Cheeger Inequalities for Directed Graphs and Hypergraphs using Reweighted Eigenvalues*

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ABSTRACT

We derive Cheeger inequalities for directed graphs and hypergraphs using the reweighted eigenvalue approach that was recently developed for vertex expansion in undirected graphs. The goal is to develop a new spectral theory for directed graphs and an alternative spectral theory for hypergraphs.

The first main result is a Cheeger inequality relating the vertex expansion of a directed graph to the vertex-capacitated maximum reweighted second eigenvalue. This provides a combinatorial characterization of the fastest mixing time of a directed graph by vertex expansion, and builds a new connection between reweighted eigenvalued, vertex expansion, and fastest mixing time for directed graphs.

The second main result is a stronger Cheeger inequality relating the edge conductance of a directed graph to the edge-capacitated maximum reweighted second eigenvalue. This provides a certificate for a directed graph to be an expander and a spectral algorithm to find a sparse cut in a directed graph, playing a similar role as Cheeger's inequality in certifying graph expansion and in the spectral partitioning algorithm for undirected graphs.

We also use this reweighted eigenvalue approach to derive the improved Cheeger inequality for directed graphs, and furthermore to derive several Cheeger inequalities for hypergraphs that match and improve the existing results. These are supporting results that this provides a unifying approach to lift the spectral theory for undirected graphs to more general settings.

CCS CONCEPTS

• **Mathematics of computing → Spectra of graphs**; *Hypergraphs*; *Approximation algorithms*; • **Theory of computation →** Random projections and metric embeddings; **Graph algorithms analysis**; **Approximation algorithms analysis**; *Semidefinite programming*.

KEYWORDS

Cheeger inequalities, directed graphs, hypergraphs, reweighted eigenvalues, mixing time, spectral analysis

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1 INTRODUCTION

Cheeger's inequality [3, 4, 11, 13] is a fundamental result in spectral graph theory that connects the edge expansion property of an undirected graph $G = (V, E)$ to the second eigenvalue of its associated matrix:

$$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2} \tag{1.1}$$

where $\phi(G)$ is the edge conductance of G and λ_2 is the second smallest eigenvalue of its normalized Laplacian matrix[1]. There are two important applications of Cheeger's inequality. One is to use the second eigenvalue to study expander graphs [22] and its eigenvector for graph partitioning [38, 40]. The other is to use the edge conductance to bound the mixing time of random walks [2, 29]. Together, Cheeger's inequality connects the second eigenvalue, edge conductance, and mixing time. More recently, the spectral theory for undirected graphs is enriched by several interesting generalizations of Cheeger's inequality [5, 25, 28, 34, 39], which establish further connections between edge expansion properties of the graph to other eigenvalues of its normalized Laplacian matrix.

In contrast, the spectral theory for directed graphs has not been nearly as well developed. One issue is that the Laplacian matrix of a directed graph is not Hermitian, and so its eigenvalues are not necessarily real numbers. There are formulations [14, 19, 21, 32] that associate certain Hermitian matrices to a directed graph, and use the second eigenvalue of these matrices to bound the mixing time of random walks [14, 19] (see Section 1.3 for details). But, to our knowledge, there are no known formulations that relate the expansion properties of a directed graph to the eigenvalues of an associated matrix[2]. The main goal of this paper is to provide such formulations using "reweighted eigenvalues" and to develop a spectral theory for directed graphs that is comparable to that for undirected graphs.

The notion of reweighted eigenvalue for undirected graphs was first formulated in [7] for studying the fastest mixing time problem on reversible Markov chains. In this formulation, we are given an undirected graph $G = (V, E)$, and the task is to find a reweighted graph $G' = (V, E, w)$ with edge weight $w(uv)$ for $uv \in E$ and weighted degree one for each vertex, that maximizes the second

[1] See Section 2 for various definitions that are not stated in this introduction.
[2] The only formulation that we know about expansion properties of a directed graph is a *nonlinear* Laplacian operator in [41, 42]. See Section 1.3 for details.

eigenvalue λ_2^* of its normalized Laplacian matrix. It was known [37] that the vertex expansion $\psi(G)$ is an upper bound on λ_2^*, but only recently [23, 26, 36] was it established that there is a Cheeger-type inequality relating these two quantities:

$$\lambda_2^* \lesssim \psi(G) \lesssim \sqrt{\lambda_2^* \cdot \log \Delta} \qquad (1.2)$$

where Δ is the maximum degree of a vertex in G. This inequality connects the reweighted second eigenvalue and vertex expansion and fastest mixing time, in a similar way that Cheeger's inequality connects the second eigenvalue and edge conductance and mixing time. This reweighted eigenvalue approach was extended in [26] to develop a spectral theory for undirected vertex expansion, by proving that several generalizations of Cheeger's inequality [25, 28, 34, 39] have close analogs in connecting vertex expansion properties to other reweighted eigenvalues.

1.1 Our Results

We formulate reweighted eigenvalues for directed graphs and hypergraphs. The main idea is to reduce the study of expansion properties in directed graphs and hypergraphs to the basic setting of edge conductances in undirected graphs. We show that this provides an intuitive and unifying approach to lift the spectral theory for undirected graphs to more general settings.

1.1.1 Cheeger Inequality for Directed Vertex Expansion. Classical spectral theory connects (i) undirected edge conductance, (ii) second eigenvalue, and (iii) mixing time of random walks on undirected graphs. We present a new spectral formulation that connects (i) directed vertex expansion, (ii) reweighted second eigenvalue, and (iii) fastest mixing time of random walks on directed graphs.

Definition 1.1 (Directed Vertex Expansion). *Let $G = (V, E)$ be a directed graph and $\pi : V \to \mathbb{R}_{\geq 0}$ be a weight function on the vertices. For a subset $S \subseteq V$, let $\partial^+(S) := \{v \notin S \mid \exists u \in S \text{ with } uv \in E\}$ be the set of out-neighbors of S, and $\pi(S) := \sum_{v \in S} \pi(v)$ be the weight of S. The directed vertex expansion of a set $S \subseteq V$ and of the graph G are defined as*

$$\vec{\psi}(S) := \frac{\min\left\{\pi\left(\partial^+(S)\right), \pi\left(\partial^+(\overline{S})\right)\right\}}{\min\left\{\pi(S), \pi(\overline{S})\right\}} \quad \text{and} \quad \vec{\psi}(G) := \min_{\emptyset \neq S \subset V} \vec{\psi}(S).$$

where $\overline{S} := V - S$ is the complement set of S. Note that $\vec{\psi}(S) \leq 1$ for all $S \subseteq V$ as $\partial^+(\overline{S}) \subseteq S$.

Remark 1.2. *When specialized to undirected graphs (by considering the bidirected graph), the current definitions are slightly different from that in [26, 36]; see Section 2. We remark that the two definitions of $\psi(G)$ are within a factor of 2 of each other. The current definitions have the advantages that $\psi(S) \leq 1$ and are more convenient in the proofs.*

To certify that a directed graph $G = (V, E)$ has large vertex expansion, our idea is to find the best reweighted *Eulerian* subgraph $G' = (V, E, w)$ of G with edge weight $w(uv)$ for $uv \in E$ and weighted degrees $\sum_{u \in V} w(uv) = \sum_{u \in V} w(vu) = \pi(v)$ for $v \in V$, and then use the edge conductance of G' as a lower bound on the vertex expansion of G. Since the weighted directed graph G' is Eulerian, the edge conductance of G' is equal to the edge conductance of the underlying undirected graph G'' with edge weight $w''(uv) = \frac{1}{2}\big(w(uv) + w(vu)\big)$. Now, as the graph G'' is undirected, we can use Cheeger's inequality to lower bound the edge conductance of G'' by the second smallest eigenvalue of its normalized Laplacian matrix. This leads to the following formulation of the reweighted second eigenvalue for directed vertex expansion (see Proposition 3.1 for more about this reduction).

Definition 1.3 (Maximum Reweighted Spectral Gap with Vertex Capacity Constraints). *Given a directed graph $G = (V, E)$ and a weight function $\pi : V \to \mathbb{R}_{\geq 0}$, the maximum reweighted spectral gap with vertex capacity constraints is defined as*

$$\begin{aligned}
\vec{\lambda}_2^{v*}(G) \; := \; & \max_{A \geq 0} \lambda_2\left(I - \Pi^{-\frac{1}{2}}\left(\frac{A + A^T}{2}\right)\Pi^{-\frac{1}{2}}\right) \\
& \text{subject to } \; A(u,v) = 0 && \forall uv \notin E \\
& \sum_{v \in V} A(u,v) = \sum_{v \in V} A(v,u) && \forall u \in V \\
& \sum_{v \in V} A(u,v) = \pi(u) && \forall u \in V
\end{aligned}$$

where A is the adjacency matrix of the reweighted Eulerian subgraph and $\Pi := \operatorname{diag}(\pi)$ is the diagonal degree matrix of A. Then $\frac{1}{2}(A + A^T)$ is the adjacency matrix of the underlying undirected graph of the reweighted Eulerian subgraph, $\mathcal{L} := I - \frac{1}{2}\Pi^{-1/2}(A + A^T)\Pi^{-1/2}$ is its normalized Laplacian matrix, and $\lambda_2(\mathcal{L})$ is the second smallest eigenvalue of $\mathcal{L}$.

To ensure that the optimization problem for $\vec{\lambda}_2^{v*}(G)$ is always feasible, we assume that the graph has a self-loop at each vertex. In the context of Markov chains, this corresponds to allowing a non-zero holding probability on each vertex.

Our first main result is a Cheeger-type inequality that relates $\vec{\lambda}_2^{v*}(G)$ and $\vec{\psi}(G)$, proving that the directed vertex expansion is large if and only if the reweighted eigenvalue is large.

Theorem 1.4 (Cheeger Inequality for Directed Vertex Expansion). *For any directed graph $G = (V, E)$ and any weight function $\pi : V \to \mathbb{R}_{\geq 0}$,*

$$\vec{\lambda}_2^{v*}(G) \lesssim \vec{\psi}(G) \lesssim \sqrt{\vec{\lambda}_2^{v*}(G) \cdot \log \frac{\Delta}{\vec{\psi}(G)}} \lesssim \sqrt{\vec{\lambda}_2^{v*}(G) \cdot \log \frac{\Delta}{\vec{\lambda}_2^{v*}(G)}},$$

where Δ is the maximum (unweighted) degree of a vertex of G.

Since directed vertex expansion is more general than undirected vertex expansion and (1.2) is tight up to a constant factor [26], we know that the $\log \Delta$ term in Theorem 1.4 is necessary. However, we do not know whether the $\log(1/\vec{\psi}(G))$ term in Theorem 1.4 is necessary or not.

The Fastest Mixing Time Problem: The notion of reweighted eigenvalue for undirected graphs was first formulated in [7] for studying the fastest mixing time problem on *reversible* Markov chains. It turns out that the reweighted eigenvalue $\vec{\lambda}_2^{v*}(G)$ in Definition 1.3 can be used to study the fastest mixing time problem on *general* Markov chains.

Definition 1.5 (Fastest Mixing Time on General Markov Chain). *Given a directed graph $G = (V, E)$ and a probability distribution π on V, the fastest mixing time problem is defined as*

$$\begin{aligned} \tau^*(G) \;:=\; & \min_{P \geq 0} \tau(P) \\ \text{subject to} \;\; & P(u,v) = 0 && \forall uv \notin E \\ & \sum_{v \in V} P(u,v) = 1 && \forall u \in V \\ & \sum_{u \in V} \pi(u) \cdot P(u,v) = \pi(v) && \forall v \in V \end{aligned}$$

where P is the transition matrix of the Markov chain. The constraints are to ensure that P has nonzero entries only on the edges of G, that P is a row stochastic matrix, and that the stationary distribution of P is π. The objective is to minimize the mixing time $\tau(P)$ to the stationary distribution π; see Section 2 for definitions of random walks and mixing times.

For the fastest mixing time problem on reversible Markov chains, we are given an undirected graph $G = (V, E)$ and a probability distribution π, and the last set of constraints in Definition 1.5 is replaced by the stronger requirement that $\pi(u) \cdot P(u,v) = \pi(v) \cdot P(v,u)$ for all $uv \in E$. With this stronger requirement, P has real eigenvalues and it is well known that $\tau(P) \lesssim \frac{1}{1-\alpha_2(P)} \cdot \log(\frac{1}{\pi_{\min}})$, where $\alpha_2(P)$ is the second largest eigenvalue of P and $\pi_{\min} := \min_{v \in V} \pi(v)$. Thus, the reweighted eigenvalue formulation in [7] is to find such a transition matrix P that maximizes the spectral gap $1 - \alpha_2(P)$, which can be solved by a semidefinite program and can be used as a proxy to upper bounding the fastest mixing time.

For general Markov chains, P may have complex eigenvalues, and there was no known efficient formulation for the fastest mixing time problem. We observe that the reweighted spectral gap $\vec{\lambda}_2^{v*}(G)$ in Definition 1.3 provides such a formulation through the results in [14, 19]. An interesting consequence of Theorem 1.4 is a combinatorial characterization of the fastest mixing time of general Markov chains, showing that small directed vertex expansion is the only obstruction of fastest mixing time.

Theorem 1.6 (Fastest Mixing Time and Directed Vertex Expansion). *For any directed graph $G = (V, E)$ with maximum total degree Δ, and for any probability distribution π on V,*

$$\frac{1}{\vec{\psi}(G)} \cdot \frac{1}{\log(1/\pi_{\min})} \lesssim \tau^*(G) \lesssim \frac{1}{\vec{\psi}(G)^2} \cdot \log \frac{\Delta}{\vec{\psi}(G)} \cdot \log \frac{1}{\pi_{\min}}.$$

Together, Theorem 1.4 and Theorem 1.6 connect the reweighted second eigenvalue, directed vertex expansion, and fastest mixing time on directed graphs, in a similar way that classical spectral graph theory connects the second eigenvalue, undirected edge conductance, and mixing time on undirected graphs.

1.1.2 Cheeger Inequality for Directed Edge Conductance. Two key applications of Cheeger's inequality are to use the second eigenvalue to certify whether an undirected graph is an expander graph, and to provide a spectral algorithm for graph partitioning that is useful in many areas. We present a new Cheeger inequality for directed graphs for these purposes.

Definition 1.7 (Directed Edge Conductance [41, 42]). *Let $G = (V, E)$ be a directed graph and $w : E \to \mathbb{R}_{\geq 0}$ be a weight function on the edges. For a subset $S \subseteq V$, let $\delta^+(S) := \{uv \in E \mid u \in S \text{ and } v \notin S\}$ be the set of outgoing edges of S and $w(\delta^+(S))$ be the total edge weight on $\delta^+(S)$, and $\mathrm{vol}_w(S) := \sum_{v \in S} \sum_{u \in V} (w(uv) + w(vu))$ be the volume of S. The directed edge conductance of a set $S \subseteq V$ and of the graph G are defined as*

$$\vec{\phi}(S) := \frac{\min\left\{w(\delta^+(S)), w(\delta^+(\overline{S}))\right\}}{\min\left\{\mathrm{vol}_w(S), \mathrm{vol}_w(\overline{S})\right\}} \quad \text{and} \quad \vec{\phi}(G) := \min_{\emptyset \neq S \subset V} \vec{\phi}(S).$$

We use the same approach to prove a Cheeger-type inequality for directed edge conductance[3]. To certify that a directed graph $G = (V, E)$ with a weight function $w : E \to \mathbb{R}_{\geq 0}$ has large edge conductance, we find the best reweighted Eulerian subgraph G' with edge weight $w'(u,v) \leq w(u,v)$ for each $uv \in E$, and use the edge conductance of G' (with respect to the volumes using w) to provide a lower bound on the edge conductance of G. Then, the edge conductance of G' is reduced to the edge conductance of the underlying undirected graph G'' with edge weight $w''(uv) = \frac{1}{2}(w'(uv) + w'(vu))$, and the second smallest eigenvalue of the normalized Laplacian matrix of G'' is used to provide a lower bound on the edge conductance of G''. See Proposition 3.2 for a proof.

Definition 1.8 (Maximum Reweighted Spectral Gap with Edge Capacity Constraints). *Given a directed graph $G = (V, E)$ and a weight function $w : E \to \mathbb{R}_{\geq 0}$, the maximum reweighted spectral gap with edge capacity constraints is defined as*

$$\begin{aligned} \vec{\lambda}_2^{e*}(G) \;:=\; & \max_{A \geq 0} \lambda_2\left(D^{-\frac{1}{2}}\left(D_A - \frac{A + A^T}{2}\right)D^{-\frac{1}{2}}\right) \\ \text{subject to} \;\; & A(u,v) = 0 && \forall uv \notin E \\ & \sum_{v \in V} A(u,v) = \sum_{v \in V} A(v,u) && \forall u \in V \\ & A(u,v) \leq w(uv) && \forall uv \in E \end{aligned}$$

where A is the adjacency matrix of the reweighted Eulerian subgraph, D_A is the diagonal degree matrix of $(A + A^T)/2$ with $D_A(v,v) = \sum_{u \in V} \frac{1}{2}(A(u,v) + A(v,u))$, and D is the diagonal degree matrix of G with $D(v,v) = \sum_{u \in V}(w(uv) + w(vu))$ equal to the total weighted degree of v in G.

Our second main result is a stronger Cheeger-type inequality that relates $\vec{\lambda}_2^{e*}(G)$ and $\vec{\phi}(G)$.

Theorem 1.9 (Cheeger Inequality for Directed Edge Conductance). *For any directed graph $G = (V, E)$ and any weight function $w : E \to \mathbb{R}_{\geq 0}$,*

$$\vec{\lambda}_2^{e*}(G) \lesssim \vec{\phi}(G) \lesssim \sqrt{\vec{\lambda}_2^{e*}(G) \cdot \log \frac{1}{\vec{\phi}(G)}} \lesssim \sqrt{\vec{\lambda}_2^{e*}(G) \cdot \log \frac{1}{\vec{\lambda}_2^{e*}(G)}}.$$

An important point about Theorem 1.9 is that there is no dependence on the maximum degree of G as in Theorem 1.4 or on the number of vertices of G as in [1, 42]. As a consequence, $\vec{\lambda}_2^{e*}(G)$ is a polynomial time-computable quantity that can be used to certify whether a directed graph has constant edge conductance. This is similar to the role of the second eigenvalue in Cheeger's inequality

[3] The reader may wonder whether it is possible to reduce directed edge conductance to directed vertex expansion, and use Theorem 1.4 to obtain a Cheeger-type inequality for directed edge conductance. This is indeed possible, but the result obtained in this way will have a dependency on the maximum total degree Δ as in Theorem 1.4, while the result that we present in Theorem 1.9 has no such dependency.

to certify whether an undirected graph has constant edge conductance.

Also, as in the proof of Cheeger's inequality, the proof of Theorem 1.9 provides a polynomial time algorithm to return a set S with $\vec{\phi}(S) \leq \sqrt{\vec{\phi}(G) \log 1/\vec{\phi}(G)}$. Since many real-world networks are directed (see [41]), we hope that this "spectral" algorithm will find applications in clustering and partitioning for directed graphs, as the classical spectral partitioning algorithm does in clustering and partitioning for undirected graphs [38, 40].

1.1.3 Generalizations of Cheeger Inequality for Directed Graphs. For undirected graphs, there are several interesting generalizations of Cheeger's inequality: Trevisan's result [39] that relate λ_n to bipartite edge conductance, the higher-order Cheeger's inequality [28, 34] that relates λ_k to k-way edge conductance, and the improved Cheeger's inequality [25] that relates λ_2 and λ_k to edge conductance. Using reweighted eigenvalues for vertex expansion, close analogs of these results were obtained in [26], relating λ_n^* to bipartite vertex expansion, λ_k^* to k-way vertex expansion, and λ_2^* and λ_k^* to vertex expansion.

We study whether there are close analogs of these results for directed graphs, using reweighted eigenvalues for directed vertex expansion in Definition 1.1 and directed edge conductance in Definition 1.7. Perhaps surprisingly, we show that the natural analogs of Trevisan's result and higher-order Cheeger's inequality do not hold, but we obtain analogs of the improved Cheeger's inequality for directed vertex expansion and directed edge conductance. See the full version of the paper for more details.

1.1.4 Cheeger Inequalities for Hypergraph Edge Conductance. We also formulate reweighted eigenvalues for hypergraphs and use them to derive Cheeger-type inequalities for hypergraphs, as supporting results that reweighted eigenvalues provide a unifying approach to study expansion properties in different settings.

Definition 1.10 (Hypergraph Edge Conductance [8, 33])**.** *Let $H = (V, E)$ be a hypergraph and $w : E \to \mathbb{R}_{\geq 0}$ be a weight function on the hyperedges. For a subset $S \subseteq V$, let $\delta(S) := \{e \in E \mid e \cap S \neq \emptyset \text{ and } e \cap \overline{S} \neq \emptyset\}$ be the edge boundary of S and $w(\delta(S))$ be the total edge weight of $\delta(S)$, and let $\mathrm{vol}_w(S) := \sum_{v \in S} \sum_{e: v \in e} w(e)$ be the volume of S. The hypergraph edge conductance of a set $S \subseteq V$ and of the graph G are defined as*

$$\phi(S) := \frac{w\big(\delta(S)\big)}{\min\big\{\mathrm{vol}_w(S), \mathrm{vol}_w(\overline{S})\big\}} \quad \text{and} \quad \phi(H) := \min_{\emptyset \neq S \subset V} \phi(S).$$

The idea is to consider the "clique-graph" of the hypergraph H, and find the best reweighted subgraph of the clique-graph to certify the edge conductance of H, subject to the constraint that the total weight of the "clique-edges" of a hyperedge e is bounded by $w(e)$.

Definition 1.11 (Maximum Reweighted Spectral Gap for Hypergraphs)**.** *Given a hypergraph $H = (V, E)$ and weight function $w : E \to \mathbb{R}_{\geq 0}$, the maximum reweighted spectral gap for H is defined as*

$$\begin{aligned} \gamma_2^*(H) \; &:= \; \max_{A \geq 0} \lambda_2\Big(D^{-\frac{1}{2}}\big(D_A - A\big)D^{-\frac{1}{2}}\Big) \\ &\text{subject to} \quad \sum_{u,v \in e} c(u,v,e) \leq w(e) \qquad \forall e \in E \\ &\qquad A(u,v) = \sum_{e \in E: u,v \in e} c(u,v,e) \qquad \forall u,v \in V. \end{aligned}$$

In this formulation, there is a clique-edge variable $c(u,v,e)$ for each pair of vertices u, v in a hyperedge e, with the constraints that the total weight of the clique-edges in e is bounded by $w(e)$. Then, A is the adjacency matrix of the reweighted subgraph of the clique-graph with edge weight $A(u,v)$ equal to the sum of the weight of the clique-edges involving u and v, D_A is the diagonal degree matrix of A with $D_A(v,v) = \sum_{u \in V} A(u,v)$, and D is the diagonal degree matrix of H with $D(v,v) = \sum_{e \in E: v \in e} w(e)$ equal to the weighted degree of v in H.

There is a spectral theory for hypergraphs based on a continuous time diffusion process with several Cheeger-type inequalities proven [8, 33]. We show that the reweighted eigenvalue approach can be used to provide a simpler and more intuitive way to obtain similar results.

Theorem 1.12 (Cheeger Inequality for Hypergraph Edge Conductance). *For any hypergraph $H = (V, E)$ and any weight function $w : E \to \mathbb{R}_{\geq 0}$,*

$$\gamma_2^*(H) \lesssim \phi(H) \lesssim \sqrt{\gamma_2^*(H) \cdot \log(r)}$$

where r is the maximum size of a hyperedge of H.

We also obtain generalizations of Cheeger's inequalities for hypergraphs using other reweighted eigenvalues such as $\gamma_k^*(H)$ and a new result about improved Cheeger inequality for hypergraphs. We will mention these results and compare the two approaches in Section 1.3.

1.2 Techniques

Conceptually, our contribution is to come up with new spectral formulations for expansion properties in directed graphs and hypergraphs, and to show that the reweighted eigenvalue approach provides a unifying method to reduce expansion problems in more general settings to the basic setting of edge conductances in undirected graphs.

Technically, the proofs are based on the framework developed in [23, 26, 36] in relating reweighted eigenvalues to undirected vertex expansion in (1.2). We briefly describe this framework and then highlight some new elements in the proofs for directed graphs. There are two main steps in proving (1.2). The first step is to construct the dual SDP for the reweighted eigenvalue, and to do random projection to obtain a 1-dimensional solution to the dual program. The second step is to analyze the threshold rounding algorithm for the 1-dimensional solution.

For directed graphs, we identify a key parameter for our analysis.

Definition 1.13 (Asymmetric Ratio of Directed Graphs)**.** *Given an edge-weighted graph $G = (V, E, w)$, the asymmetric ratio of a set $S \subseteq V$ and of the graph G are defined as*

$$\alpha(S) := \frac{w(\delta^+(S))}{w(\delta^+(\overline{S}))} \quad \text{and} \quad \alpha(G) := \max_{\emptyset \neq S \subset V} \alpha(S).$$

Given a vertex-weighted graph $G = (V, E, \pi)$, we define the π-induced weight of an edge $uv \in E$ as $w_\pi(uv) = \min\{\pi(u), \pi(v)\}$, and the asymmetric ratio of a set $S \subseteq V$ and of the graph are defined as above using the edge weight function w_π.

We note that the asymmetric ratio of an edge-weighted graph was defined in [18] with the name "α-balanced" and was used in the analysis of oblivious routing in directed graphs. The asymmetric ratio is a measure of how close a directed graph is to an undirected graph for our purpose, as when $\alpha(G) = 1$ the directed graph is Eulerian and so its edge conductance is the same as the edge conductance of the underlying undirected graph.

This parameter is defined to satisfy two useful properties. The first is that it can be used to prove more refined Cheeger inequalities that

$$\vec{\phi}(G) \lesssim \sqrt{\vec{\lambda}_2^{e*}(G) \cdot \log \alpha(G)}$$
$$\text{and} \quad \vec{\psi}(G) \lesssim \sqrt{\vec{\lambda}_2^{v*}(G) \cdot \log\left(\Delta \cdot \alpha(G)\right)}. \qquad (1.3)$$

The second is that it can be related to the directed edge conductance and directed vertex expansion such that $\alpha(G) \leq 1/\vec{\phi}(G)$ in Lemma 3.5 and $\alpha(G) \leq \Delta/\vec{\psi}(G)$ in Lemma 3.6. Combining the two properties gives Theorem 1.9 and Theorem 1.4.

We highlight two new elements in the proofs of (1.3), one in dimension reduction and one in threshold rounding. In the dimension reduction step, the Johnson-Lindenstrauss lemma can be used to project to a 1-dimensional solution with a factor of $\log |V|$ loss as in [36]. For undirected vertex expansion, this was improved to a factor of $\log \Delta$ loss in two ways: one is the Gauassian projection method in [26, 35], while the other is a better analysis of dimension reduction for maximum matching in [23]. For directed edge conductance and directed vertex expansion, the SDP is more complicated and we do not know how to extend the Gaussian projection method to improve on the $\log |V|$ loss; see the full paper for discussions. Instead, we extend the approach in [23] to prove that random projections only lose a factor of $\log \alpha(G)$ with high probability. When the asymmetric ratio is small, we use Hoffman's result in Lemma 3.8 about bounded-weighted circulations to prove a "large optimal property" of the SDPs (see Lemma 3.9), and use it to adapt the proof in [23] for maximum weighted Eulerian subgraphs; see Section 3.4 for details.

In the threshold rounding step of the 1-dimensional solution, we consider the dual SDP of $\vec{\lambda}_2^{v*}(G)$ and $\vec{\lambda}_2^{e*}(G)$ as in [26]. Unlike the dual SDP for undirected vertex expansion, these dual SDPs (see Lemma 3.18) has some negative terms from some vertex potential function $r : V \to \mathbb{R}$. The new idea in our threshold rounding is to not only consider the ordering defined by the vertex embedding function $f : V \to \mathbb{R}$ as usual, but to consider the two orderings defined by $f \pm r$ and show that threshold rounding will work on one of these two orderings. This idea also leads to a cleaner and nicer proof of the hard directions than that in [26], e.g. without the preprocessing and postprocessing steps; see Section 3.5 for details.

The generalizations of Cheeger inequalities for directed graphs and all Cheeger-type inequalities for hypergraphs are based on the same proofs of the corresponding results in [26] with no new ideas involved. We believe these results show that the reweighted eigenvalue approach provides a unifying method to lift the spectral theory for undirected edge conductance to obtain new results in more general settings in a systematic way.

Finally, we note that the maximum degree Δ for undirected vertex expansion, the asymmetric ratio $\alpha(G)$ for directed edge conductance and directed vertex expansion, and the maximum hyperedge size r for hypergraph edge conductance all play the same role as a measure of how close the respective problem is to the basic problem of undirected edge conductance. The trivial reductions to undirected edge conductance lose a factor of Δ for undirected vertex expansion, a factor of $\alpha(G)$ for directed edge-conductance (by just ignoring the directions), and a factor of r for hypergraph edge conductance (by just considering the clique graph). But the reductions through the reweighted eigenvalue approach only lose a factor of $\log \Delta$ in (1.2), a factor of $\log \alpha(G)$ in (1.3), and a factor of $\log r$ in Theorem 1.12 respectively.

1.3 Related Work

There has been considerable interest in developing a spectral theory for directed graphs and hypergraphs, with many papers that we cannot review them all here. We describe the most relevant ones and compare to our work.

Nonlinear Laplacian for Directed Graphs: Yoshida [41] introduced a nonlinear Laplacian operator for directed graphs and used it to define the following second eigenvalue

$$\lambda_G = \inf_{x \perp \mu_G} \frac{\sum_{uv \in E} \left(\left[x_u/\sqrt{d_u} - x_v/\sqrt{d_v} \right]^+ \right)^2}{\sum_{u \in V} x_u^2}$$

where μ_G denotes the first eigenvector, $[a-b]^+$ denotes $\max\{a - b, 0\}$, and d_u is the total degree of u. He considered the same directed edge conductance as in Definition 1.7 and proved the Cheeger inequality that $\lambda_G/2 \leq \vec{\phi}(G) \leq 2\sqrt{\lambda_G}$, but did not give an approximation algorithm for computing λ_G in [41]. Later, Yoshida [42] gave an SDP approximation algorithm for computing λ_G, and this gives a polynomial time computable quantity $\widetilde{\lambda}_G$ that satisfies $\widetilde{\lambda}_G \lesssim \vec{\phi}(G) \lesssim \sqrt{\widetilde{\lambda}_G \cdot \log |V|}$. We note that this is comparable but improved by our result[4] for $\vec{\phi}(G)$ in (1.3), and cannot be used for certifying constant directed edge conductance as in Theorem 1.9. We also note that this result is dominated by the SDP-based $O(\sqrt{\log |V|})$-approximation algorithm for $\vec{\phi}(G)$ in [1] that we describe below. To our knowledge, this is the only spectral formulation known in the literature that relates to directed edge conductance, and no spectral formulation was known for directed vertex expansion. We also believe that the reweighted eigenvalue approach is simpler and more intuitive than the nonlinear Laplacian operator approach.

Approximation Algorithms Using Semidefinite Programming: In [1], Agarwal, Charikar, Makarychev and Makarychev gave an SDP-based $O(\sqrt{\log |V|})$-approximation algorithm for the directed sparsest cut problem on a directed graph $G = (V, E)$, where the objective is to find a set S that minimizes $|\delta^+(S)|/\min\{|S|, |\overline{S}|\}$. We note that in the unweighted case, directed vertex expansion and directed edge conductance can be reduced to directed sparsest cut via standard reductions. In the weighted case, the SDP for directed sparsest cut can be slightly modified to obtain a $O(\sqrt{\log |V|})$-approximation algorithm for directed edge conductance. To our knowledge, it was not known that the SDP in [1] can be used to certify whether a directed graph has constant edge conductance

[4] We remark that we can use the Johnson-Lindenstrauss lemma to do the dimension reduction step as in [36], and this would give $\vec{\phi}(G) \lesssim \sqrt{\vec{\lambda}_2^{e*}(G) \cdot \log |V|}$ as well.

as in Theorem 1.9, as the analysis using triangle inequalities based on [6] has a $\sqrt{\log|V|}$ factor loss. We show that the SDP in [1] is stronger than the SDP for directed edge conductance in Proposition 3.4. Therefore, using the new analysis through asymmetric ratio in this paper, we also prove that the SDP in [1] provides a polynomial time computable quantity to certify constant directed edge conductance as in Theorem 1.9. See the full paper for proofs and discussion.

Cheeger Constant for Directed Graphs: Fill [19] and Chung [14] defined some symmetric matrices for directed graphs, and related their eigenvalues to Cheeger's constant and to mixing time. Their formulations are similar to each other, but Chung's formulation is closer and more consistent with ours, as her work is also based on an Eulerian reweighted subgraph (which was called a circulation in [14]) that we describe below.

Given a directed graph $G = (V, E)$ with a weight function $w : E \to \mathbb{R}_{\geq 0}$, let P be the transition matrix of the ordinary random walks on G with $P(u, v) = w(uv)/\sum_{v \in V} w(uv)$ for $uv \in E$. Suppose G is strongly connected, then there is a unique stationary distribution $\pi : V \to \mathbb{R}_+$ of the random walks on G such that $\pi^T P = \pi^T$. Let $\Pi := \text{diag}(\pi)$. Fill [19] defined the product and the sum matrices as $M(P) := P\Pi^{-1}P^T\Pi$ and $A(P) := (P + \Pi^{-1}P^T\Pi)/2$. Chung [14] noted that if the weight of an edge uv is defined as $f(u, v) = \pi(u) \cdot P(u, v)$, then the weighted directed graph $G' = (V, E, f)$ is Eulerian such that $\sum_{u:uv \in E} f(u, v) = \sum_{w:vw \in E} f(v, w)$ for all $v \in V$. Then she used the underlying weighted undirected graph to define the Laplacian of a directed graphs as

$$\begin{aligned} \widetilde{\mathcal{L}} &:= I - \left(\Pi^{1/2}P\Pi^{-1/2} + \Pi^{-1/2}P^T\Pi^{1/2}\right)/2 \\ &= I - \Pi^{-\frac{1}{2}}\left(F + F^T\right)\Pi^{-\frac{1}{2}}/2 \end{aligned} \tag{1.4}$$

where $F = \Pi P$ is the adjacency matrix of G'. Note that the spectrums of $A(P)$ and $\widetilde{L}$ are essentially the same, as $P + \Pi^{-1}P^T\Pi$ and $\Pi^{1/2}P\Pi^{-1/2} + \Pi^{-1/2}P^T\Pi^{1/2}$ are similar matrices. Note also that $\widetilde{\mathcal{L}}$ is exactly the same as the normalized Laplacian matrix in the objective function in Definition 1.3. The Cheeger constant of a directed graph [14, 19] is defined as

$$h(G) := \min_{S:S\neq\emptyset, S\neq V} h(S) \text{ where } h(S) = \frac{\sum_{u,v:u\in S, v\notin S} \pi(u)P(u,v)}{\min\{\pi(S), \pi(\overline{S})\}}, \tag{1.5}$$

and Chung [14] proved that $\lambda_2(\widetilde{\mathcal{L}})/2 \leq h(G) \leq \sqrt{2\lambda_2(\widetilde{\mathcal{L}})}$.

The main difference between our formulations and Chung's formulation is that we search for an *optimal* reweighting while Chung used a specific vertex-based reweighing by the stationary distribution. We note that the Cheeger constant in (1.5) could be very different from the directed edge conductance in Definition 1.7 and the directed vertex expansion in Definition 1.1; see the full paper for some examples. We remark that many subsequent works used Cheeger constant as the objective for clustering and partitioning for directed graphs, and these examples illustrate their limitations in finding sets of small directed edge conductance or directed vertex expansion, which are much more suitable notions for clustering and partitioning (see [41] for related discussions).

Mixing Time and Fastest Mixing Time: A main result in [14, 19] is to use the second eigenvalue of $M(P), A(P)$, or $\lambda_2(\widetilde{\mathcal{L}})$ to bound the mixing time of the ordinary random walks on G. We state the result using Chung's formulation as it is closer to our formulation in Definition 1.1.

Theorem 1.14 (Bounding Mixing Time by Second Eigenvalue of Directed Graphs [14, 19]). *Let G be a strongly connected directed graph $G = (V, E)$ with a weight function $w : E \to \mathbb{R}_{\geq 0}$, and P be the transition matrix of the ordinary random walks on G with $P(u, v) = w(uv)/\sum_{v \in V} w(uv)$ for $uv \in E$. Then the mixing time of the lazy random walks of G with transition matrix $(I + P)/2$ to the stationary distribution π is*

$$\tau\left(\frac{I+P}{2}\right) \lesssim \frac{1}{\lambda_2(\widetilde{\mathcal{L}})} \cdot \log\left(\frac{1}{\pi_{\min}}\right)$$

where $\lambda_2(\widetilde{\mathcal{L}})$ is the second smallest eigenvalue of the Laplacian in (1.4) and $\pi_{\min} = \min_{v \in V} \pi(v)$.

We will use Theorem 1.14 to bound the fastest mixing time for general Markov chains in Theorem 1.6. The fastest mixing time problem of reversible Markov chains was introduced by Boyd, Diaconis, and Xiao [7]. This is a well-motivated problem in the study of Markov chains and has generated considerable interest (see the references in [36]), but there were no known combinatorial characterization of the fastest mixing time for quite some time. Recently, Oleskar-Taylor and Zanetti [36] discovered a new Cheeger-type inequality relating reweighted second eigenvalue λ_2^* and vertex expansion, and used it to give a combinatorial characterization of the fastest mixing time of reversible Markov chains by the vertex expansion of the graph. Theorem 1.6 is a significant generalization of their result to general Markov chains, and we believe it is of independent interest.

Other Cheeger-Type Inequalities for Directed Graphs: Chan, Tang and Zhang [10] gave a higher-order Cheeger inequality for directed graphs. Roughly speaking, they showed that there are k disjoint subsets $S_1, \ldots, S_k \subseteq V$ with $\lambda_k(\widetilde{\mathcal{L}}) \lesssim h(S_i) \lesssim k^2 \cdot \sqrt{\lambda_k(\widetilde{\mathcal{L}})}$ for $1 \leq i \leq k$, where $h(S_i)$ is the Cheeger constant in (1.5) and $\lambda_k(\widetilde{\mathcal{L}})$ is the k-th smallest eigenvalue of the Laplacian in (1.4). The proof is a direct application of the higher-order Cheeger inequality for undirected graphs on the reweighted subgraph by the stationary distribution. We show an example (see the full paper) that rules out the possibility of having a higher-order Cheeger inequality for directed graphs relating $\lambda_k(\widetilde{\mathcal{L}})$ to k-way directed edge conductance.

Other Hermitian Matrices of Directed Graphs: Besides the matrices in [14, 19], there are other Hermitian matrices associated to a directed graph studied in the literature. Guo and Mohar [21] and Liu and Li [32] defined the Hermitian adjacency matrix H of a directed graph as $H(u, v) = 1$ if both $uv, vu \in E$, $H(u, v) = \imath$ if $uv \in E$ and $vu \notin E$ where $\imath$ is the imaginary unit, $H(u, v) = -\imath$ if $uv \notin E$ and $vu \in E$, and $H(u, v) = 0$ if both $uv, vu \notin E$. There are also other Hermitian matrices defined for clustering directed graphs [17, 27] and for the Max-2-Lin problem [30]. We confirm that there are no known relations between the eigenvalues of these Hermitian matrices and the expansion properties of a directed graph.

Directed Laplacian Solver Using Eulerian Reweighting: We note that the idea of reducing the problem for a directed graph to an Eulerian directed graph was also used in directed Laplacian solvers [15, 16]. As in [14], they also use the same reweighting by the stationary distribution to obtain an Eulerian graph from a directed graph. (Furthermore, they introduced a notion of spectral sparsification of Eulerian directed graphs.) We believe that the idea of reducing to Eulerian directed graphs and the concept of asymmetric ratio will find more applications in solving problems on directed graphs.

Spectral Theory for Hypergraphs: Louis [33] and Chan, Louis, Tang, Zhang [8] developed a spectral theory for hypergraphs. They defined a continuous time diffusion process on a hypergraph $H = (V, E)$ and used it to define a *nonlinear* Laplacian operator and its eigenvalues $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_{|V|}$. Then they derived a Cheeger inequality $\frac{1}{2}\gamma_2 \leq \phi(H) \leq \sqrt{2\gamma_2}$, where $\phi(H)$ is the hypergraph edge conductance of H in Definition 1.10. But the quantity γ_2 is not polynomial time computable, and a semidefinite programming relaxation of γ_2 was used to output a set of edge conductance $O(\sqrt{\phi(H) \cdot \log r})$ where r is the maximum size of a hyperedge. They also proved an analog of higher-order Cheeger inequality for hypergraph edge conductance, such that for any $\epsilon \geq 1/k$ there are disjoint subsets $S_1, \ldots, S_{(1-\epsilon)k}$ with

$$\phi(S_i) \lesssim k^{2.5} \cdot \epsilon^{-1.5} \cdot \log k \cdot \log\log k \cdot \log r \cdot \sqrt{\widetilde{\gamma}_k} \qquad (1.6)$$

for all $i \leq (1-\epsilon)k$, where $\widetilde{\gamma}_k$ can be thought of as a relaxation of γ_k. They also gave an improved approximation algorithm for the small-set hypergraph edge conductance problem.

Using the reweighted eigenvalue approach, we can define the maximum reweighted k-th eigenvalue γ_k^* as in Definition 1.11, and proved the following analog of higher-order Cheeger inequality for hypergraph edge conductance: For any $\epsilon \geq 1/k$, there are disjoint subsets $S_1, \ldots, S_{(1-\epsilon)k}$ with

$$\phi(S_i) \lesssim \sqrt{k} \cdot \epsilon^{-4} \cdot \log k \cdot \sqrt{\log r} \cdot \sqrt{\gamma_k^*}$$

for all $i \leq (1-\epsilon)k$. This bound is comparable to that in [8] when $\epsilon \approx 1/k$, and is an improvement when $\epsilon = \Theta(1)$ by a factor of more than k^2. This also improves the approximation algorithm for the small-set hypergraph edge conductance problem in [8] by a factor of more than k. In addition, we also prove an analog of the improved Cheeger's inequality [25] for hypergraphs. See Section 5 of the full paper for these results.

Compared to the spectral theory in [8, 33] for hypergraphs using the continuous time diffusion process, we believe that the reweighted eigenvalue approach is simpler and more intuitive. The definitions of the hypergraph diffusion process and its eigenvalues are quite technically involved and required considerable effort to make rigorous [9]. The reweighted eigenvalue approach allows us to recover and improve their results on hypergraph partitioning, and also to obtain a new result. Since their spectral theory for hypergraph partitioning is gaining more attention in machine learning lately (e.g. [31]), we believe that it would be beneficial to have an alternative approach that is easier to understand and to prove new results and to have efficient implementations.

Finally, as a technical remark, we note that some careful reweighting schemes are crucially used in the construction of the diffusion process [8, 33], and also in recent exciting developments in hypergraph spectral sparsification [12, 24] (called balanced weight assignments). This suggests that the concept of *reweighting* is central to these recent developments, and it would be very interesting to find connections between the different reweighting methods used in this work and these previous works.

2 PRELIMINARIES

Notations and basic facts about undirected graphs and hypergrpahs, directed graphs, random walks, spectral graph theory, and semidefinite programming that are not present in Section 1 can be found in the full paper.

3 CHEEGER INEQUALITIES FOR DIRECTED GRAPHS

We prove the two main results Theorem 1.4 and Theorem 1.9 in this section. First, we prove the easy directions of the two results in Section 3.1, and write the semidefinite programs for the reweighted eigenvalues in Section 3.2. Then, we show some properties of the asymmetric ratio in Section 3.3, and use these properties and the proof in [23] to analyze a random projection algorithm to construct 1-dimensional spectral solutions to the semidefinite programs in Section 3.4. Then, we analyze a new threshold rounding algorithm for a 1-dimensional solution to the dual programs, and prove the hard direction of the two results in Section 3.5. Finally, we show Theorem 1.6 about fastest mixing time using [14, 19] in Section 3.6. We omit here a subsection in the full paper about the relations with some previous work mentioned in Section 1.3.

3.1 Easy Directions by Reductions

There are two ways to prove the easy directions in Theorem 1.4 and Theorem 1.9. A standard way is to construct a solution to $\vec{\lambda}_2^{v*}(G)$ or $\vec{\lambda}_2^{e*}(G)$ with small objective value when the directed vertex expansion or the directed edge conductance is small. Instead, we use the reduction idea discussed in the introduction to prove the easy directions, as this is how we came up with the formulations and the reduction is the main theme in this paper.

Proposition 3.1 (Easy Direction for Directed Vertex Expansion)**.** *For any directed graph $G = (V, E)$ with weight function $\pi : V \to \mathbb{R}_{\geq 0}$, it holds that $\vec{\lambda}_2^{v*}(G) \leq 2\vec{\psi}(G)$.*

Proof. The idea is to reduce directed vertex expansion of G to the directed edge conductance of the reweighted Eulerian subgraph defined by A in Definition 1.3, and then reduce to the underlying undirected graph defined by $\frac{1}{2}(A + A^T)$ and use classical Cheeger's inequality to lower bound its edge conductance by the second eigenvalue of its normalized Laplacian matrix.

Let $w(uv) := A(u, v)$ be the edge weight in the Eulerian reweighted subgraph for $uv \in E$. For any nonempty $S \subset V$, by Definition 1.1 of directed vertex expansion and Definition 1.7 of directed edge conductance,

$$\vec{\psi}(S) = \frac{\min\left\{\pi(\partial^+(S)), \pi(\partial^+(\overline{S}))\right\}}{\min\{\pi(S), \pi(\overline{S})\}} \geq \frac{2 \cdot \min\left\{w(\delta^+(S)), w(\delta^-(S))\right\}}{\min\{\mathrm{vol}_w(S), \mathrm{vol}_w(\overline{S})\}} = 2\vec{\phi}(S)$$

where we use the degree constraints in Definition 1.3 to establish that $w(\delta^+(S)) \leq \pi(\partial^+(S))$ and $w(\delta^-(S)) \leq \pi(\partial^+(\overline{S}))$ (note that they are not necessarily equalities because of the self-loops), and $\mathrm{vol}_w(S) = 2\pi(S)$ for every nonempty $S \subset V$.

As the edge-weighted directed graph $G' = (V, E, w)$ is Eulerian, it holds that $w(\delta^+(S)) = w(\delta^-(S))$ for every nonempty $S \subset V$, and thus the directed edge conductance of G' is equal to half the edge conductance of the underlying undirected graph G'' with edge weight $w''(uv) = \frac{1}{2}\big(w(uv) + w(vu)\big)$, because

$$2\vec{\phi}(S) = \frac{\min\left\{w(\delta^+(S)), w(\delta^-(S))\right\}}{\frac{1}{2}\cdot\min\{\mathrm{vol}_w(S), \mathrm{vol}_w(\overline{S})\}} = \frac{w''(\delta(S))}{\min\{\mathrm{vol}_{w''}(S), \mathrm{vol}_{w''}(\overline{S})\}} = \phi(S).$$

As the graph G'' is undirected, we can use Cheeger's inequality in (1.1) to lower bound the edge conductance of G'' by the second smallest eigenvalue of its normalized Laplacian matrix $\mathcal{L}(A) := I - \frac{1}{2}\Pi^{-1/2}(A + A^T)\Pi^{-1/2}$. Therefore, for any nonempty $S \subset V$, $\vec{\psi}(S) \geq 2\vec{\phi}(S) = \phi(S) \geq \lambda_2(\mathcal{L}(A))/2$. Since this holds for any nonempty $S \subset V$ and any weighted Eulerian subgraph defined by A satisfying the constraints in Definition 1.3, we conclude that $2\vec{\psi}(G) \geq \max_A \lambda_2(\mathcal{L}(A)) = \vec{\lambda}_2^{v*}(G)$. □

The proof of the easy direction of Theorem 1.9 is similar, but with a subtle difference in handling the denominator. Refer to the full paper for the proof.

Proposition 3.2 (Easy Direction for Directed Edge Conductance)**.** *For any directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}_{\geq 0}$, it holds that $\vec{\lambda}_2^{e*}(G) \leq 2\vec{\phi}(G)$.*

3.2 Semidefinite Programs

We show that the optimization problems of reweighted eigenvalues can be formulated exactly as semidefinite programs, and so they can be approximated arbitrarily well in polynomial time. The construction is similar to that of the semidefinite program for undirected vertex expansion in [7, 37], but von Neumann minimax theorem is used instead of SDP duality.

Proposition 3.3 (SDP for Reweighted Second Eigenvalue with Vertex Capacity Constraints)**.** *Given a directed graph $G = (V, E)$ and a weight function $\pi : V \to \mathbb{R}_{\geq 0}$, the optimization problem in Definition 1.3 can be written as*

$$\begin{aligned}
\vec{\lambda}_2^{v*}(G) := \min_{f:V\to\mathbb{R}^n} \max_{A\geq 0}\ & \frac{1}{2}\sum_{uv\in E} A(u,v)\cdot\|f(u)-f(v)\|^2 \\
\text{subject to}\ & A(u,v) = 0 && \forall uv \notin E \\
& \sum_{v\in V} A(u,v) = \sum_{v\in V} A(v,u) && \forall u \in V \\
& \sum_{v\in V} A(v,u) = \pi(u) && \forall u \in V \\
& \sum_{v\in V} \pi(v)\cdot f(v) = \vec{0} \\
& \sum_{v\in V} \pi(v)\cdot\|f(v)\|^2 = 1.
\end{aligned}$$

Proof. Let $\mathcal{L} := I - \frac{1}{2}\Pi^{-1/2}(A + A^T)\Pi^{-1/2}$ be the normalized Laplacian matrix in the objective function $\max_A \lambda_2(\mathcal{L})$ in Definition 1.3. By Rayleigh quotient,

$$\lambda_2(\mathcal{L}) = \min_{f\perp\Pi\vec{1}} \frac{\sum_{(u,v)\in\binom{V}{2}} \frac{1}{2}\big(A(u,v)+A(v,u)\big)\cdot|f(u)-f(v)|^2}{\sum_v \pi(v) f(v)^2}.$$

Then we write $f \perp \Pi\vec{1}$ as the second last constraint and normalize the denominator to 1 as the last constraint. Note that the SDP relaxation where we replace $f : V \to \mathbb{R}$ by $f : V \to \mathbb{R}^n$ is an exact relaxation. Moreover, after the SDP relaxation, the feasible domain becomes convex and so we can apply von Neumann minimax theorem to switch the order of $\max_A \min_f$ in Definition 1.3 to $\min_f \max_A$ as in the statement of this lemma. □

The same construction is used for $\vec{\lambda}_2^{e*}(G)$ in Definition 1.8 and the proof is omitted.

Proposition 3.4 (SDP for Reweighted Second Eigenvalue with Edge Capacity Constraints)**.** *Given a directed graph $G = (V, E)$ and a weight function $w : E \to \mathbb{R}_{\geq 0}$, the optimization problem in Definition 1.8 can be written as*

$$\begin{aligned}
\vec{\lambda}_2^{e*}(G) := \min_{f:V\to\mathbb{R}^n} \max_{A\geq 0}\ & \frac{1}{2}\sum_{uv\in E} A(u,v)\cdot\|f(u)-f(v)\|^2 \\
\text{subject to}\ & A(u,v) = 0 && \forall uv \notin E \\
& \sum_{v\in V} A(u,v) = \sum_{v\in V} A(v,u) && \forall u \in V \\
& A(u,u) \leq w(uv) && \forall uv \in E \\
& \sum_{v\in V} d_w(v)\cdot f(v) = \vec{0} \\
& \sum_{v\in V} d_w(v)\cdot\|f(v)\|^2 = 1.
\end{aligned}$$

We will use these semidefinite programs to prove the two main results.

3.3 Asymmetric Ratio

A key parameter in our proofs is the asymmetric ratio $\alpha(G)$ in Definition 1.13. This parameter satisfies two useful properties. One is that $\alpha(G)$ can be used to bound the directed edge conductance and directed vertex expansion. Another is that directed graphs with bounded asymmetric ratio satisfy the "large optimal property" that we will describe in Section 3.3.2, which can be used in the proof in [23] to provide a better analysis of the random projection algorithm for dimension reduction of the SDP solutions.

3.3.1 Asymmetric Ratio and Expansion Properties. The relation between asymmetric ratio of edge-weighted graph and directed edge conductance is simple.

Lemma 3.5 (Asymmetric Ratio and Directed Edge Conductance)**.** *For any directed graph $G = (V, E)$ and any weight function $w : E \to \mathbb{R}_{\geq 0}$, it holds that $\alpha(G) \leq 1/\vec{\phi}(G)$.*

The relation between asymmetric ratio of vertex-weighted graph and directed vertex expansion is less trivial and has a dependency on the maximum total degree Δ.

Lemma 3.6 (Asymmetric Ratio and Directed Vertex Expansion)**.** *For any directed graph $G = (V, E)$ and any weight function $\pi : V \to \mathbb{R}_{\geq 0}$, it holds that $\alpha(G) \lesssim \Delta/\vec{\psi}(G)$.*

The proofs of Lemma 3.5 and Lemma 3.6 can be found in the full paper.

3.3.2 Asymmetric Ratio and Large Optimal Property. Consider the semidefinite programs for $\vec{\lambda}_2^{v*}(G)$ and $\vec{\lambda}_2^{e*}(G)$ in Proposition 3.3 and Proposition 3.4. When the geometric embedding $f : V \to \mathbb{R}^n$ in the outer minimization problem is fixed, the inner maximization problem is simply to find a maximum weighted Eulerian subgraph A with vertex capacity constraints in Proposition 3.3 and with edge capacity constraints in Proposition 3.4. The following are trivial upper bounds on the optimal values of the inner maximization problems.

Claim 3.7 (Maximum Weighted Eulerian Subgraph with Capacity Constraints)**.** *Given a directed graph $G = (V, E)$ and an embedding $f : V \to \mathbb{R}^n$, let $\nu_f^{v*}(G)$ and $\nu_f^{e*}(G)$ be the objective values of the inner maximization problem in Proposition 3.3 and Proposition 3.4 respectively. Then*

$$\nu_f^{v*}(G) \leq \frac{1}{2}\sum_{uv\in E} w_\pi(uv)\cdot\|f(u)-f(v)\|^2$$

$$\text{and}\quad \nu_f^{e*}(G) \leq \frac{1}{2}\sum_{uv\in E} w(uv)\cdot\|f(u)-f(v)\|^2,$$

where $w_\pi(uv) = \min\{\pi(u), \pi(v)\}$ be the π-induced edge weight function defined in Definition 1.13.

In the undirected vertex expansion problem [23, 36], when $\pi(v) = 1$ for all $v \in V$, the inner maximization problem is exactly the maximum weighted fractional matching problem. Jain, Pham and Vuong [23] used the fact that any graph with maximum degree Δ has an edge coloring with at most $\Delta + 1$ colors to show that the inner maximization problem has a solution with weight at least $1/(\Delta + 1)$ fraction of the trival upper bound. They then used this "large optimal property" to analyze a dimension reduction algorithm for maximum weighted matching; see Section 3.4.

We observe that the asymmetric ratio $\alpha(G)$ in Definition 1.13 can be used to play the same role as Δ to establish the large optimal property for the maximum weighted Eulerian subgraph problems in Claim 3.7. The proof uses the following characterization of asymmetric ratio by Hoffman (see also [18, Theorem 2.3]), rephrased using our terminologies.

Lemma 3.8 (Hoffman's Circulation Lemma)**.** *Let $G = (V, E)$ be a directed graph with a weight function $w : E \to R_{\geq 0}$. Then G has asymmetric ratio at most α if and only if there exists an Eulerian reweighting A of G such that*

$$\sum_{v:uv\in E} A(u,v) = \sum_{v:vu\in E} A(v,u) \text{ for all } u \in V$$

$$\text{and}\quad w(uv) \leq A(u,v) \leq \alpha\cdot w(uv) \text{ for all } uv \in E.$$

The large optimal property in terms of asymmetric ratio is a simple consequence of Hoffman's circulation lemma.

Lemma 3.9 (Large Optimal Property)**.** *Given a directed graph $G = (V, E)$ and an embedding $f : V \to \mathbb{R}^n$, let $\nu_f^{v*}(G)$ and $\nu_f^{e*}(G)$ be the objective values of the inner maximization problem in Proposition 3.3 and Proposition 3.4 respectively. Then*

$$\nu_f^{v*}(G) \geq \frac{1}{2\Delta\cdot\alpha(G)}\sum_{uv\in E} w_\pi(uv)\cdot\|f(u)-f(v)\|^2$$

$$\text{and}\quad \nu_f^{e*}(G) \geq \frac{1}{2\alpha(G)}\sum_{uv\in E} w(uv)\cdot\|f(u)-f(v)\|^2.$$

Proof. First, consider $\nu_f^{e*}(G)$ in Proposition 3.4 with weight function $w : E \to \mathbb{R}_{\geq 0}$. Let A be an Eulerian reweighting of G with weight function w given in Lemma 3.8. As $w(uv) \leq A(u,v) \leq \alpha(G)\cdot w(uv)$ for $uv \in E$, the scaled-down subgraph $A/\alpha(G)$ satisfies the edge capacity constraints and is a feasible solution to the inner maximization problem in Proposition 3.4, with objective value $\frac{1}{2}\sum_{uv\in E}\frac{A(u,v)}{\alpha(G)}\cdot\|f(u)-f(v)\|^2 \geq \frac{1}{2\alpha(G)}\sum_{uv\in E} w(uv)\cdot\|f(u)-f(v)\|^2$.

The lower bound on $\nu_f^{v*}(G)$ can be similarly proven. □

We will use Lemma 3.9 in the analysis of the dimension reduction step in the next subsection.

3.4 Dimension Reduction

The goal in this subsection is to obtain a good low-dimensional solution to the semidefinite programs in Proposition 3.3 and Proposition 3.4. See the full paper for omitted proofs and discussions.

Definition 3.10 (Low-Dimensional Solutions to Semidefinite Programs)**.** *Define*

$$\vec{\lambda}_v^{(k)}(G) := \min_{f:V\to\mathbb{R}^k}\max_{A\geq 0}\frac{1}{2}\sum_{uv\in E} A(u,v)\cdot\|f(u)-f(v)\|^2$$

to be the objective value of the SDP in Proposition 3.3 when restricting f to be a k-dimensional embedding and subjecting to the same constraints. Define $\vec{\lambda}_e^{(k)}(G)$ similarly as the objective value of the SDP in Proposition 3.4 when restricting f to be a k-dimensional embedding subjecting to the same constraints.

The main result that we will prove in this subsection is that there is a good 1-dimensional solution when the asymmetric ratio of the graph is small.

Theorem 3.11 (One Dimensional Solutions to Semidefinite Programs). *Let $\vec{\lambda}_v^{(k)}(G)$ and $\vec{\lambda}_e^{(k)}(G)$ be as defined in Definition 3.10. Then*

$$\vec{\lambda}_v^{(1)}(G) \lesssim \log\left(\Delta\cdot\alpha(G)\right)\cdot\vec{\lambda}_2^{v*}(G) \text{ and } \vec{\lambda}_e^{(1)}(G) \lesssim \log\alpha(G)\cdot\vec{\lambda}_2^{e*}(G).$$

The proof consists of two stages. First, by adapting the dimension reduction theorem for maximum matchings in [23], we obtain the following main technical result. The main ingredient of the proof is the Large Optimal Property in Lemma 3.9.

Theorem 3.12 (Dimension Reduction for Maximum Weighted Eulerian Subgraphs). *Let $\vec{\lambda}_v^{(k)}(G)$ and $\vec{\lambda}_e^{(k)}(G)$ be as defined in Definition 3.10. There exists a constant C such that*

$$\vec{\lambda}_v^{\left(C\cdot\log(\Delta\cdot\alpha(G))\right)}(G) \lesssim \vec{\lambda}_2^{v*}(G) \quad\text{and}\quad \vec{\lambda}_e^{\left(C\cdot\log\alpha(G)\right)}(G) \lesssim \vec{\lambda}_2^{e*}(G).$$

Next, by choosing the best coordinate from a k-dimensional embedding, one can achieve the following bound.

Lemma 3.13 (One Dimensional Solution from k-Dimensional Solution)**.** *Let $\vec{\lambda}_v^{(k)}(G)$ and $\vec{\lambda}_e^{(k)}(G)$ be as defined in Definition 3.10. Then*

$$\vec{\lambda}_v^{(1)}(G) \leq k\cdot\vec{\lambda}_v^{(k)}(G) \quad\text{and}\quad \vec{\lambda}_e^{(1)}(G) \leq k\cdot\vec{\lambda}_e^{(k)}(G)$$

Theorem 3.11 follows immediately from Theorem 3.12 and Lemma 3.13.

3.5 Rounding Algorithms

The main goal in this subsection is to show how to find a set of small directed vertex expansion (respectively directed edge conductance) from a solution to $\vec{\lambda}_v^{(1)}(G)$ (respectively $\vec{\lambda}_e^{(1)}(G)$).

Theorem 3.14 (Rounding One Dimensional Solution). *For any vertex-weighted directed graph $G = (V, E, \pi)$,*

$$\vec{\psi}(G) \lesssim \sqrt{\vec{\lambda}_v^{(1)}(G)}.$$

For any edge-weighted directed graph $G = (V, E, w)$,

$$\vec{\phi}(S) \lesssim \sqrt{\vec{\lambda}_e^{(1)}(G)}.$$

Assuming Theorem 3.14, we can complete the proofs of the two main results.

Proof of Theorem 1.4 and Theorem 1.9. The easy directions are proved in Proposition 3.1 and Proposition 3.2. For the hard directions, first we solve the semidefinite programs for $\vec{\lambda}_2^{v*}(G)$ in Proposition 3.3 and $\vec{\lambda}_2^{e*}(G)$ in Proposition 3.4. Then, we use the dimension reduction result in Theorem 3.11 to obtain 1-dimensional solutions to the semidefinite programs with $\vec{\lambda}_v^{(1)}(G) \lesssim \log(\Delta \cdot \alpha(G)) \cdot \vec{\lambda}_2^{v*}(G)$ and $\vec{\lambda}_e^{(1)}(G) \lesssim \log \alpha(G) \cdot \vec{\lambda}_2^{e*}(G)$. Then, we apply the rounding result in Theorem 3.14 to establish that

$$\vec{\psi}(G) \lesssim \sqrt{\log(\Delta \cdot \alpha(G)) \cdot \vec{\lambda}_2^{v*}(G)} \text{ and } \vec{\phi}(G) \lesssim \sqrt{\log \alpha(G) \cdot \vec{\lambda}_2^{e*}(G)}. \tag{3.1}$$

Finally, we use the inequality $\alpha(G) \lesssim \Delta/\vec{\psi}(G)$ in Lemma 3.6 and $\alpha(G) \leq 1/\vec{\phi}(G)$ in Lemma 3.5 to obtain the final forms in Theorem 1.4 and Theorem 1.9. □

We remark that all the steps in the proofs of the two main results can be implemented in polynomial time, and so these give efficient "spectral" algorithms to find a set of small directed vertex expansion or small directed edge conductance.

3.5.1 Proof Structure and Auxiliary Programs. The programs $\vec{\lambda}_v^{(1)}$ and $\vec{\lambda}_e^{(1)}$ can be considered "ℓ_2^2 programs" because the embedded distance across an edge is the squared ℓ_2 distance $\|f(u) - f(v)\|^2$. To prove Theorem 3.14, we first obtain a solution to the following ℓ_1 versions of $\vec{\lambda}_v^{(1)}$ and $\vec{\lambda}_e^{(1)}$.

Definition 3.15 (ℓ_1 Version of $\vec{\lambda}_v^{(1)}$). *Given a vertex-weighted directed graph $G = (V, E, \pi)$, let*

$$\begin{aligned} \eta_v(G) := \min_{f:V\to\mathbb{R}} \max_{A\geq 0} \ & \frac{1}{2} \sum_{uv\in E} A(u,v) \cdot |f(u) - f(v)| \\ \text{subject to } \ & A(u,v) = 0 && \forall uv \notin E \\ & \sum_{v\in V} A(u,v) = \sum_{v\in V} A(v,u) && \forall u \in V \\ & \sum_{v\in V} A(v,u) = \pi(u) && \forall u \in V \\ & \sum_{v\in V} \pi(v) \cdot f(v) = 0 \\ & \sum_{v\in V} \pi(v) \cdot |f(v)| = 1. \end{aligned}$$

Definition 3.16 (ℓ_1 Version of $\vec{\lambda}_e^{(1)}$). *Given an edge-weighted directed graph $G = (V, E, w)$, let*

$$\begin{aligned} \eta_e(G) := \min_{f:V\to\mathbb{R}} \max_{A\geq 0} \ & \frac{1}{2} \sum_{uv\in E} A(u,v) \cdot |f(u) - f(v)| \\ \text{subject to } \ & A(u,v) = 0 && \forall uv \notin E \\ & \sum_{v\in V} A(u,v) = \sum_{v\in V} A(v,u) && \forall u \in V \\ & A(u,u) \leq w(uv) && \forall uv \in E \\ & \sum_{v\in V} d_w(v) \cdot f(v) = 0 \\ & \sum_{v\in V} d_w(v) \cdot |f(v)| = 1. \end{aligned}$$

We will prove in Section 3.5.2 that there is a square root loss by going from ℓ_2^2 to ℓ_1.

Proposition 3.17 (Reductions from ℓ_2^2 to ℓ_1). *For any vertex-weighted directed graph $G = (V, E, \pi)$,*

$$\eta_v(G) \lesssim \sqrt{\vec{\lambda}_v^{(1)}(G)}.$$

For any edge-weighted directed graph $G = (V, E, w)$,

$$\eta_e(G) \lesssim \sqrt{\vec{\lambda}_e^{(1)}(G)}.$$

For threshold rounding, we construct the duals of $\eta_v(G)$ and $\eta_e(G)$ using linear programming duality in the inner maximization problems.

Lemma 3.18 (Dual Program of $\eta_v(G)$). *Given a vertex-weighted directed graph $G = (V, E, \pi)$, let $\xi_v(G)$ be defined as*

$$\begin{aligned} \min_{f:V\to\mathbb{R}} \min_{\substack{q:V\to\mathbb{R}_{\geq 0} \\ r:V\to\mathbb{R}}} \ & \sum_{v\in V} \pi(v) \cdot q(v) \\ \text{subject to } \ & q(v) \geq |f(u) - f(v)| - r(u) + r(v) && \forall uv \in E \\ & \sum_{v\in V} \pi(v) \cdot f(v) = 0 \\ & \sum_{v\in V} \pi(v) \cdot |f(v)| = 1. \end{aligned}$$

Then $\xi_v(G) = 2\eta_v(G)$.

Proof. To write the dual program, we consider the equivalent program of $\vec{\lambda}_v^{(1)}(G)$, where we remove the self-loops and replace the constraint $\sum_{v\in V} A(v,u) = \pi(u)$ by $\sum_{v\in V} A(v,u) \leq \pi(u)$. Then we multiply the objective of $\eta_v(G)$ by a factor of 2 (to avoid the factor 1/2 carrying around). Then we associate a dual variable $q(u) \geq 0$ to each constraint $\sum_{v\in V} A(v,u) \leq \pi(u)$, and a dual variable $r(u)$ to each constraint $\sum_{v\in V} A(u,v) = \sum_{v\in V} A(v,u)$. The result follows from standard linear programming duality. □

The dual program of $\eta_e(G)$ is constructed in the same way and the proof is omitted.

Lemma 3.19 (Dual Program of $\eta_e(G)$). *Given an edge-weighted directed graph $G = (V, E, w)$, let $\xi_e(G)$ be defined as*

$$\min_{f:V\to\mathbb{R}} \min_{\substack{q:E\to\mathbb{R}_{\geq 0}\\ r:V\to\mathbb{R}}} \sum_{uv\in E} w(uv)\cdot q(uv)$$
$$\text{subject to } q(uv) \geq |f(u)-f(v)| - r(u) + r(v) \quad \forall uv \in E$$
$$\sum_{v\in V} d_w(v)\cdot f(v) = 0$$
$$\sum_{v\in V} d_w(v)\cdot |f(v)| = 1.$$

Then $\xi_e(G) = 2\eta_e(G)$.

In Section 3.5.3, we will present a threshold rounding algorithm to return a set of small directed vertex expansion (respectively directed edge conductance) from a solution to $\xi_v(G)$ (respectively $\xi_e(G)$), with only a constant factor loss.

Proposition 3.20 (Threshold Rounding). *For any vertex-weighted directed graph* $G = (V, E, \pi)$*,*

$$\vec{\psi}(G) \lesssim \xi_v(G).$$

For any edge-weighted directed graph $G = (V, E, w)$*,*

$$\vec{\phi}(G) \lesssim \xi_e(G).$$

Note that Theorem 3.14 follows immediately from Proposition 3.17 and Proposition 3.20, so it remains to prove the two propositions in Section 3.5.2 and Section 3.5.3.

3.5.2 *Reduction from ℓ_2^2 to ℓ_1.*

We prove the first inequality in Proposition 3.17 about directed vertex expansion.

Let $G = (V, E, \pi)$ be a vertex-weighed directed graph. Let $f : V \to \mathbb{R}$ be a solution to $\vec{\lambda}_v^{(1)}(G)$ with objective value λ_f, with A being an optimal solution to the inner maximization problem (which can be computed by linear programming). Our goal is to construct a solution to $\eta_v(G)$ in Definition 3.15 with objective value $O\big(\sqrt{\lambda_f}\big)$.

To this end, define $g : V \to \mathbb{R}$ by

$$g(u) := \begin{cases} (f(u)+c)^2 & \text{if } f(u)+c > 0 \\ -(f(u)+c)^2 & \text{otherwise ,} \end{cases}$$

where $c \in \mathbb{R}$ is chosen so as to satisfy the constraint $\sum_u \pi(u)\cdot g(u) = 0$ in Definition 3.15. Note that such c exists and is unique.

We would like to prove that $1 \leq \sum_u \pi(u)\cdot|g(u)| \leq 2$, so that scaling g down by a factor of at most 2 will satisfy the constraint $\sum_u \pi(u)\cdot|g(u)| = 1$ in Definition 3.15. The argument uses the normalization constraints $\sum_u \pi(u)f(u) = 0$ and $\sum_u \pi(u)|f(u)| = 1$ and we defer the details to the full paper.

Now we bound the objective value of the ℓ_1 program in Definition 3.15 using g as a solution. Let B be an optimal solution to the inner maximization problem in Definition 3.15 after fixing g. Assuming the inequality $|g(u)-g(v)|^2 \leq 2(f(u)-f(v))^2\big(|g(u)|+|g(v)|\big)$ (the proof is technical yet straightforward, so we defer it to the full version), the objective value to the ℓ_1 program is

$$\frac{1}{2}\sum_{uv\in E} B(u,v)\cdot|g(u)-g(v)|$$
$$\lesssim \sum_{uv\in E} B(u,v)\sqrt{(f(u)-f(v))^2\big(|g(u)|+|g(v)|\big)}$$
$$\leq \sqrt{\sum_{uv\in E} B(u,v)(f(u)-f(v))^2}\cdot \sqrt{\sum_{uv\in E} B(u,v)\big(|g(u)|+|g(v)|\big)}$$
$$= \sqrt{\sum_{uv\in E} B(u,v)(f(u)-f(v))^2}\cdot \sqrt{\sum_{u\in V} |g(u)|\cdot\Big(\sum_{v:uv\in E} B(u,v) + \sum_{v:vu\in E} B(v,u)\Big)}$$
$$= \sqrt{\sum_{uv\in E} B(u,v)(f(u)-f(v))^2}\cdot \sqrt{2\sum_{u\in V}\pi(u)\cdot|g(u)|}$$
$$\lesssim \sqrt{\sum_{uv\in E} B(u,v)(f(u)-f(v))^2}$$
$$\leq \sqrt{\sum_{uv\in E} A(u,v)(f(u)-f(v))^2}$$
$$\lesssim \sqrt{\lambda_f},$$

where the second inequality is by Cauchy-Schwarz, the second equality is by the degree constraints in Definition 3.15, and the second last inequality is because A is an optimal solution to the inner maximization problem when f is fixed. Therefore, we conclude that g (after normalizing to satisfy $\sum_{u\in V}\pi(u)\cdot|g(u)| = 1$) is a solution to $\nu_v(G)$ with objective value $O\big(\sqrt{\lambda_f}\big)$. This completes the proof of the first inequality about directed vertex expansion in Proposition 3.17.

The proof of the second inequality about directed edge conductance is the same (with $\pi(u)$ replaced by $d_w(u)$) and is omitted.

3.5.3 *Threshold Rounding.*

Finally, we prove Proposition 3.20. Again, we first prove the first inequality in Proposition 3.20 about directed vertex expansion. Let $G = (V, E, \pi)$ be a vertex-weighted directed graph. Let (f, q, r) be a feasible solution to $\xi_v(G)$ in Lemma 3.18 with objective value ξ_f. Our goal is to construct a nonempty set $S \subset V$ with $\psi(S) \lesssim \xi_f$.

The algorithm is a threshold rounding algorithm, where each vertex u is mapped to some $g(u) \in [0,\infty)$ and the output is a set $S_t := \{u \in V \mid g(u) > t\}$ for some threshold t. In previous threshold rounding algorithms for Cheeger-type inequalities, only the embedding function $f : V \to \mathbb{R}$ is used as the function g to produce the output set, so in particular only one ordering of the vertices is considered.

The new twist in our algorithm is that we would consider a few candidate choices for $g(u)$. They will all ensure that the threshold rounding would produce a set with small expected directed vertex boundary, and we will choose one that gives large expected set size.

To this end, define the following four functions:

- $g_1(u) := \max\{0, f(u) + r(u) - c_1\}$
- $g_2(u) := \max\{0, f(u) - r(u) - c_2\}$
- $g_3(u) := \max\{0, -f(u) + r(u) + c_2\}$
- $g_4(u) := \max\{0, -f(u) - r(u) + c_1\}$,

where c_1 is a π-weighted median of $f(u) + r(u)$, chosen so that $\max(\pi(\mathrm{supp}(g_1)), \pi(\mathrm{supp}(g_4))) \leq \pi(V)/2$.

Similarly, c_2 is a π-weighted median of $f(u) - r(u)$, chosen so that $\max(\pi(\text{supp}(g_2)), \pi(\text{supp}(g_3))) \leq \pi(V)/2$.

Numerator: We bound the size of the outer boundary of either S_t or $\overline{S_t}$ for uniformly random t, depending on whether the coefficient of $r(u)$ is -1 or $+1$ in the function g_i.

On the one hand, if we consider g_1 (similar for g_3), then we would bound the expected outer boundary size of $\overline{S_t}$ as:

$$\begin{aligned}
&\int_0^\infty \pi(\partial^+(\overline{S_t}))\, dt \\
&= \sum_v \pi(v) \int_0^\infty \mathbb{1}[v \in \partial^+(\overline{S_t})]\, dt \\
&= \sum_v \pi(v) \int_0^\infty \mathbb{1}[\exists\, u \text{ with } uv \in E \text{ and } g_1(u) \leq t < g_1(v)]\, dt \\
&= \sum_v \pi(v) \max_{u:uv\in E} \{g_1(v) - g_1(u)\} \\
&\leq \sum_v \pi(v) \max_{u:uv\in E} \{(f(v) + r(v)) - (f(u) + r(u))\} \\
&\leq \sum_v \pi(v) \max_{u:uv\in E} \{|f(u) - f(v)| + r(v) - r(u)\} \\
&\leq \sum_v \pi(v) \cdot q(v).
\end{aligned}$$

On the other hand, if we consider the function g_2 (similar for g_4), then we similarly bound the expected outer boundary size of S_t as

$$\int_0^\infty \pi(\partial^+(S_t))\, dt \leq \sum_v \pi(v) \cdot q(v).$$

To summarize, when we do threshold rounding with respect to any of g_1, g_2, g_3, g_4, it holds that

$$\int_0^\infty \min\left\{\pi(\partial^+(S_t)), \pi(\partial^+(\overline{S_t}))\right\} dt \leq \sum_v \pi(v) q(v).$$

Denominator: For the function g_i, the expected size of S_t is given by

$$\int_0^\infty \pi(S_t)\, dt = \sum_u \pi(u) \int_0^\infty \mathbb{1}[g_i(u) > t]\, dt = \sum_u \pi(u) \cdot g_i(u).$$

Therefore, our goal is to show that there exists $1 \leq i \leq 4$ with $\sum_u \pi(u) g_i(u) \geq \Omega(1)$. To do so, we will show that

$$\sum_{i=1}^4 \sum_u \pi(u) \cdot g_i(u) \geq \Omega(1).$$

By the definitions of g_i, for any $u \in V$,

$$\begin{aligned}
&g_1(u) + g_4(u) = |(f(u) + r(u)) - c_1| \\
\text{and}\quad &g_2(u) + g_3(u) = |(f(u) - r(u)) - c_2|.
\end{aligned}$$

Thus it suffices to show that

$$\sum_u \pi(u) \cdot (|(f(u) + r(u)) - c_1| + |(f(u) - r(u)) - c_2|) \geq \frac{1}{2}. \quad (3.2)$$

To this end, we note that either $\sum_u \pi(u)|f(u) + r(u)| \geq 1$ or $\sum_u \pi(u)|f(u) - r(u)| \geq 1$, because

$$\begin{aligned}
&\sum_u \pi(u) \left(|f(u) + r(u)| + |f(u) - r(u)|\right) \\
&= \sum_u \pi(u) \cdot 2\max(|f(u)|, |r(u)|) \geq 2 \sum_u \pi(u)|f(u)| = 2.
\end{aligned}$$

Assume without loss that $\sum_u \pi(u) \cdot r(u) = 0$ (as we can shift every $r(u)$ by the same amount without changing anything). Then both $\sum_u \pi(u)(f(u) + r(u)) = 0$ and $\sum_u \pi(u)(f(u) - r(u)) = 0$.

Consider first the case where $\sum_u \pi(u)|(f + r)(u)| \geq 1$; the other case is treated similarly. Then, since $\sum_u \pi(u)((f + r)(u)) = 0$ and $\sum_u \pi(u)|(f + r)(u)| \geq 1$, it follows that

$$\sum_{u:(f+r)(u)\leq 0} \pi(u)|(f + r)(u)| = \sum_{u:(f+r)(u)\geq 0} \pi(u)|(f + r)(u)| \geq \frac{1}{2}.$$

If $c_1 \geq 0$, then

$$\begin{aligned}
\sum_u \pi(u)|(f + r)(u) - c_1| &\geq \sum_{u:(f+r)(u)\leq 0} \pi(u)|(f + r)(u) - c_1| \\
&\geq \sum_{u:(f+r)(u)\leq 0} \pi(u)|(f + r)(u)| \geq \frac{1}{2};
\end{aligned}$$

similar if $c_1 < 0$. (3.2) follows.

Conclusion: There exists $g = g_i$ for some $1 \leq i \leq 4$, such that if we use this function for threshold rounding,

- $\int_0^\infty \min\left\{\pi(\partial^+(S_t)), \pi(\partial^+(\overline{S_t}))\right\} dt \leq \sum_v \pi(v) \cdot q(v) = \xi_f$;
- $\int_0^\infty \pi(S_t)\, dt \geq 1/8$;
- $\pi(S_t) \leq \pi(V)/2$ always.

Hence, we can return some $S = S_t$, whence $0 < \pi(S) \leq \pi(V)/2$ and

$$\begin{aligned}
\vec{\psi}(S) &= \frac{\min\left\{\pi(\partial^+(S)), \pi(\partial^+(\overline{S}))\right\}}{\min\left\{\pi(S), \pi(\overline{S})\right\}} \\
&= \frac{\min(\pi(\partial^+(S)), \pi(\partial^+(\overline{S})))}{\pi(S)} \leq 8\xi_f.
\end{aligned}$$

The proof of the second inequality about directed edge conductance is essentially the same and is omitted.

3.6 Fastest Mixing Time

The goal of this subsection is to prove Theorem 1.6 that

$$\frac{1}{\vec{\psi}(G)} \cdot \frac{1}{\log(1/\pi_{\min})} \lesssim \tau^*(G) \lesssim \frac{1}{\vec{\psi}(G)^2} \cdot \log \frac{\Delta}{\vec{\psi}(G)} \cdot \log \frac{1}{\pi_{\min}}.$$

There are two parts of the proof. First, we upper bound the fastest mixing time using Theorem 1.14 by Fill [19] and Chung [14]. Second, we lower bound the fastest mixing time using a combinatorial argument and the ∞-norm mixing time that we will define.

Proof of Theorem 1.6. Recall that in the setting of the theorem, π is not only a weight function, but a probability distribution. We assume the graph is strongly connected and so $\vec{\lambda}_2^{v*}(G) > 0$.

To prove the upper bound, we prove that $\tau^*(G) \lesssim \left(\vec{\lambda}_2^{v*}(G)\right)^{-1} \cdot \log(\pi_{\min}^{-1})$, and then the result will follow from Theorem 1.4. Let A be an optimal reweighted Eulerian subgraph in Definition 1.3. Let $P := \Pi^{-1}A$ be the transition matrix of the ordinary random walk corresponding to the reweighted subgraph A. Observe that $P := \Pi^{-1}A$ is a feasible solution to Definition 1.5, and so is $(I+P)/2$. Therefore, by Theorem 1.14,

$$\tau^*(G) \leq \tau\left(\frac{I + P}{2}\right) \lesssim \frac{1}{\lambda_2(\widetilde{\mathcal{L}})} \cdot \log\left(\frac{1}{\pi_{\min}}\right) = \frac{1}{\vec{\lambda}_2^{v*}(G)} \cdot \log\left(\frac{1}{\pi_{\min}}\right),$$

where the last inequality is because $\widetilde{\mathcal{L}} = I - \Pi^{-\frac{1}{2}}(A + A^T)\Pi^{-\frac{1}{2}}/2$ as defined in (1.4) and $\lambda_2(\widetilde{\mathcal{L}}) = \vec{\lambda}_2^{v*}(G)$ by Definition 1.1.

To prove the lower bound, we consider the ∞-norm ϵ-mixing time defined as

$$\tau_\epsilon^\infty(P) := \min\left\{t : \max_{p_0:V\to\mathbb{R}_{\geq 0}} \max_{v\in V}\left\{1 - \frac{p_t(v)}{\pi(v)}\right\} < \epsilon\right\},$$

where p_0 is an initial distribution on V and p_t denotes $p_0 P^t$. We will prove that for any feasible solution P to Definition 1.5,

$$\frac{1}{\vec{\psi}(G)} \lesssim \tau_{1/e}^\infty(P), \tag{3.3}$$

and this would imply that

$$\frac{1}{\vec{\psi}(G)} \lesssim \max_P \tau_{1/e}^\infty(P) \leq \max_P \tau_{1/e}(P)\cdot\log\left(\frac{1}{\pi_{\min}}\right) = \tau^*(G)\cdot\log\left(\frac{1}{\pi_{\min}}\right),$$

proving the lower bound, where the second inequality is by [20, Proposition 2.47(f)] relating τ_ϵ^∞ and τ_ϵ using sub-multiplicity of mixing time.

To prove (3.3), let P be an arbitrary feasible solution to Definition 1.5, and $S \subset V$ be a nonempty subset such that $\vec{\psi}(G) = \vec{\psi}(S)$. We will use S to define an initial distribution $p_0 : V \to \mathbb{R}_{\geq 0}$ such that

$$\Delta_\infty(p_t, \pi) := \max_{v\in V}\left\{1 - \frac{p_t(v)}{\pi(v)}\right\} > \frac{1}{e}$$

for any $t \leq 1/(4\vec{\psi}(S))$, and this would imply that $\tau_{1/e}^\infty(P) > 1/(4\vec{\psi}(S))$.

To define p_0, we assume without loss of generality that $\pi(S) \leq 1/2$ and consider two cases.

(1) $\pi(\partial^+(S)) \leq \pi(\partial^+(\overline{S}))$. In this case, we set

$$p_0(u) = \begin{cases}\pi(u)/\pi(S), & \text{if } u \in S;\\ 0, & \text{otherwise.}\end{cases}.$$

We will show that $p_t(S) := \sum_{v\in S} p_t(v) \geq 1 - t\cdot\vec{\psi}(S)$ for all $t \geq 0$. Note that, by induction, $p_t(v) \leq \pi(v)/\pi(S)$ for all $v \in V$ and $t \geq 0$, as

$$p_{t+1}(v) = \sum_{u\in V} p_t(u)\cdot P(u,v) \leq \sum_{u\in V}\frac{\pi(u)}{\pi(S)}\cdot P(u,v) = \frac{\pi(v)}{\pi(S)}.$$

It follows that at step $t+1$, the total amount of probability mass escaping from S is at most

$$\sum_{v\in\partial^+(S)} p_t(v) \leq \frac{\pi(\partial^+(S))}{\pi(S)} = \vec{\psi}(S).$$

Hence, for any $t \leq 1/(4\vec{\psi}(S))$, we have $p_t(\overline{S}) \leq \frac{1}{4} \leq \frac{1}{2}\cdot\pi(\overline{S})$, and so

$$\Delta_\infty(p_t,\pi) \geq \max_{v\in\overline{S}}\left\{1 - \frac{p_t(v)}{\pi(v)}\right\} \geq 1 - \frac{p_t(\overline{S})}{\pi(\overline{S})} \geq \frac{1}{2} > \frac{1}{e}.$$

(2) $\pi(\partial^+(S)) > \pi(\partial^+(\overline{S}))$. In this case, we define

$$p_0(u) = \begin{cases}\pi(u)/\pi(\overline{S}), & \text{if } u \notin S;\\ 0, & \text{otherwise.}\end{cases}.$$

We will show that $p_t(S) \leq 2t\cdot\pi(S)\cdot\vec{\psi}(S)$. Again, by induction, $p_t(v) \leq \pi(v)/\pi(\overline{S})$ for all $v \in V$ and $t \geq 0$.
It follows that in step $t+1$, the total amount of probability mass entering S is at most

$$\sum_{v\in\partial^+(\overline{S})} p_t(v) \leq \frac{\pi(\partial^+(\overline{S}))}{\pi(\overline{S})} \leq 2\pi(S)\cdot\frac{\pi(\partial^+(\overline{S}))}{\pi(S)} = 2\pi(S)\cdot\vec{\psi}(S).$$

Hence, for $t \leq 1/(4\vec{\psi}(S))$, we have $p_t(S) \leq \pi(S)/2$, and so

$$\Delta_\infty(p_t,\pi) \geq \max_{v\in S}\left\{1 - \frac{p_t(v)}{\pi(v)}\right\} \geq 1 - \frac{p_t(S)}{\pi(S)} \geq \frac{1}{2} > \frac{1}{e}.$$

This completes the proof of the lower bound and hence Theorem 1.6. □

4 CONCLUDING REMARKS

In this paper, we show that the reweighted eigenvalue approach can be extended substantially to derive Cheeger inequalities for directed graphs and hypergraphs. Most notably, this develops into an interesting new spectral theory for directed graphs, which is much closer to the spectral theory for undirected graphs than what are previously known. We hope that this spectral theory will find more applications in practice, in clustering and partitioning of directed graphs and hypergraphs.

Technically, the reweighted eigenvalue approach provides an intuitive and unifying method to reduce the study of expansion properties in more general settings to the basic setting of edge conductance in undirected graphs. We believe that this approach can be used to lift more results in spectral graph theory for undirected graphs to more general settings, as the ideas are consistent with recent works on directed Laplacian solvers and hypergraph spectral sparsification that we mentioned in Section 1.3.

There are some concrete open problems. The most obvious one is to prove tight bounds for the two main results Theorem 1.4 and Theorem 1.9, to settle whether the dependency on the asymmetric ratio can be completely removed or not [5]. Another one is to formulate and prove higher-order Cheeger inequality and bipartite Cheeger inequality for directed graphs. An important one for applications is to design fast algorithms (ideally near-linear time algorithms) for computing reweighted eigenvalues.

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[5] Note that the dimension reduction result for directed edge conductance is tight, and so a positive result removing the $\log\alpha(G)$ factor in Theorem 1.9 would probably need substantial new ideas. We incline to believe that the $\log\alpha(G)$ factor in Theorem 1.9 cannot be completely removed, but we do not have an example supporting this belief. We are less sure about what the right bound should be for Theorem 1.4.

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