{P}_k(v)$ such that every vertex in R is balanced-extendible.

Now consider case (ii). Notice that we must have that $v \in G_1$, since $v \in S$ and we assume $S \subseteq Y$ and $Y \subset V(G_1)$. Then $V(\mathcal{G}) - Y$ is a R-isolating cut which does not contain v in $\mathcal{G}$. By using exactly the same argument as in Lemma 3, we can show that this contradicts with the fact that $\mathcal{G}$ is a minimal counterexample. So case (ii) cannot happen either, and this completes the proof. □

5.2 Construction and properties of G'

The case when $|S \cup R| = 2$ is trivial. Henceforth, we assume that $|S \cup R| \geq 3$. Our goal is to show that $\mathcal{G}$ has k edge-disjoint double S-subgraphs that extend $\mathcal{P}_k(v)$ of v such that every vertex in R is balanced-extendible. Let W be the set of neighbours of v in $V(\mathcal{G}) - S - R$ and B be the set of neighbours of v in $S \cup R$. By Lemma 2, each $w_i \in W$ is incident with exactly three edges and adjacent to exactly three vertices, so we let $N_{\mathcal{G}}(w_i) := \{v, x_i, y_i\}$ and call $\{x_i, y_i\}$ a **couple**. By Lemma 6, x_i and y_i are in $S \cup R$. For each $b_i \in B$, we denote by $c(b_i)$ the number of multiple edges between v and b_i.

Let G' be $\mathcal{G} - v - W$. Let Z be a minimum $(S \cup R - v)$-cut of G' and $\{C_1, \ldots, C_l\}$ be the connected components of $G' - Z$. We let $S_i := S \cap V(C_i)$, $R_i := R \cap V(C_i)$ and $B_i := B \cap V(C_i)$. Also, $c(B_i)$ denotes the sum of the $c(b)$ for $b \in B_i$ and X_i denotes the collection of couples with both vertices in C_i. By the minimality of Z, each edge e in Z connects two vertices in different components, and we call it a **crossing edge**. Similarly, a couple $\{x_i, y_i\}$ is a **crossing couple** if x_i and y_i are in different components, and we denote the collection of crossing couples by X_C.

Lemma 7. *$(S \cup R - v)$ is at most $(6k-1)$-edge-connected in G'.*

Proof. Since $|S \cup R| \geq 3$, $|S \cup R - v| \geq 2$. If $(S \cup R - v)$ is $6k$-edge-connected in G', then by Theorem 2, there are $2k$ edge-disjoint $(S \cup R - v)$-subgraphs

$\{H_1'', \ldots, H_{2k}''\}$ in G'. Notice that since the union of two edge-disjoint $(S \cup R - v)$-subgraphs is a double $(S \cup R - v)$-subgraph, by setting $H_i' := H_{2i-1}' \cup H_{2i}'$, $\{H_1', \ldots, H_k'\}$ are k edge-disjoint double $(S \cup R - v)$-subgraphs of G'. Now, let $H_i := H_i' \cup \{vb_j | vb_j \in E_i(v)\} \cup \{vw_j, w_j x_j | vw_j \in E_i(v)\}$. So, $E_i(v) \subseteq H_i$, and $H_i - v$ is a double $(S \cup R - v)$-subgraph that spans $N_{E_i}(v)$. Also, since $|E_i(v)| \geq 2$, H_i is a double $(S \cup R)$-subgraph of $\mathcal{G}$. By Definition 1, $\{H_1, \ldots, H_k\}$ are k edge-disjoint double $(S \cup R)$-subgraphs of $\mathcal{G}$ that extend $\mathcal{P}_k(v)$, a contradiction. □

Lemma 8. *$G' - Z$ has 2 connected components.*

Proof. We just need to show that G' has at most 2 connected components, then the statement that $G' - Z$ has 2 connected components follows from the minimality of Z. Notice that from our construction of G' from $\mathcal{G}$, the set of neighbours of every vertex in $V(\mathcal{G}) - S - R$ that remained in G' is the same as in $\mathcal{G}$. Since $\mathcal{G}$ is connected, no component in G' contains only vertices in $V(\mathcal{G}) - S - R$. Therefore, it remains to show that there are at most two components in G' that contain vertices in $S \cup R$.

Suppose there are three connected components containing vertices in $S \cup R$. Let $u_1, u_2, u_3 \in S \cup R$ be vertices in C_1, C_2, C_3 respectively. Since u_1, u_2, u_3 have at least $(Q-2)k$ edge-disjoint paths to v and v is of degree Qk, there exists a vertex $w \in N_{\mathcal{G}}(v)$ such that u_1, u_2, u_3 all have a path to w in $\mathcal{G}$. If $w \in S \cup R$, then clearly u_1, u_2, u_3 are still connected in G', a contradiction. If $w \in V(\mathcal{G}) - S - R$, then w is of degree 3 by Lemma 2. Then there must exist a pair of vertices, say u_1 and u_2, both have a path to the same neighbour of w in G', a contradiction. □

5.3 Diverging paths and common paths

Consider a vertex $u \neq v$ where $u \in S$. Since S is Qk-edge-connected in $\mathcal{G}$, by Menger's theorem, there are Qk edge-disjoint paths, denoted by $P(u) := \{P_1(u), \ldots, P_{Qk}(u)\}$, from u to v. Note that since v is of degree exactly Qk, each path in $P(u)$ uses exactly one edge in $E(v)$. Furthermore, since w_i is of degree 3 by Lemma 2, each w_i is used by exactly one path in $P(u)$. Similarly, for $u \neq v$ where $u \in R$. There are $(Q-2)k$ edge-disjoint paths from u to v denoted by $P(u) := \{P_1(u), \ldots, P_{(Q-2)k}(u)\}$. Again, we may assume that each path use exactly one edge in $E(v)$ and at most one vertex in W. Consider $P_i(u)$ induced in G', denoted by $P_i'(u)$. Let $P'(u) := \{P_1'(u), \ldots, P_{Qk}'(u)\}$ for $u \in S$, and similarly $P'(u) := \{P_1'(u), \ldots, P_{(Q-2)k}'(u)\}$ for $u \in R$. Notice that $P'(u)$ contains edge-disjoint paths in G', and we call them the **diverging paths** from u.

We plan to use the diverging paths from a and b for any two vertices $a, b \in S \cup R$ in the same component of $G' - Z$ to establish the connectivity of $S \cup R$ in each component of $G' - Z$. We say v_1 and v_2 have λ **common paths** if there are λ edge-disjoint paths starting from v_1, λ edge-disjoint paths starting from v_2, and an one-to-one mapping of the paths from v_1 to the paths from v_2 so that each pair of paths in the mapping ends in the same vertex. The following lemma gives a lower bound on the number of edge-disjoint paths between two vertices based on the number of their common paths.

Lemma 9. *[9] If v_1 and v_2 have $2\lambda+1$ common paths in G, then there exist $\lambda+1$ edge-disjoint paths from v_1 to v_2 in G.*

5.4 Both components of $G'-Z$ contain vertices in S

In this subsection, we consider the case that both components contain some vertices in S. The lemmas in this section all share this assumption.

Lemma 10. *If both components of $G'-Z$ contain some vertices in S, then there are at least $Qk-2|Z|$ crossing couples, that is, $|X_C| \geq Qk-2|Z|$.*

Proof. Let $u_1 \in S$ be in C_1. In G', u_1 has at least $c(B_2)+|X_2|$ edge-disjoint paths in $P'(u_1)$ to C_2. Since Z is an edge-cut in G', it follows that $c(B_2)+|X_2| \leq |Z|$. Similarly, by considering a vertex $u_2 \in S$ in C_2, we have $c(B_1)+|X_1| \leq |Z|$. By Lemma 8, there are only two components in $G'-Z$. So $Qk = |X_C|+|X_1|+|X_2|+c(B_1)+c(B_2)$, and we have $|X_C| \geq Qk-2|Z|$. □

Lemma 11. *If both components of $G'-Z$ contain some vertices in S, then $S_i \cup R_i$ is $(Q/2-14)k$-edge-connected in C_i of $G'-Z$.*

Proof. Consider any two vertices $a, b \in S_i \cup R_i$ in C_i. In G', $P'(a)$ has at least $|X_C|-2k$ paths to different crossing couples. Among those paths, at most $|Z|$ of them may use edges in Z. So, in $G'-Z$, a has at least $|X_C|-2k-|Z|$ edge-disjoint paths such that each starts at a and ends in different crossing couples, and similarly for b. Therefore, in $G'-Z$, a and b have at least $(|X_C|-2k-|Z|)+(|X_C|-2k-|Z|)-(|X_C|) = |X_C|-4k-2|Z| \geq Qk-4k-4|Z| \geq Qk-28k$ common paths in C_i; the second last and the last inequality hold because of Lemma 10 and Lemma 7, respectively. By Lemma 9, a and b are $(Q/2-14)k$-edge-connected in C_i. □

Lemma 12. *If both components of $G'-Z$ contain some vertices in S, then $\mathcal{G}$ has k edge-disjoint double S-subgraphs $\{H_1, \ldots, H_k\}$ that extend $\mathcal{P}_k(v)$ such that every vertex in R is balanced-extendible.*

Proof. (Sketch) We pick arbitrarily $\min\{k, |Z|\}$ edges in Z and call them the *connecting edges*. For each connecting edge e with a endpoint $w \in V(\mathcal{G})-S-R$ in C_i, we remove one edge e' in C_i which is incident with w (by Lemma 6, the other endpoint of e' must be black), and we call e' a *reserve edge* of e. Let the resulting component be C'_i. Since we remove at most k edges and $S_i \cup R_i$ is $(Q/2-14)k$-edge-connected in C_i by Lemma 11, each $S_i \cup R_i$ is $(Q/2-15)k$-edge-connected in C'_i. In particular, each $S_i \cup R_i$ is $6k$-edge-connected in C'_i. By Theorem 2, there are $2k$ edge-disjoint $(S_i \cup R_i)$-subgraphs in C'_i. So there are k edge-disjoint double $(S_i \cup R_i)$-subgraphs $\{H^i_1, \ldots, H^i_k\}$ in each C'_i for $i \in \{1, 2\}$.

Now we set $H_j := H^1_j \cup H^2_j \cup \{vb_i | vb_i \in E_j(v)\} \cup \{vw_i, w_ix_i, w_iy_i | vw_i \in E_j(v)\}$ for $1 \leq j \leq k$. Notice that $E_j(v) \subseteq E(H_j)$ and $H_j - v$ spans $N_{E_j}(v)$ for $1 \leq j \leq k$. Suppose there is a crossing couple $\{x_i, y_i\}$ such that $vw_i \in E_j(v)$, then H_j is also connected and thus is a $(S \cup R)$-subgraph of $\mathcal{G}$ that $E_j(v) \subseteq E(H_j)$ and $H_j - v$ is a

$(S \cup R - v)$-subgraph that spans $N_{E_j}(v)$. Let's assume that $\{vw_1, \ldots, vw_{|X_C|}\}$ be the set of edges such that the corresponding couples are crossing. By Lemma 10, $|X_C| \geq Qk - 2|Z|$. Since $\mathcal{P}_k(v)$ is a balanced edge-subpartition, $|E_i(v)| \geq 2$ for $1 \leq i \leq k$. So, there are at most $\min\{k, |Z|\}$ classes of $\mathcal{P}_k(v)$ with no edges in $\{vw_1, \ldots, vw_{Qk-2|Z|}\}$. Hence there are at most $\min\{k, |Z|\}$ of H_j's, say $\{H_1, \ldots, H_{\min\{k,|Z|\}}\}$, are not connected by the crossing couples. Now, by adding each connecting edge and its reserve edge (if any) to a different H_j that has not been connected by a crossing couple, $\{H_1, \ldots, H_k\}$ are k edge-disjoint double $(S \cup R)$-subgraphs of $\mathcal{G}$ that extend $\mathcal{P}_k(v)$. (We skip the not-so-interesting case that $|S_i| = 1$ for some $i \in \{1, 2\}$, which needs to be handled separately.) □

5.5 One component contains only vertices in R

Without loss of generality, we assume that C_1 contains vertices in S and C_2 contains only vertices in R. Lemma 13 and Lemma 14 are counterparts of Lemma 10 and Lemma 11, we omit the proofs here. However, it should be pointed out that we only have a weaker bound on the number of crossing couples in Lemma 13, and hence the strategy in Lemma 12 cannot be used. In Lemma 15, we use a different strategy to construct the desired subgraphs.

Lemma 13. *If C_2 contains only vertices in R, then there are at least $(Q-2)k - 2|Z|$ crossing couples, that is, $|X_C| \geq (Q-2)k - 2|Z|$.*

Lemma 14. *If C_2 contains only vertices in R, then $S_1 \cup R_1$ is at least $(Q/2 - 15)k$-edge-connected in C_1 of $G' - Z$.*

Lemma 15. *If C_2 contains only vertices in R, then $\mathcal{G}$ has k edge-disjoint double S-subgraphs that extend $\mathcal{P}_k(v)$ such that every vertex in R is balanced-extendible.*

Proof. Let $E' := \{e_1, \ldots, e_{|X_2|+c(B_2)}\}$ be the set of edges incident to v in $\mathcal{G}$ so that either (i) the other endpoint of e_i is in B_2 or (ii) the other endpoint of e_i has both of its neighbour in C_2. Let $u_1 \in S$. From $P(u_1)$ in $\mathcal{G}$, there are $|X_2| + c(B_2)$ edge-disjoint paths from u_1 to v such that each uses exactly one edge in E'. From these paths, in $\mathcal{G} - C_1$, there are $|X_2| + c(B_2)$ edge-disjoint paths $P := \{P_1, \ldots, P_{|X_2|+c(B_2)}\}$ with the following property: each P_i starts from v and ends in some vertex of C_1, and $e_i \in P_i$. Since each vertex in $w \in V(C_1) - S_1 - R_1$ is of degree 3, by the minimality of $|Z|$, it has at most one neighbour in C_2. Therefore, each $w \in V(C_1)$ can be in at most one path in P. Now, for each $u_i \in V(C_1) - S_1 - R_1$, we remove one edge e' in C_1 which is incident with w and set $P'_i := P_i \cup \{e'\}$ (by Lemma 6, the other endpoint of e' must be in $S_1 \cup R_1$); otherwise, $P'_i := P_i$. So $P' := \{P'_1, \ldots, P'_{|X_2|+c(B_2)}\}$ are edge-disjoint paths with the following property: each P'_i starts from v and ends in some vertex $u_i \in S_1 \cup R_1$, and $e_i \in P'_i$. In constructing P' from P, we remove at most $|X_2| + c(B_2) \leq |Z| \leq 6k$ edges from C_1. Let the resulting component be C'_1. Since $S_1 \cup R_1$ is $(Q/2-15)k$-edge-connected in C_1 by Lemma 14, $S_1 \cup R_1$ is $(Q/2-21)k$-edge-connected in C'_1. In particular, C'_1 is $S_1 \cup R_1$ is $6k$-edge-connected in C'_1. By Theorem 2, there are $2k$ edge-disjoint $(S_1 \cup R_1)$-subgraphs in C'_1. So there are k edge-disjoint double $(S_1 \cup R_1)$-subgraphs $\{H_1^1, \ldots, H_k^1\}$ in C_1'. Now, we set $H_j := H_j^1 \cup \{vb_i | vb_i \in E_j(v)\} \cup \{vw_i, w_ix_i, w_iy_i | vw_i \in E_j(v)\} \cup \{P_i | e_i \in E_j(v)\}$. Then, it is straightforward to check $\{H_1, \ldots, H_k\}$ are k edge-disjoint double S-subgraphs that extend $\mathcal{P}_k(v)$ so that every vertex in R is balanced-extendible. Indeed, every vertex in R_1 is balanced-extendible because it is of degree at least 2 in each H_i^1 and thus H_i; while every vertex in R_2 is balanced-extendible because it has degree at most $12k$ in H_j, as it is used by at most $6k$ paths from P'. This completes the proof of the lemma. □

Putting Lemma 12 and Lemma 15 together shows that a minimal counterexample $\mathcal{G}$ of Theorem 5 does not exist, and this completes the proof of Theorem 5. □

6 Concluding Remarks

As far as the algorithmic aspects go, it is straightforward to check that the proof yields a polynomial time constant factor approximation algorithm for the Steiner Forest Packing problem. Also, using the same technique as in [9], the approximation algorithm can be extended to the capacitated version of the Steiner Forest Packing, where each edge e has an capacity c_e so that at most c_e forests can use e (for the original problem, $c_e = 1$ for all $e \in E(G)$) .

The following is a general problem that captures the Steiner Forest Packing problem. Given an undirected multigraph G and a connectivity requirement r_{uv} for each pair of vertices $u, v \in V(G)$, find a largest collection of edge-disjoint subgraphs of G such that in each subgraph there are r_{uv} edge-disjoint paths from u to v for all $u, v \in V(G)$. Since connectivity is transitive, it is not difficult to see that (see [1]) Theorem 1 is equivalent to the following:

Theorem 6. *Given an undirected multigraph G and a connectivity requirement $r_{uv} \in \{0, 1\}$ for $u, v \in V(G)$. If there are $Qk \cdot r_{uv}$ edge-disjoint paths for all $u, v \in V(G)$, then there are k edge-disjoint forests such that in each forest there is r_{uv} path between u, v for all $u, v \in V(G)$.*

I conjecture that Theorem 6 can be generalized to arbitrary non-negative integer connectivity requirements:

Conjecture 1. Given an undirected multigraph G and a connectivity requirement r_{uv} for each pair of vertices $u, v \in V(G)$. There exists a universal constant c so that the following holds. If there are $ck \cdot r_{u,v}$ edge-disjoint paths for all $u, v \in V(G)$, then there are k edge-disjoint subgraphs $H_1, \ldots, H_k$ in G such that in each subgraph there are r_{uv} edge-disjoint paths between u and v for all $u, v \in V(G)$.

It would be interesting to first verify Conjecture 1 for Eulerian graphs (with $c = 2?$). This may also yield insights into the generalized Steiner network problem, for which the only constant factor approximation algorithm is due to Jain's iterative rounding technique [5].

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