Chapter 22

Log-Concave Polynomials

In this chapter, we take a different perspective to view a 0-local-spectral expander as a strongly log-concave polynomial. Then we see two related notions of polynomials, sector-stable polynomials and fractionally log-concave polynomials, and their connections to spectral independence.

22.1 Log-Concave Polynomials

The polynomial approach is actually the original approach that was used to solve the matroid expansion conjecture [ALOV19] that we saw in Chapter 20.

Definition 22.1 (Strongly Log-Concave Distribution)**.** *Let $\mu : \{0,1\}^n \to \mathbb{R}$ be a probability distribution and $g_\mu(x) = \sum_{S\subseteq[n]} \mu(S) \cdot \prod_{i\in S} x_i$ be its generating polynomial as defined in Definition 16.1. We say μ is a log-concave distribution if $\log g_\mu$ is a concave function at the point $\vec{1}$.*

We say μ is a strongly log-concave distribution if for any $k \geq 0$ and any sequence of integers $1 \leq i_1, \ldots, i_k \leq n$,

$$\big(\partial_{x_{i_1}} \cdots \partial_{x_{i_k}} g_\mu\big)(x_1, \ldots, x_n)$$

is log-concave at the point $\vec{1}$.

In Chapter 19, given a d-homogeneous probability distribution μ, we use μ to define a pure $(d-1)$-dimensional weighted simplicial complex (X_μ, Π) as in Definition 19.5, with $\Pi_{d-1} := \mu$ being the distribution on the maximal faces of dimension $d-1$. In this chapter, give a d-homogeneous probability distribution, we use μ to define a d-homogeneous generating polynomial g_μ as in Definition 22.1. The connection between the weighted simplicial complex (X_μ, Π) and the generating polynomial g_μ is through the Hessian matrix of $\log g_\mu$ at the point $\vec{1}$.

Exercise 22.2 (Hessian Matrix of a Polynomial)**.** *The Hessian matrix of $\log p$ is*

$$\nabla^2 \log p = \frac{p \cdot (\nabla^2 p) - (\nabla p)(\nabla p)^T}{p^2}.$$

A basic result in convex analysis is that $\log p$ is concave at a point x if and only if $\nabla^2 \log p$ is negative semidefinite at x.

The main observation in [ALOV19] is that the Hessian matrix of g_μ is closely related to the random walk matrix of the empty link of X_μ.

Theorem 22.3 (Strongly Log-Concave Polynomial and 0-Local-Spectral Expander [ALOV19]). *Let $\mu : \{0,1\}^n \to \mathbb{R}$ be a probability distribution, g_μ be its generating polynomial as in Definition 22.1, and X_μ be its weighted simplicial complex as in Definition 19.5. Then g_μ is strongly log-concave if and only if X_μ is a 0-local-spectral expander.*

Proof. The main step is to show that the "random walk matrix" of the Hessian matrix of g_μ at point $\vec{1}$ and the random walk matrix of the empty link of X_μ are the same. On one hand, let $H := \nabla^2 g_\mu|_{x=\vec{1}}$ be the Hessian matrix of g_μ at the point $\vec{1}$. Note that $g_\mu|_{x=\vec{1}} = 1$,

$$\big(\nabla g_\mu\big)\big|_{x=\vec{1}}(i) = \partial_{x_i} g_\mu\big|_{x=\vec{1}} = \Pr_{S\sim\mu}\big[i \in S\big] \quad \text{and} \quad \big(\nabla^2 g_\mu\big)\big|_{x=\vec{1}}(i,j) = \partial_{x_i}\partial_{x_j} g_\mu\big|_{x=\vec{1}} = \Pr_{S\sim\mu}\big[\{i,j\} \subseteq S\big].$$

Let D_H be the degree matrix of H. Then, as μ is d-homogeneous, for any $i \in [n]$,

$$D_H(i,i) = \sum_{j=1}^n H(i,j) = \sum_{j=1}^n \Pr_{S\sim\mu}\big[\{i,j\} \subseteq S\big] = (d-1) \Pr_{S\sim\mu}\big[i \in S\big].$$

On the other hand, let the random walk matrix of the empty link be W. For $i \neq j \in [n]$, by Definition 19.12 and Definition 19.6,

$$W(i,j) = \frac{\Pi\big(\{i,j\}\big)}{2\Pi\big(\{i\}\big)} = \frac{\binom{d}{2}^{-1} \cdot \Pr_{S\sim\mu}\big[\{i,j\} \subseteq S\big]}{2\binom{d}{1}^{-1} \cdot \Pr_{S\sim\mu}\big[i \in S\big]} = \frac{\Pr_{S\sim\mu}\big[\{i,j\} \subseteq S\big]}{(d-1) \cdot \Pr_{S\sim\mu}\big[i \in S\big]}.$$

Therefore,

$$W = D_H^{-1} H.$$

With this identity, we next show that g_μ is log-concave if and only if $\lambda_2(W) \leq 0$. In one direction, if g_μ is log-concave, then $\nabla^2 \log g_\mu$ is negative semidefinite by Definition 22.1. Note that this implies that $H = \nabla^2 g_\mu$ has at most one positive eigenvalue by the identity in Exercise 22.2. Check that it follows that $W = D_H^{-1} H$ also has at most one positive eigenvalue, and thus $\lambda_2(W) \leq 0$.

In the other direction, if W has at most one positive eigenvalue, then note that W has exactly one positive eigenvalue, as $\vec{1}$ is an eigenvector of W with eigenvalue 1. Check that this implies that $\lambda_1(W - \vec{1}\Pi_0^T) \leq 0$, and then it follows that $D_H(W - \vec{1}\Pi_0^T) = H - \frac{d-1}{d}(\nabla g_\mu)(\nabla g_\mu)^T$ is negative semidefinite. By the identity in Exercise 22.2, this implies that $\nabla^2 \log p$ is negative semidefinite.

Finally, observe that there is a one-to-one correspondence between the differentiated polynomials $\big(\partial_{x_{i_1}} \cdots \partial_{x_{i_k}} g_\mu\big)(x_1, \ldots, x_n)$ and the links $X_{\{i_1,\ldots,i_k\}}$ of the simplicial complex. Thus, by the arguments above, the differentiated polynomial at point $\vec{1}$ is log-concave if and only if the second eigenvalue of the random walk matrix of the corresponding link is at most 0. Therefore, we conclude that g_μ is strongly log-concave if and only if X_μ is a 0-local-spectral expander. □

Corollary 22.4 (Matroid Polynomial is Strongly Log-Concave). *The generating polynomial of the uniform distribution on matroid bases is strongly log-concave.*

This result was proved earlier in [AOV18] using advanced techniques from Hodge theory for matroids [AHK18], so the techniques from high-dimensional expanders provide a more elementary and simpler proof. It is very interesting to see a correspondence between the concepts in polynomials and the concepts in high-dimensional expanders (see [ALOV19] for more).

Mason's Ultra Log-Concavity Conjecture

An important consequence of the polynomial perspective is a proof of the conjecture that the rank sequence of a matroid is ultra log-concave.

Theorem 22.5 (Mason's Conjecture [ALOV18, BH20]). *For a matroid M on n elements with m_k independent sets of size k, the sequence $m_0, m_1, \ldots, m_n$ is ultra log-concave such that for $1 < k < n$,*

$$\left(\frac{m_k}{\binom{n}{k}}\right)^2 \geq \frac{m_{k-1}}{\binom{n}{k-1}} \cdot \frac{m_{k+1}}{\binom{n}{k+1}}.$$

The question that whether the sequence $m_0, m_1, \ldots, m_n$ is log-concave was a long standing open problem in combinatorics from the 70s, and was first proved in [AHK18] using Hodge theory for matroids. The proof of the stronger ultra log-concavity in [ALOV18, Ove20] is short and elementary, and should be readily understandable for readers who followed the course thus far. There is another proof of ultra log-concavity in [BH20] using a closely related notion called Lorentzian polynomials. We remark that Gurvits also studied log-concave polynomials earlier in his work on generalizations of permanent problems.

22.2 Sector-Stable Polynomials

Given that the matroid expansion conjecture can be solved from both the high-dimensional expander perspective and the strongly log-concave polynomial perspective, and that the high-dimensional expander approach can be extended further as in Chapter 21, one may wonder whether the polynomial approach can also be extended further and possibly in different directions.

A very interesting recent paper by Alimohammadi, Anari, Shirajur, Vuong [AASV21] proposed two notions for polynomials called sector-stability and fractional log-concavity. We discuss sector-stable polynomials in this section, and fractional log-concave polynomials in the next section.

Definition 22.6 (Sector-Stable Polynomials). *The open sector of aperature $\alpha\pi$ centered around the positive real axix is denoted by*

$$\Gamma_\alpha := \left\{ \exp(x + \imath y) \mid x \in \mathbb{R}, y \in \left(-\frac{\alpha\pi}{2}, \frac{\alpha\pi}{2} \right) \right\}$$

A polynomial $g(z_1, \ldots, z_n)$ is Γ_α-stable if

$$z_1, \ldots, z_n \in \Gamma_\alpha \quad \Longrightarrow \quad g(z_1, \ldots, z_n) \neq 0.$$

Note that Γ_1 is the right half-plane, and Γ_1-stability is called Hurwitz-stability. The following exercise shows that it is a generalization of real-stable polynomials for homogeneous polynomials.

Exercise 22.7. *Show that a homogeneous polynomial is Hurwitz-stable if and only if it is $\mathcal{H}$-stable in Definition 13.8.*

A very interesting theorem in [AASV21] is a connection between sector-stability and spectral independence. The proof is very nice and elegant, using some elementary complex analysis.

Theorem 22.8 (Sector-Stability Implies Spectral Independence). *Suppose that $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ is a probability distribution whose generating polynomial is Γ_α-stable. Then the ℓ_1-norm of any row in the correlation matrix Ψ in Definition 21.5 is bounded by*

$$\sum_j \left| \Pr_{S\sim\mu}[j \in S \mid i \in S] - \Pr_{S\sim\mu}[j \in S] \right| \leq \frac{2}{\alpha}.$$

A consequence is that $\lambda_{\max}(\Psi) \leq \frac{2}{\alpha}$.

Sector Stability Preserving Operations

Before we see some examples of sector-stable polynomials, we first record some sector-stability preserving operations from [AASV21].

Exercise 22.9 (Sector-Stability Preserving Operations). *Show that the following operations preserve Γ_α-sector stability.*

1. *Specialization: $g(z_1, \ldots, z_n) \to g(a, z_2, \ldots, z_n)$, where $a \in \bar{\Gamma}_\alpha$.*
2. *Scaling: $g(z_1, \ldots, z_n) \to g(\lambda_1 z_1, \lambda_2 z_2, \ldots, \lambda_n z_n)$, where $\lambda_i \geq 0$ for $1 \leq i \leq n$.*
3. *Dual: $g \to g^*$, where $g(z) = \sum_{S\subseteq[n]} c_S \cdot z^S$ and $g^*(z_1, \ldots, z_n) := \sum_{S\subseteq[n]} c_S \cdot z^{[n]\setminus S}$.*

Exercise 22.10 (Homogenization). *If multi-affine polynomial $g(z_1, \ldots, z_n) := \sum_{S\subseteq[n]} c_S \cdot z^S$ is Γ_α-stable, then its homogenization*

$$g^{\text{hom}}(z_1, \ldots, z_n, w_1, \ldots, w_n) := \sum_{S\subseteq[n]} c_S \cdot z^S \cdot w^{[n]\setminus S}$$

is multi-affine, homogeneous of degree n, and $\Gamma_{\alpha/2}$-stable.

Proposition 22.11 (Partial Derivative). *If $g(z_1, \ldots, z_n)$ is a multiaffine polynomial, then $\partial_{z_i} g$ is sector stable for $1 \leq i \leq n$.*

Theorem 22.12 (Truncation). *If $g(z_1, \ldots, z_n)$ is Γ_1-stable, then g_k is either identically zero or $\Gamma_{1/2}$-stable, where g_k is the truncation of g that keeps only the degree k terms.*

Applications

One application in [AASV21] is to sample matchings of a given size in planar graphs. The starting point is a theorem by Heilman and Lieb.

Theorem 22.13 (Monomer-Dimer Polynomial). *Given a graph $G = (V, E)$ with edge weight w_e for $e \in E$, the polynomial*

$$\sum_{M\subseteq E: M \text{ matching}} \prod_{e\in M} w(e) \prod_{v: v\notin M} z_v$$

is Γ_1-stable, where $\{z_v\}_{v\in V}$ are the variables in this polynomial.

This polynomial is not homogeneous, and homogenization does not preserve Hurwitz-stabiliy, nor truncation to matchings of a given size. Applying Theorem 22.12, however, they can say that the truncation to matchings of a given size is still $\Gamma_{1/2}$-stable, so that they can apply Theorem 22.8 to prove that the Markov chain on the set of "monomers" is fast mixing. For any class of graphs that counting matchings is polynomial time solvable, including planar graphs and bounded genus graphs, their results can be used to approximate sample and count matchings of a given size.

Another application is in non-asymmetric determinantal point process.

Theorem 22.14 (Non-Symmetric k-DPPs)**.** *For any matrix $L \in \mathbb{R}^{n\times n}$ satisfying $L + L^T \succcurlyeq 0$ and a number k, the polynomial*

$$g(z_1, \ldots, z_n) = \sum_{S\in\binom{[n]}{k}} \det(L_{S,S}) \prod_{i\in S} z_i$$

is $\Gamma_{1/2}$-stable

The following result shows some limitation on the class of sector-stable polynomials for combinatorial problems.

Lemma 22.15 (Bounded Length of Sector-Stable Distributions)**.** *If $\mu : \{0,1\}^n \to \mathbb{R}$ is a $\Gamma_{1/k}$-sector-stable distribution, then the length of edges of its Newton polytope* $\mathrm{newt}(\mu)$ *is at most $2k$, where*

$$\mathrm{newt}(\mu) := \mathrm{conv}\big(\{S : \mu(S) > 0\}\big).$$

This result shows that if the polytope of the combinatorial problem has unbounded edge length (such as the matching polytope and the arborescence polytope), then the corresponding generating polynomial cannot be sector-stable.

22.3 Fractionally Log-Concave Polynomials

The class of fractionally log-concave polynomials is a generalization of the class of log-concave polynomials.

Definition 22.16 (Fractionally Log-Concave Polynomials)**.** *A polynomial $g_\mu(z_1, \ldots, z_n)$ is called α-fractionally log-concave for $\alpha \in [0,1]$ if $\log g_\mu(z_1^\alpha, \ldots, z_n^\alpha)$ is concave when viewed as a function over $\mathbb{R}^n_{\geq 0}$.*

The key observation is that the Hessian matrix of $\log g_\mu(z_1^\alpha, \ldots, z_n^\alpha)$ at the point $\vec{1}$ is closely related to the correlation matrix in Definition 21.5. The proof is similar to that in Exercise 22.2 and Theorem 22.3.

Proposition 22.17 (α-Fractionally Log-Concavity and Spectral Independence)**.** *Let $\mu : \{0,1\}^n \to \mathbb{R}$ be a probability distribution and g_μ be its generating polynomial. Then g_μ is α-fractionally log-concave at the point $\vec{1}$ if and only if $\lambda_{\max}(\Psi) \leq \frac{1}{\alpha}$ where Ψ is the correlation matrix of μ as defined in Definition 21.5.*

Proof. Let $H := \nabla^2 \log g_\mu(z_1^\alpha, \ldots, z_n^\alpha)|_{z=\vec{1}}$. Let $\Pr[i] := \Pr_{S\sim\mu}[i \in S]$ and similarly $\Pr[i \wedge j] := \Pr_{S\sim\mu}[i \in S \wedge j \in S]$. Check that a similar calculation as in Exercise 22.2 gives

$$H_{i,j} = \begin{cases} \alpha(\alpha-1)\Pr[i] - \alpha^2 \Pr[i]^2 & \text{if } i = j \\ \alpha^2\big(\Pr[i \wedge j] - \Pr[i]\cdot\Pr[j]\big) & \text{if } i \neq j \end{cases}$$

Let $D := \operatorname{diag}(\Pr[i])$ be the diagonal matrix of marginal probability. It follows from Definition 21.5 that

$$\Psi = \frac{1}{\alpha^2} D^{-1} H + \frac{1}{\alpha} I.$$

This implies that $\lambda_{\max}(\Psi) \leq \frac{1}{\alpha}$ if and only if $\lambda_{\max}(D^{-1}H) \leq 0$ if and only if $H \preccurlyeq 0$ if and only if g_μ is α-fractionally log-concave at the point $\vec{1}$. □

Recall from Theorem 18.11 that a homogeneous real-stable polynomial is log-concave, and thus log-concavity is a generalization of real-stability that does not involve root locations. Using Proposition 22.17 and Theorem 22.8, we see that fractionally log-concavity is a generalization of sector-stability that does not involve root locations.

Theorem 22.18 (Sector-Stability Implies Fractionally Log-Concavity)**.** *For $\alpha \in [0, \frac{1}{2}]$, if g_μ is $\Gamma_{2\alpha}$-stable, then g_μ is α-fractionally log-concave.*

Proof. Theorem 22.8 proves that $\Gamma_{2\alpha}$-stability of g_μ implies that $\lambda_{\max}(\Psi) \leq \frac{1}{\alpha}$, and thus implies that g_μ is α-fractionally log-concave at the point $\vec{1}$ by Proposition 22.17.

Note that sector-stability is preserved under the change of variables $z_i \to \lambda_i z_i$ when $\lambda_1, \ldots, \lambda_n$ are positive reals by Exercise 22.9. This allows us to map any point in $\mathbb{R}^n_{\geq 0}$ to the point $\vec{1}$, and to use the above argument to show that g_μ is α-fractionally log-concave at any point in $\mathbb{R}^n_{\geq 0}$. □

While fractional log-concavity at the point $\vec{1}$ is equivalent to a bound on the eigenvalues of the correlation matrix Ψ, it does not imply a bound for the conditioned distributions μ_S. However, fractional log-concavity at all points in $\mathbb{R}^n_{\geq 0}$ does, because the polynomial for conditional distributions μ_S can be obtained as the following limit:

$$g_{\mu_S} \propto \lim_{\lambda\to\infty} \frac{g_\mu\left(\lambda z_1, \ldots, \lambda z_{|S|}, \ldots, z_n\right)}{\lambda^{|S|}}.$$

Scaling the variables or the polynomial, and taking limits all preserve fractional log-concavity.

Theorem 22.19 (Fractional Log-Concavity Implies Spectral Independence)**.** *If $\mu : \binom{[n]}{k} \to \mathbb{R}_{\geq 0}$ has a α-fractionally log-concave generating polynomial, then the correlation matrix of every conditioned distribution μ_S has maximum eigenvalue $\frac{1}{\alpha}$. It follows that μ is $\frac{1}{\alpha}$-spectrally independent as defined in Definition 21.8.*

22.4 References

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