On disjoint common bases in two matroids

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Abstract

We prove two results on packing common bases of two matroids. First, we show that the computational problem of common base packing reduces to the special case where one of the matroids is a direct sum of uniform matroids. Second, we give a counterexample to a conjecture of Chow, which proposed a sufficient condition for the existence of a common base packing. Chow's conjecture is a generalization of Rota's basis conjecture.

1 Introduction

The problem of packing bases in a matroid was considered in classical work of Edmonds [6] [7, Application 2]. He characterized when such a packing is possible and gave efficient algorithms to find such a packing. We consider the following generalization.

Problem 1. Let $\mathbf{M}_1 = (S, \mathcal{I}_1)$ and $\mathbf{M}_2 = (S, \mathcal{I}_2)$ be matroids on ground set S, where $\mathcal{I}_1$ and $\mathcal{I}_2$ are the respective families of independent sets. A set $B \subseteq S$ that is both a base of $\mathbf{M}_1$ and of $\mathbf{M}_2$ is called a ***common base***. The problem is to decide if S can be partitioned into common bases.

The computational complexity of Problem 1 is unclear. In particular, the answers to the following questions are unknown.

- If each matroid is given by an oracle that tests independence in the matroid, is there an algorithm that solves the problem using a number of queries that is polynomial in $|S|$?
- If each matroid is linear and given by an explicit matrix representing the matroid, is there an algorithm that solves the problem using a number of steps that is polynomial in the size of this matrix?

Two well-studied special cases of this problem include edge-coloring bipartite graphs and packing arborescences in digraphs. For these two special cases, both of these questions have a positive answer; this follows from results of Kőnig [14], Tarjan [19] and Lovász [15]. The latter two results give an efficient, constructive proof of a min-max relation originally proved by Edmonds [9].

Another problem related to packing common bases is Rota's basis conjecture.

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Conjecture 2 (Rota, 1989). Let $\mathbf{M} = (T, \mathcal{I})$ be a matroid of rank n. Let $A_1, \ldots, A_n$ be a partition of T into bases of $\mathbf{M}$. Then there are disjoint bases $B_1, \ldots, B_n$ such that $|A_i \cap B_j| = 1$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$.

Rota's conjecture is stated in the work of Huang and Rota [13, Conjecture 4] and remains open. It can be restated in a way that emphasizes its connection to Problem 1. Let $\mathbf{M}_1$ be a matroid of rank n and let $A_1, \ldots, A_n$ be disjoint bases of $\mathbf{M}$. Let $\mathbf{M}_2$ be the direct sum of uniform rank-1 matroids on the sets $A_1, \ldots, A_n$. The conjecture asserts that the solution to Problem 1 for $\mathbf{M}_1$ and $\mathbf{M}_2$ is "yes".

Recently, Chow [1] proposed the following generalization of Rota's conjecture.

Conjecture 3. Let $\mathbf{M} = (T, \mathcal{I})$ be a matroid of rank n with the property that T can be partitioned into b bases, where $3 \leq b \leq n$. Let $I_1, \ldots, I_n \in \mathcal{I}$ be disjoint independent sets, each of size at most b. Then there exists a partition of T into sets $A_1, \ldots, A_n$ such that $I_i \subseteq A_i$ and $|A_i| = b$ for every $i = 1, \ldots, n$, and there exist disjoint bases $B_1, \ldots, B_b$ such that $|A_i \cap B_j| = 1$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, b$.

For the remainder of this paper, we will only consider the special case of Chow's conjecture in which $|I_i| = b$ and hence $A_i = I_i$ for every $i = 1, \ldots, n$.

Obviously Chow's conjecture implies Rota's conjecture, by setting $b = n$. A stronger statement is also true: Chow [1] proved that, for *every* value of b, his conjecture implies Rota's conjecture. In particular, this suggests an approach to proving Rota's conjecture, which is to prove Chow's conjecture for the special case $b = 3$. Note that Conjecture 3 is not true if $b = 2$, as is shown by a well-known instance based on the graphic matroid of the complete graph K_4. See, e.g., [5], [16, Exercise 12.3.11(ii)] or [18, Section 42.6c].

This paper contains two related results. First, we give a reduction from Problem 1 for arbitrary $\mathbf{M}_1$ and $\mathbf{M}_2$ to the same problem for new matroids $\mathbf{M}'_1$ and $\mathbf{M}'_2$ where $\mathbf{M}'_2$ is a direct sum of uniform matroids. As will be clear later, it is not possible to apply the reduction twice so that both matroids become direct sums of uniform matroids. Our reduction is efficiently computable, implying the following statement.

Theorem 4. Problem 1 can be solved in polynomial time if and only if this is true under the additional assumption that one of the matroids is a direct sum of uniform matroids.

This shows that the computational difficulty of Problem 1 does not stem from the interaction of two potentially complicated matroids — the problem is equally difficult when one of the matroids is very simple.

Our second result disproves Chow's conjecture.

Theorem 5. Conjecture 3 is false for every b such that $2 \leq b \leq n/3$.

In fact, we give two proofs of Theorem 5. Chow [1] mentioned that a variant of Conjecture 3, when $I_1, I_2, \ldots, I_n$ are not required to be independent in the matroid, is not true. Our first proof, given in Section 4, shows that Conjecture 3 can be reduced to this variant, and thus any counterexample to the variant (such as the one in Appendix A) can be transformed to a counterexample of Conjecture 3. The second proof, given in Section 5, uses a connection between packing common bases and packing dijoins. We note that Chow's conjecture remains open when $b > n/3$; in particular, Rota's conjecture remains open.

By combining our two results, we obtain the following refinement.

Corollary 6. Problem 1 can be solved in polynomial time if and only if this is true under the additional assumption that $\mathbf{M}_2$ is a direct sum of uniform matroids whose blocks are each independent in $\mathbf{M}_1$.

2 Preliminaries

We begin with some terminology which will be useful throughout this paper. Let $\mathbf{M} = (T, \mathcal{I})$ be a direct sum of uniform rank-1 matroids. In the field of combinatorial optimization, such a matroid is commonly called a ***partition matroid*** [18, pp. 659]. Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be the partition of T induced by this direct sum; these sets A_i are called the ***blocks*** of $\mathbf{M}$. The independent sets of $\mathbf{M}$ are

$$\mathcal{I} \;=\; \{\, I \,:\, |I \cap A_i| \leq 1 \;\forall i = 1, \ldots, n \,\}.$$

It is convenient to denote this matroid $\mathbf{U}_1(\mathcal{A})$.

A matroid is called a ***generalized partition matroid*** if it is the direct sum of uniform matroids of arbitrary rank. Again assuming that $\mathcal{A} = \{A_1, \ldots, A_n\}$ is a partition of T, we let $\mathbf{U}_k(\mathcal{A})$ denote the generalized partition matroid whose independent sets are

$$\{\, I \,:\, |I \cap A_i| \leq k \;\forall i = 1, \ldots, n \,\}.$$

The set of integers $\{1, \ldots, k\}$ is denoted $[k]$. For a finite set S, we will use its Cartesian product with $[k]$, namely

$$S \times [k] \;=\; \{\, (s, i) \,:\, s \in S,\, i \in [k] \,\}.$$

For brevity, we also write this as $S^{[k]}$. Any subset $A \subseteq S$ is extended to a subset $A^{[k]} \subseteq S^{[k]}$ by taking $A^{[k]} = A \times [k]$. Similarly, for any $s \in S$, let $s^{[k]} = \{s\} \times [k]$. Conversely, the projection onto S of any subset $B \subseteq S^{[k]}$ is

$$\pi(B) \;=\; \{\, s \in S \,:\, \exists x \in [k] \;\text{ s.t. }\; (s, x) \in B \,\}.$$

For any matroid $\mathbf{M} = (S, \mathcal{I})$, we define $\mathbf{M}^{[k]}$ to be the matroid on the ground set $S^{[k]}$ whose independent sets are

$$\left\{\, I \subseteq S^{[k]} \,:\, \pi(I) \in \mathcal{I},\; |I \cap s^{[k]}| \leq 1 \;\; \forall s \in S \,\right\}.$$

In other words, every element of S has been replaced by k parallel elements. Note that the rank of $\mathbf{M}^{[k]}$ is the same as the rank of $\mathbf{M}$.

The direct sum of two matroids $\mathbf{M}_1$ and $\mathbf{M}_2$ on disjoint ground sets is denoted $\mathbf{M}_1 \oplus \mathbf{M}_2$. The dual of a matroid $\mathbf{M}$ is denoted $\mathbf{M}^*$. For simplicity we write $\mathbf{M}^{*[k]}$ to denote $(\mathbf{M}^*)^{[k]}$.

3 Packing common bases and partition matroids

In this section we prove Theorem 4. Suppose we are given two matroids $\mathbf{M}_1$ and $\mathbf{M}_2$ on a ground set S. We will show how to construct two new matroids, one of which is a partition matroid, such that $\mathbf{M}_1$ and $\mathbf{M}_2$ can be partitioned into common bases if and only if the new matroids can be partitioned into common bases. The essence of our proof is to generalize an observation of Edmonds [8, claims 104–106]. He constructs two new matroids, one of which is a partition matroid, such that a common base of $\mathbf{M}_1$ and $\mathbf{M}_2$ exists if and only if a common base of the new matroids exists.

We may assume that $\mathbf{M}_1$ and $\mathbf{M}_2$ contain no loops, that their rank is the same number r, that they have at least one common base, and that $|S|$ is a multiple of r, say $|S| = (k+1) \cdot r$. These assumptions can easily be tested in polynomial time, and if they do not hold then the solution to Problem 1 is "no".

The two new matroids are defined on the ground set $S \cup S^{[k]}$. Let $\hat{\mathcal{S}} = \{ \hat{s} : s \in S \}$ be the partition of $S \cup S^{[k]}$ where $\hat{s} = \{s\} \cup s^{[k]}$ for each $s \in S$. To visualize this, one can view the elements of $S \cup S^{[k]}$ as being written in an $|S| \times (k+1)$ array, where the elements in S are written in the first column and the elements of $\hat{s}$ are written in a row for every $s \in S$. The new matroids are

$$\mathbf{M} \;=\; \mathbf{M}_1 \oplus \mathbf{M}_2^{*[k]} \qquad \text{and} \qquad \mathbf{U}_1(\hat{\mathcal{S}}).$$

One may easily verify that both of these matroids have rank $|S|$.

Claim 7. The common bases of $\mathbf{M}$ and $\mathbf{U}_1(\hat{\mathcal{S}})$ are precisely the subsets $B \subseteq S \cup S^{[k]}$ satisfying

$$|B \cap \hat{s}| = 1 \;\; \forall s \in S \qquad \text{and} \qquad B \cap S \text{ is a common base of } \mathbf{M}_1 \text{ and } \mathbf{M}_2. \tag{1}$$

Proof. Recall that r is the rank of both $\mathbf{M}_1$ and $\mathbf{M}_2$. Let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_2^*$ respectively denote the base families of $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_2^*$.

Clearly the bases of $\mathbf{U}_1(\hat{\mathcal{S}})$ are the subsets $B \subseteq S \cup S^{[k]}$ for which $|B \cap \hat{s}| = 1$ for every $s \in S$. Of these subsets, the bases of $\mathbf{M}$ are exactly those for which $B \cap S \in \mathcal{B}_1$ and $\pi(B \cap S^{[k]}) \in \mathcal{B}_2^*$. Note that $\pi(B \cap S^{[k]}) = S \setminus B$. Since $S \setminus B \in \mathcal{B}_2^*$ is equivalent to $B \cap S \in \mathcal{B}_2$, this establishes (1). ∎

Corollary 8. If $B_1, \ldots, B_{k+1}$ is a partition of $S \cup S^{[k]}$ into common bases of $\mathbf{M}$ and $\mathbf{U}_1(\hat{\mathcal{S}})$, then $B_1 \cap S, \ldots, B_{k+1} \cap S$ is a partition of S into common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$.

Claim 9. Given a partition $B_1, \ldots, B_{k+1}$ of S into common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$, we can construct a partition $B'_1, \ldots, B'_{k+1}$ of $S \cup S^{[k]}$ into common bases of $\mathbf{M}$ and $\mathbf{U}_1(\hat{\mathcal{S}})$.

Proof. The idea is simple: we extend each B_j into a common base of $\mathbf{M}$ and $\mathbf{U}_1(\hat{\mathcal{S}})$ by picking one element from $s^{[k]}$ for each $s \in S \setminus B_j$. Visualizing the new ground set as an array, the process is: for each row containing no element of B_j, we pick an arbitrary element in that row, excluding the first element, since it lies in S.

More formally, we will partition $S^{[k]}$ into $C_1, \ldots, C_{k+1}$ such that the following properties are satisfied.

$$\pi(C_j) = S \setminus B_j \qquad \text{and} \qquad |C_j \cap s^{[k]}| \leq 1 \;\; \forall s \in S.$$

Then we will set $B'_j = B_j \cup C_j$. The resulting sets B'_j will satisfy (1), so by Claim 7 they are common bases of $\mathbf{M}$ and $\mathbf{U}_1(\hat{\mathcal{S}})$. The construction of the sets C_j is by a simple greedy approach that proceeds by sequentially constructing C_1, then C_2, etc. To construct C_j, for each element $s \in S \setminus B_j$ we add to C_j an arbitrary element in $s^{[k]} \setminus \bigcup_{\ell<j} C_\ell$. Such an element exists because the sets B_j are a partition of S, so for every $s \in S$, we have $|\{ j : s \notin B_j \}| = k = |s^{[k]}|$. ∎

Claim 7 and Claim 9 together imply Theorem 4.

4 A counterexample to Chow's Conjecture

In [1], Chow stated that the following variant of Conjecture 3 is not true.

Conjecture 10. Let $\mathbf{M} = (S, \mathcal{I})$ be a matroid of rank m with the property that S can be partitioned into b bases, where $3 \leq b \leq m$. Let $A_1, \ldots, A_m$ be disjoint sets, each of size b. Then there are disjoint bases $B_1, \ldots, B_b$ such that $|A_i \cap B_j| = 1$ for every $i = 1, \ldots, m$ and $j = 1, \ldots, b$.

Using a reduction similar to that in Theorem 4, we show that any counterexample to Conjecture 10 yields a counterexample to Conjecture 3. Since counterexamples to Conjecture 10 are known, this yields counterexamples to Conjecture 3 via our reduction. The precise statement that we prove is the following theorem.

Theorem 11. Let $\mathbf{M}_1 = (S, \mathcal{I}_1)$ be a matroid with rank m, no loops and $|S| = (k+1) \cdot m$. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a partition of S where each $|A_i| = k+1$. With a slight abuse of notation, define

$$\mathcal{A}^{[k]} \;=\; \left\{ A_1^{[k]}, \ldots, A_m^{[k]} \right\} \qquad \text{and} \qquad \mathbf{M} \;=\; \mathbf{M}_1 \oplus \mathbf{U}_k(\mathcal{A}^{[k]}).$$

As above, let $\hat{\mathcal{S}} = \{\, \hat{s} \,:\, s \in S \,\}$ be the partition of $S \cup S^{[k]}$ where $\hat{s} = \{s\} \cup s^{[k]}$ for each $s \in S$. Then the following statements hold.

$$\mathbf{M}_1 \text{ and } \mathbf{U}_1(\mathcal{A}) \text{ have } k+1 \text{ disjoint common bases if and only if } \mathbf{M} \text{ and } \mathbf{U}_1(\hat{\mathcal{S}}) \text{ do,} \tag{2a}$$

$$\text{every set in } \hat{\mathcal{S}} \text{ is independent in } \mathbf{M}, \text{ and} \tag{2b}$$

$$\text{if } \mathbf{M}_1 \text{ has } k+1 \text{ disjoint bases then } \mathbf{M} \text{ has } k+1 \text{ disjoint bases.} \tag{2c}$$

Proof. Statement (2b) is straightforward, so we begin by proving (2c). Since $\mathbf{M}$ is a direct sum, it suffices to show that S can be partitioned into $k+1$ bases of $\mathbf{M}_1$ and that $S^{[k]}$ can be partitioned into $k+1$ bases of $\mathbf{U}_k(\mathcal{A}^{[k]})$. The first condition holds by assumption. We now prove the second condition in a manner similar to Claim 9. Since $|A_i| = k+1$ for every i, there exists a partition of S into bases $B_1, \ldots, B_{k+1}$ of $\mathbf{U}_1(\mathcal{A})$. We will greedily construct a partition of $S^{[k]}$ into $C_1, \ldots, C_{k+1}$ such that $|C_j \cap A_i^{[k]}| = k$ for each i and each j, implying that each C_j is a base of $\mathbf{U}_k(\mathcal{A})$. To construct C_j, for each element $s \in S \setminus B_j$ we add to C_j an arbitrary element in $s^{[k]} \setminus \bigcup_{\ell < j} C_\ell$. Such an element exists because the sets B_j are a partition of S, so for every $s \in S$, we have $|\{\, j \,:\, s \notin B_j \,\}| = k = |s^{[k]}|$.

To prove (2a) we require the following claim, which is similar to Claim 7.

Claim 12. The common bases of $\mathbf{M}$ and $\mathbf{U}_1(\hat{\mathcal{S}})$ are precisely the subsets $B \subseteq S \cup S^{[k]}$ satisfying

$$|B \cap \hat{s}| = 1 \;\; \forall s \in S \qquad \text{and} \qquad B \cap S \text{ is a common base of } \mathbf{M}_1 \text{ and } \mathbf{U}_1(\mathcal{A}).$$

Proof. The common bases of $\mathbf{M}$ and $\mathbf{U}_1(\hat{\mathcal{S}})$ are the subsets $B \subseteq S \cup S^{[k]}$ satisfying

$$|B \cap \hat{s}| \;=\; 1 \;\; \forall s \in S \tag{3a}$$

$$B \cap S \text{ is a base of } \mathbf{M}_1 \tag{3b}$$

$$|B \cap A_i^{[k]}| \;=\; k \;\; \forall i. \tag{3c}$$

The main point is that, under the assumption that (3a) holds, (3c) is equivalent to

$$|B \cap A_i| \;=\; 1 \;\; \forall i.$$

This last condition is equivalent to $B \cap S$ being a base of $\mathbf{U}_1(\mathcal{A})$. □

Now we prove (2a). If $B_1, \ldots, B_{k+1}$ are disjoint common bases of $\mathbf{M}$ and $\mathbf{U}_1(\hat{\mathcal{S}})$ then, by Claim 12, $B_1 \cap S, \ldots, B_{k+1} \cap S$ are disjoint common bases of $\mathbf{M}_1$ and $\mathbf{U}_1(\mathcal{A})$. Conversely, suppose that $B_1, \ldots, B_{k+1}$ are disjoint common bases of $\mathbf{M}_1$ and $\mathbf{U}_1(\mathcal{A})$. Then the argument of Claim 9 shows that we can construct $k+1$ disjoint common bases of $\mathbf{M}$ and $\mathbf{U}_1(\hat{\mathcal{S}})$. ■

Proof (of Theorem 5). Suppose that we have a counterexample to Conjecture 10 consisting of a matroid $\mathbf{M}_1$ together with the sets $A_1, \ldots, A_m$, each of which has $|A_i| = b$. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$. Then each of $\mathbf{M}_1$ and $\mathbf{U}_1(\mathcal{A})$ can be partitioned into b bases, but they cannot be partitioned into b common bases.

Let $k = b-1$. Construct the matroids $\mathbf{M}$ and $\mathbf{U}_1(\hat{\mathcal{S}})$ as in Theorem 11. Then $\mathbf{M}$ has rank $|S|$ and it can be partitioned into b bases, by (2c). Furthermore $\mathbf{U}_1(\hat{\mathcal{S}})$ is a partition matroid whose blocks are each independent in $\mathbf{M}$, by (2b). Since $\mathbf{M}_1$ and $\mathbf{U}_1(\mathcal{A})$ do not have $k+1$ disjoint bases, neither do $\mathbf{M}$ and $\mathbf{U}_1(\hat{\mathcal{S}})$, by (2a). Thus $\mathbf{M}$ and $\mathbf{U}_1(\hat{\mathcal{S}})$ give a counterexample to Conjecture 3.

McDiarmid showed a counterexample (briefly described in Appendix A) to Conjecture 10 for any $b \geq 2$ with $m = 3$ and $|S| = 3b$. Thus our construction shows that Conjecture 3 is false for any $b \geq 2$ and $n = 3b$. By Chow's theorem [1], this implies that Conjecture 3 is false whenever $2 \leq b \leq n/3$. ■

Theorem 4 describes a polynomial-time reduction from an arbitrary instance of Problem 1 to an instance in which one of the matroids is a partition matroid. Theorem 11 describes a polynomial-time reduction from an instance of Problem 1 in which one of the matroids is a partition matroid to another instance in which one of the matroids is a partition matroid whose blocks are independent in the other matroid. Composing these two reductions proves Corollary 6.

5 Chow's Conjecture and Dijoins

In this section we give an alternative proof of Theorem 5. The proof is based on a connection between dijoins and common matroid bases, due to Frank and Tardos [11], and Schrijver's counterexample on packing dijoins [17].

Let $D = (V, A)$ be a directed graph. For any set $U \subseteq V$, we let $\delta^{\text{in}}(U)$ be the set of arcs entering U and let $\delta^{\text{out}}(U)$ be the set of arcs leaving U. Define $d^{\text{in}}(U) = |\delta^{\text{in}}(U)|$ and $d^{\text{out}}(U) = |\delta^{\text{out}}(U)|$. For any arc set F we also define $d_F^{\text{in}}(U) = |\delta^{\text{in}}(U) \cap F|$ and $d_F^{\text{out}}(U) = |\delta^{\text{out}}(U) \cap F|$. For any vertex $v \in V$, we use the shorthand $d_F^{\text{in}}(v)$ for $d_F^{\text{in}}(\{v\})$.

An arc set $C \subseteq A$ is called a ***directed cut*** if there exists $\emptyset \neq U \subsetneq V$ such that $C = \delta^{\text{in}}(U)$ and $d^{\text{out}}(U) = 0$. A k***-dijoin*** is an arc set $F \subseteq A$ that contains at least k arcs from each directed cut of D. A 1-dijoin is called simply a ***dijoin***. Schrijver's counterexample showed the existence of a digraph and a 2-dijoin that cannot be partitioned into two disjoint dijoins. By adding three arcs x', y', z' to Schrijver's example, we can obtain a 3-dijoin that cannot be decomposed into three dijoins. The resulting example is shown in Figure 1 and is denoted $D = (V, A)$. Let F be the set of bold arcs in this example. One may verify that F is a 3-dijoin.

Claim 13. The arc set F cannot be decomposed into three dijoins.

Proof. Let $\mathcal{D}$ be the family of all arc sets equivalent to $\{x', y', z'\}$ under the relations $x \equiv x'$, $y \equiv y'$ and $z \equiv z'$. Note that all arc sets in $\mathcal{D}$ are dijoins.

Any decomposition of F into three dijoins cannot contain a dijoin in $\mathcal{D}$ since the remainder is Schrijver's counterexample, which cannot be decomposed into two dijoins. Any dijoin not in $\mathcal{D}$ must contain at least four arcs because of the nodes of in-degree and out-degree zero represented in Figure 2(a). Any dijoin of exactly four arcs must contain two nonparallel arcs from $\{x, x', y, y', z, z'\}$, as is clear from Figure 2(a). Since F has twelve elements, each of the three disjoint dijoins must have exactly four arcs and each must contain two nonparallel arcs from $\{x, x', y, y', z, z'\}$. But Figure 2(b) shows that such an arc set cannot be a dijoin: there is a set of out-degree zero in D that it does not enter. ■

We define an arc set F' (which is not a subset of A) by taking F and adding two reversed arcs for each arc of F. For $a \in F$, these reversed arcs will be denoted by a_1^{-1} and a_2^{-1}. We obtain another counterexample for Chow's conjecture by defining a matroid with ground set F'. First define

$$\mathcal{X} \;:=\; \left\{\, X \,:\, \emptyset \neq X \subsetneq V \;\text{ and }\; d_A^{\text{out}}(X) = 0 \,\right\}$$

and define $i_F(X)$ to be the number of arcs of F with both endpoints in X. The matroid $\mathbf{M}$ is defined by its bases: a set $B \subseteq F'$ is a base if and only if $|B| = |F|$ and

$$\sum_{v \in X} d_B^{\text{in}}(v) \;\geq\; i_F(X) + 1 \qquad \forall X \in \mathcal{X}. \tag{4}$$

It was shown by Frank and Tardos [11] [18, Section 55.5] that this construction gives a matroid.

Claim 14. The ground set F' of the matroid $\mathbf{M}$ can be partitioned into three bases.

Proof. First we claim that the base polyhedron of $\mathbf{M}$ is

$$Q \;:=\; \left\{ x \,:\, x(F') = |F|, \quad \sum_{v \in X} x(\delta^{\text{in}}(v)) \geq i_F(X) + 1 \;\; \forall X \in \mathcal{X}, \quad 0 \leq x_a \leq 1 \;\; \forall a \in F' \right\}.$$

To see this, first note that, since the directed cuts form a crossing family, their complements also form a crossing family $\mathcal{C}$. We can define a crossing submodular function $f : \mathcal{C} \to \mathbb{R}$ such that (4) is equivalent to $|B \cap Z| \leq f(Z)$ for all $Z \in \mathcal{C}$. Now by following another argument in Schrijver [18] proving that $\mathbf{M}$ is a matroid (see Theorem 49.7, Equation (49.11) and Equation (44.43)) it follows that Q is indeed the base polyhedron of $\mathbf{M}$.

Since $F \subset F'$ and F is a 3-dijoin, we have

$$\sum_{v \in X} d_{F'}^{\text{in}}(v) \;=\; i_{F'}(X) + d_{F'}^{\text{in}}(X) \;\geq\; i_{F'}(X) + 3 \qquad \forall X \in \mathcal{X}.$$

Since $i_{F'}(X) = 3 i_F(X)$ for every $X \subseteq V$, we have

$$\sum_{v \in X} \frac{d_{F'}^{\text{in}}(v)}{3} \geq i_F(X) + 1 \qquad \forall X \in \mathcal{X}.$$

Let x be the the characteristic vector of F', divided by 3. Then we have shown that $x \in Q$. Since matroid base polyhedra have the integer decomposition property [18, Corollary 42.1e], this implies that F' can be partitioned into three bases. ■

Let us define the sets $I_a = \{a, a_1^{-1}, a_2^{-1}\}$ $(a \in F)$. These triplets are independent sets. To see this, note that F is a base; moreover, it satisfies inequality (4) with $i_F(X) + 3$ instead of $i_F(X) + 1$. Thus, for arbitrary distinct arcs $a, b, c \in F$, the set $(F \setminus \{a, b, c\}) \cup I_a$ is a base.

Conjecture 3 would imply that F' can be decomposed into three bases B_1, B_2, B_3 such that $|B_j \cap I_a| = 1$ for any $j \in \{1, 2, 3\}$ and $a \in F$. Suppose that this is possible; then $i_{B_j}(X) = i_F(X)$ for every $X \subseteq V$, so $\sum_{v \in X} d_{B_j}^{\text{in}}(v) \geq i_F(X) + 1$ implies that

$$d_{B_j}^{\text{in}}(X) \geq 1 \qquad \text{for every } \emptyset \neq X \subsetneq V \text{ with } d_A^{\text{out}}(X) = 0.$$

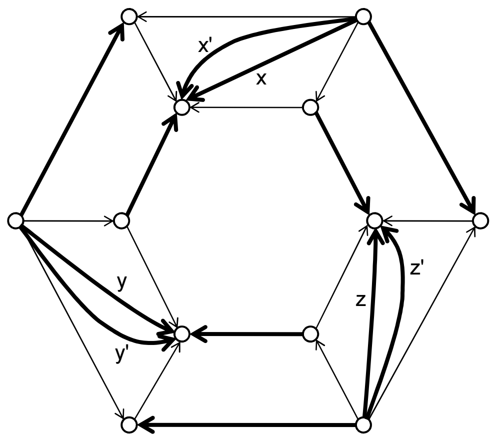


Figure 1: Schrijver's example, augmented with additional arcs x', y' and z'. The bold arcs form a 3-dijoin which cannot be decomposed into three dijoins.

In other words, B_j has at least one arc in every directed cut of D. However, the only arcs that are in directed cuts of D are the arcs of F. Thus the conjecture would imply that F can be decomposed into three dijoins, but by Claim 13 this is impossible.

This proves Theorem 5 for the case $b = 3$. By adding additional arcs parallel to x', y', z' one can extend this argument to obtain a counterexample to Conjecture 3 for all $3 \leq b \leq n/3-1$. This proves Theorem 5, with slightly weaker parameters.

Concluding Remarks

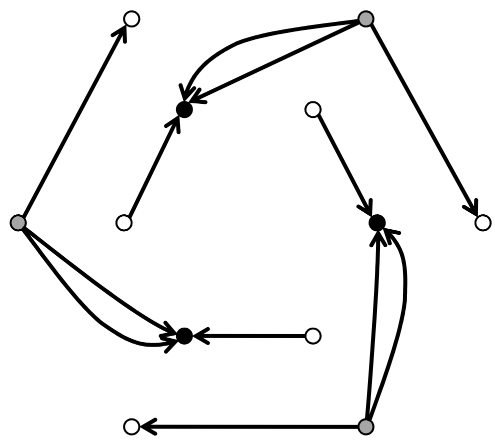
Several basic questions on disjoint common bases of two matroids remain open. One question is to determine the computational complexity of Problem 1. As we have shown, it suffices to consider the case when one of the matroids is a partition matroid. Even when the other matroid is a graphic matroid, the computational complexity is still unknown. Another question is to find a sufficient condition that guarantees the existence of k disjoint common bases. Geelen and Webb [12] showed that there are $\sqrt{n}$ disjoint common bases under the setting in Rota's conjecture.

Finding further counterexamples to Chow's conjecture may lead to an improvement of the parameters in Theorem 5, and perhaps a better understanding of Rota's conjecture. One framework that seems quite relevant for such questions is the topic of clutters [2, 3]. A ***clutter*** $\mathcal{C}$ is a pair $(V(\mathcal{C}), E(\mathcal{C}))$, where $V(\mathcal{C})$ is a finite set and $E(\mathcal{C}) = \{E_1, E_2, \ldots\}$ is an antichain in the lattice of subsets of $V(\mathcal{C})$, i.e., a family of distinct subsets of $V(\mathcal{C})$ such that $E_i \subseteq E_j$ implies $i = j$. The elements of $V(\mathcal{C})$ are called ***vertices*** and the elements of $E(\mathcal{C})$ are called ***edges***. A ***transversal*** of $\mathcal{C}$ is a subset of $V(\mathcal{C})$ that intersects all edges in $E(\mathcal{C})$. Let $\tau(\mathcal{C})$ denote the minimum cardinality of any transversal. We say that the clutter $\mathcal{C}$ ***packs*** if there exist $\tau(\mathcal{C})$ pairwise disjoint edges.

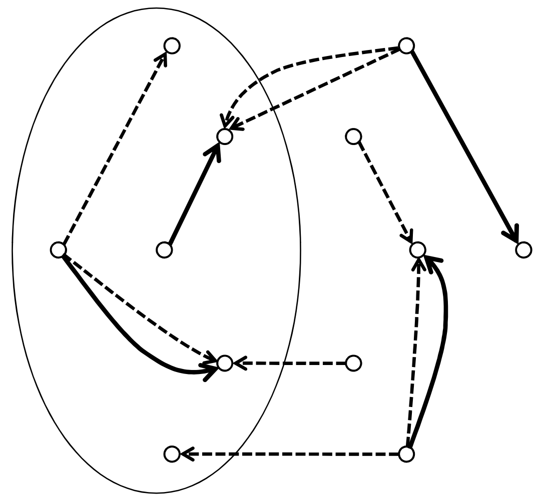
As in Conjectures 2 and 3, let $\mathbf{M} = (T, \mathcal{I})$ be a matroid of rank n with the property that T can be partitioned into b bases, where $3 \leq b \leq n$. Let $A_1, \ldots, A_n \in \mathcal{I}$ be disjoint independent sets, each of size b. Consider the clutter $\mathcal{C}$ with $V(\mathcal{C}) = T$ and

$$E(\mathcal{C}) \;=\; \{\; B \,:\, B \in \mathcal{I} \text{ and } |A_i \cap B| = 1 \;\; \forall i \in [n] \;\}\,. \tag{5}$$

Note that every $B \in E(\mathcal{C})$ is a base of $\mathbf{M}$.



(a)



(b)

Figure 2: (a) Nodes of out-degree 0 (black) and in-degree 0 (gray) in the digraph (V, F). (b) The solid arcs do not form a dijoin because of the set of out-degree 0, indicated by the oval.

Conjectures 2 and 3 are equivalent to the statement that the clutter $\mathcal{C}$ packs, since we show in Appendix B that $\tau(\mathcal{C}) = b$. Therefore any counterexample to these conjectures necessarily involves a clutter that does not pack. Characterizing clutters that do not pack seems difficult, although there has been significant work on identifying the minimal such clutters [4].

The counterexample to Conjecture 3 given in Section 4 is based on a well-known clutter[1] Q_6 that does not pack, and which underlies the K_4 counterexample described in Appendix A. The counterexample to Conjecture 3 given in Section 5 is based on another famous such clutter, known as $Q_6 \otimes \{1,3,5\}$. This clutter was developed by Schrijver [17] to disprove a conjecture of Edmonds and Giles [10] on packing dijoins.

An important class of clutters is the the class of ***ideal*** clutters [2]. One can show that the clutter $\mathcal{C}$ defined in (5) is not necessarily ideal: there is a laminar matroid on nine elements such that C_3^2 is a minor of $\mathcal{C}$. On the other hand, our two counterexamples are based on Q_6 and $Q_6 \otimes \{1,3,5\}$, which are both ideal. Is there a counterexample based on a non-ideal, non-packing clutter?

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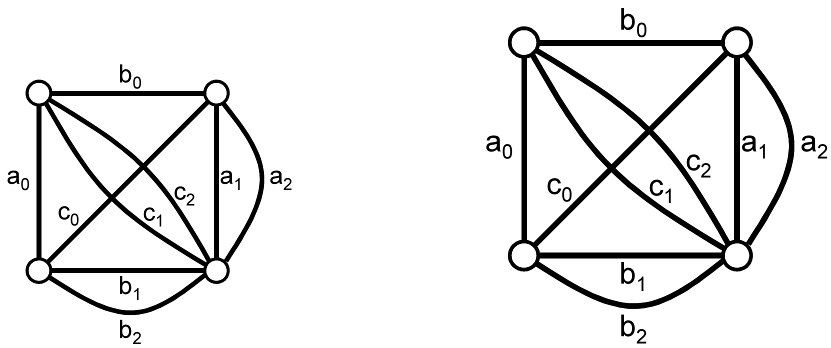

(a) (b)

Figure 3: (a) The graph K_4 with our chosen edge labeling. (b) The graph G_2 is obtained by letting a_2, b_2 and c_2 be parallel copies of a_1, b_1 and c_1, respectively.

A The K_4 counterexample

Consider K_4, the complete graph on four vertices. As shown in Figure 3(a), we write its edges as $S = \{a_0, a_1, b_0, b_1, c_0, c_1\}$, where $a_0 \cap a_1 = \emptyset$, $b_0 \cap b_1 = \emptyset$, $c_0 \cap c_1 = \emptyset$, and $\{a_1, b_1, c_1\}$ forms a spanning star. Let $\mathbf{M}_1 = (S, \mathcal{I}_1)$ be the graphic matroid of K_4. Let $\mathbf{M}_2$ be the partition matroid

$$\mathbf{U}_1(\{\{a_0, a_1\}, \{b_0, b_1\}, \{c_0, c_1\}\}).$$

It is well-known [5] that both $\mathbf{M}_1$ and $\mathbf{M}_2$ have two disjoint bases, but they do not have two disjoint common bases. The common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$ are precisely the spanning stars in K_4.

McDiarmid [1] showed how to extend this example to obtain, for any $k \geq 1$, two matroids $\mathbf{M}_1 = (S, \mathcal{I}_1)$ and $\mathbf{M}_2 = (S, \mathcal{I}_2)$ such that

- S can be partitioned into $k+1$ bases of $\mathbf{M}_1$,
- S can be partitioned into $k+1$ bases of $\mathbf{M}_2$, and
- S cannot be partitioned into $k+1$ common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$.

We now describe this extension. The example is based on the graph G_k, which is constructed from K_4 by adding new edges:

- $a_2, \ldots, a_k$ parallel to a_1,
- $b_2, \ldots, b_k$ parallel to b_1, and
- $c_2, \ldots, c_k$ parallel to c_1.

The graph G_2 is shown in Figure 3(b). Define

$$\begin{aligned} E_a &= \{a_1, \ldots, a_k\} & E_b &= \{b_1, \ldots, b_k\} & E_c &= \{c_1, \ldots, c_k\} \\ F_a &= \{a_0, \ldots, a_k\} & F_b &= \{b_0, \ldots, b_k\} & F_c &= \{c_0, \ldots, c_k\}. \end{aligned}$$

Let $\mathbf{M}_1$ be the graphic matroid of G_k. Let $\mathbf{M}_2 = \mathbf{U}_1(F_a, F_b, F_c)$. It is easy to see that the edges can be partitioned into $k+1$ bases of $\mathbf{M}_1$, or into $k+1$ bases of $\mathbf{M}_2$.

Claim 15. The edges cannot be partitioned into $k+1$ common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$.

Proof. As remarked above, the common bases of $\mathbf{M}_1$ and $\mathbf{M}_2$ are precisely the spanning stars in G_k. We consider two cases.

Case 1: $k \geq 3$. Since there are only three edges not in $E_a \cup E_b \cup E_c$, at least one of the $k+1$ common bases is contained in $E_a \cup E_b \cup E_c$. Removing this common base, the resulting graph is G_{k-1}. By induction, this instance cannot be partitioned into k common bases.

Case 2: $k = 2$. Note that there is no spanning star using exactly two edges from $E_a \cup E_b \cup E_c$. So two of the common bases use three of those edges, and the other common base uses none. But the complement of $E_a \cup E_b \cup E_c$ is not a spanning star. ■

The matroids $\mathbf{M}_1$ and $\mathbf{M}_2$ give a counterexample to Conjecture 10 for $m = 3$ and arbitrary $b \geq 2$: take $k = b-1$, and define the sets A_1, A_2, A_3 to be the blocks of the matroid $\mathbf{M}_2$. However, this does not directly yield a counterexample to Conjecture 3 for $b \geq 3$ since the sets A_1, A_2, A_3 are not independent in $\mathbf{M}_1$.

B Minimum Transversals and Rota's Conjecture

In this appendix, we determine the minimum cardinality of any transversal for the clutter defined in (5).

Claim 16. $\tau(\mathcal{C}) = b$.

Proof. Obviously $\tau(\mathcal{C}) \leq b$ as any set A_i is a transversal. So suppose there exists a transversal $D \subseteq T$ such that $|D| < b$. Let $\mathcal{A} = \{A_1, \ldots, A_n\}$. We wish to show that there is an edge that does not intersect D, which is equivalent to showing that $\mathbf{M} \setminus D$ and $\mathbf{U}_1(\mathcal{A}) \setminus D$ have a common base. Let $r_{\mathbf{M}}$ and $r_{\mathbf{U}_1(\mathcal{A})}$ respectively be the rank function of $\mathbf{M}$ and $\mathbf{U}_1(\mathcal{A})$. By the matroid intersection theorem [18, Theorem 41.1] [16, Theorem 12.3.15], it suffices to show that

$$r_{\mathbf{M}}(A) + r_{\mathbf{U}_1(\mathcal{A})}(T \setminus (D \cup A)) \;\geq\; n \quad \forall A \subseteq T \setminus D. \tag{6}$$

By Edmonds' matroid base covering theorem [18, Corollary 42.1c] [16, Theorem 12.3.12], for any set A we have $r_{\mathbf{M}}(A) \geq \lceil |A|/b \rceil$ and $r_{\mathbf{U}_1(\mathcal{A})}(A) \geq \lceil |A|/b \rceil$. Thus

$$r_{\mathbf{M}}(A) + r_{\mathbf{U}_1(\mathcal{A})}(T \setminus (D \cup A)) \;\geq\; \left\lceil \frac{|A|}{b} \right\rceil + \left\lceil \frac{|T \setminus (D \cup A)|}{b} \right\rceil \;=\; \left\lceil \frac{|A|}{b} \right\rceil + \left\lceil \frac{|T \setminus A|}{b} \right\rceil - \epsilon \;\geq\; n - \epsilon,$$

where $\epsilon \in \{0, 1\}$, since $|D| < b$.

If the last inequality is strict, then (6) must be satisfied. If last inequality holds with equality then $|A|/b$ and $|T \setminus A|/b$ are both integers, which implies that $\lceil |T \setminus (D \cup A)|/b \rceil = |T \setminus A|/b$, since $|D| < b$. Thus $\epsilon = 0$ and so (6) is satisfied. ■