Chapter 21

Spectral Independence

We will first show that Kaufman-Oppenheim Theorem 20.10 can be improved to a natural product form. Then we will see the notion of "spectral independence", a nice probabilistic formulation of this improved result without the language of high-dimensional expanders. Finally, we will mention some recent developments using this notion in analyzing random sampling algorithms for combinatorial objects.

21.1 Improved Analysis of Higher Order Random Walks

The proof of the matroid expansion conjecture shows that the techniques developed in higher order random walks provide a completely new approach to analyze mixing times of Markov chains. Unlike previous approaches such as couplings and multicommodity flows, this simplicial complex approach directly bounds the spectral gap of the random walk matrix. It is of great interest to investigate whether this approach can be extended to other problems such as independent sets and graph colorings.

First we discuss some limitations of the results in Chapter 20. Note that Theorem 20.10 can be used to establish a non-trivial spectral gap of $P_d^{\triangledown}$ only when $\lambda < \frac{1}{d(d+1)}$, which is a very strong spectral requirement of the simplicial complex. As discussed in Problem 19.19, the second eigenvalue is at most zero if and only if the graph is a complete multi-partite graph, and more generally a 0-local-spectral expander can be shown to be a weighted matroid complex. For most natural combinatorial simplicial complexes, it does not hold that $\lambda_2(G_\alpha) \leq O(\frac{1}{d^2})$ even when restricted to faces α of dimension $d-2$. This suggests that we need to sharpen the bound in Theorem 20.10 in order to apply this approach for other problems.

Small Improvement

It was observed that the comparison bound in Proposition 20.14 can be slightly improved.

Proposition 21.1 (Improved Comparison of $P_k^{\triangledown}$ and $P_k^{\wedge}$ [AL20]). *Let (X, Π) be a pure d-dimensional simplicial complex. For any $0 \leq k \leq d-1$,*

$$P_k^{\wedge} - P_k^{\triangledown} \preccurlyeq_{\Pi_k} \gamma_{k-1}(I - P_k^{\triangledown})$$

where $\gamma_j := \max_{\alpha \in X(j)} \lambda_2(W_\alpha)$ is the maximum second largest eigenvalue of the link graphs of dimension j as in Definition 19.13.

Proof. Following the proof in Proposition 20.14,

$$\langle f, (P_k^{\wedge} - P_k^{\nabla}) f\rangle_{\Pi_k} = \mathbb{E}_{\tau\sim\Pi_{k-1}} \big\langle f_\tau^{\perp}, (W_\tau - J_\tau) f_\tau^{\perp}\big\rangle_{\Pi_0^\tau} \le \mathbb{E}_{\tau\sim\Pi_{k-1}} \gamma_{k-1} \big\langle f_\tau^{\perp}, f_\tau^{\perp}\big\rangle_{\Pi_0^\tau} \tag{21.1}$$

Instead of bounding the right hand side simply by $\mathbb{E}_{\tau\sim\Pi_{k-1}} \gamma_{k-1} \big\langle f_\tau, f_\tau\big\rangle_{\Pi_0^\tau}$, we collect the dropped terms to prove the stated bound. As in Proposition 20.14, write $f_\tau = c\vec{1} + f_\tau^{\perp}$ where $\big\langle \vec{1}, f_\tau^{\perp}\big\rangle_{\Pi_0^\tau} = 0$ and $c = \langle f_\tau, \vec{1}\rangle_{\Pi_0^\tau}$ as in Equation 19.5. So, the dropped terms are

$$\mathbb{E}_{\tau\sim\Pi_{k-1}} \gamma_{k-1} \big\langle c\vec{1}, c\vec{1}\big\rangle_{\Pi_0^\tau} = \gamma_{k-1} \cdot \mathbb{E}_{\tau\sim\Pi_{k-1}} \langle f_\tau, \vec{1}\rangle_{\Pi_0^\tau}^2 = \gamma_{k-1} \cdot \mathbb{E}_{\tau\sim\Pi_{k-1}} \langle f_\tau, J_\tau f_\tau\rangle_{\Pi_0^\tau} = \gamma_{k-1} \cdot \langle f, P_k^{\nabla} f\rangle_{\Pi_k},$$

where the second equality is because $\langle f_\tau, \vec{1}\rangle_{\Pi_0^\tau}^2 = \langle f_\tau, \Pi_0^\tau\rangle^2 = \langle f_\tau, (\Pi_0^\tau)(\Pi_0^\tau)^T f_\tau\rangle = \langle f_\tau, J_\tau f_\tau\rangle_{\Pi_0^\tau}$ since $J_\tau = \vec{1}(\Pi_0^\tau)^T$ as defined in Lemma 20.13, and the last equality is by the statement in Lemma 20.13. Therefore, we conclude that

$$\begin{aligned}
\langle f, (P_k^{\wedge} - P_k^{\nabla}) f\rangle_{\Pi_k} &\le \mathbb{E}_{\tau\sim\Pi_{k-1}} \gamma_{k-1} \big\langle f_\tau^{\perp}, f_\tau^{\perp}\big\rangle_{\Pi_0^\tau} \\
&= \mathbb{E}_{\tau\sim\Pi_{k-1}} \gamma_{k-1} \big\langle f_\tau, f_\tau\big\rangle_{\Pi_0^\tau} - \mathbb{E}_{\tau\sim\Pi_{k-1}} \gamma_{k-1} \big\langle c\vec{1}, c\vec{1}\big\rangle_{\Pi_0^\tau} \\
&= \gamma_{k-1} \big\langle f, f\big\rangle_{\Pi_k} - \gamma_{k-1} \cdot \langle f, P_k^{\nabla} f\rangle_{\Pi_k} \\
&= \gamma_{k-1} \cdot \langle f, (I - P_k^{\nabla}) f\rangle_{\Pi_k},
\end{aligned}$$

where the second line is by $\big\langle f_\tau, f_\tau\big\rangle_{\Pi_0^\tau} = \big\langle c\vec{1}, c\vec{1}\big\rangle_{\Pi_0^\tau} + \big\langle f_\tau^{\perp}, f_\tau^{\perp}\big\rangle_{\Pi_0^\tau}$ using orthonormality, and the third line is using Exercise 20.15 and the calculation above. This holds for any f and thus implies the statement. □

Product Form

The small improvement in Proposition 21.1 is very simple, but what is perhaps surprising is that this is all we needed to prove a much sharper bound on $\lambda_2(P_k^{\nabla})$.

Theorem 21.2 (Improved Second Eigenvalue Bound on P_k^{∇} [AL20])**.** *Let (X, Π) be a pure d-dimensional simplicial complex. For any $0 \le k \le d$,*

$$\lambda_2(P_k^{\nabla}) \le 1 - \frac{1}{k+1} \prod_{j=-1}^{k-2} (1 - \gamma_j),$$

where $\gamma_j := \max_{\alpha\in X(j)} \lambda_2(W_\alpha)$ is as defined in Definition 19.13.

Proof. We prove by induction on k. The base case is when $k = 0$, where P_0^{∇} is a rank one matrix and so $\lambda_2(P_0^{\nabla}) \le 0$, and hence the statement trivially holds.

Now, assume the statement holds for k, and we would like to prove the induction step. By Proposition 21.1, $P_k^{\wedge} \preccurlyeq_{\Pi_k} \gamma_{k-1} \cdot I + (1 - \gamma_{k-1}) P_k^{\nabla}$, which implies by Exercise 20.16 and Exercise 20.9 that

$$\lambda_2(P_k^{\wedge}) \le \gamma_{k-1} + (1 - \gamma_{k-1}) \cdot \lambda_2(P_k^{\nabla}) \le 1 - \frac{1}{k+1} \prod_{i=-1}^{k-1} (1 - \gamma_i),$$

where the last inequality is by the induction hypothesis. Recall from Definition 20.8 that

$$P_k^{\wedge} = \frac{k+2}{k+1}\Big(P_k^{\triangle} - \frac{I}{k+2}\Big) \quad \implies \quad P_k^{\triangle} = \frac{k+1}{k+2}P_k^{\wedge} + \frac{I}{k+2}.$$

Therefore, the second largest eigenvalue of $P_k^{\triangle}$ is

$$\lambda_2(P_k^{\triangle}) \le \frac{k+1}{k+2}\Big(1 - \frac{1}{k+1}\prod_{i=-1}^{k-1}(1-\gamma_i)\Big) + \frac{1}{k+2} = 1 - \frac{1}{k+2}\prod_{i=-1}^{k-1}(1-\gamma_i).$$

The induction step follows from Exercise 20.6 that $P_{k+1}^{\triangledown}$ and $P_k^{\triangle}$ have the same spectrum. □

Implications

We discuss some implications of the product form in Theorem 21.2. A basic result is that a simplicial complex X has $\lambda_2(P_d^{\triangledown}) < 1$ if and only if $\lambda_2(G_\alpha) < 1$ for every face α of dimension up to $d-2$. Theorem 21.2 provides a quantitative generalization of this result. The product form matches the combinatorial intuition that we replace the complete graphs in the links of $P_k^{\triangledown}$ by expander graphs in the links of $P_k^{\wedge}$ as described in Chapter 20, and so we expect that the spectral gap decreases by a multiplicative factor but is always non-zero.

Combining with Oppenheim's trickling down Theorem 19.15, Theorem 21.2 provides the following convenient bound for the second eigenvalue of higher order random walks in a black box fashion.

Exercise 21.3. *Let (X, Π) be a pure d-dimensional simplicial complex. For any $0 \le k \le d$, suppose $\gamma_{k-2} \le \frac{1}{k+1}$ and G_α is connected for every face α up to dimension $k-2$, then*

$$\lambda_2(P_k^{\triangledown}) \le 1 - \frac{1}{(k+1)^2}.$$

In particular, this implies that the down-up walk $P_d^{\triangledown}$ is fast mixing for any $O\big(\frac{1}{d}\big)$-local-spectral expander, which is an improvement of Theorem 20.10 where it requires the simplicial complex to be a $O\big(\frac{1}{d^2}\big)$-local-spectral expander. See [AL20] for an application of Exercise 21.3 in sampling a random independent set of size up to $n/(2\Delta)$ where n is the number of vertices and Δ is the maximum degree of the input graph.

Another consequence is that the following type of eigenvalue profile is enough to guarantee polynomial mixing time.

Exercise 21.4 (Improving Profile)**.** *Let (X, Π) be a pure d-dimensional simplicial complex. If there is a constant $0 < c < 1$ such that*

$$(\gamma_{-1}, \gamma_0, \ldots, \gamma_{d-2}) = \Big(\frac{c}{d}, \frac{c}{d-1}, \ldots, \frac{c}{1}\Big),$$

then

$$\lambda_2(P_d^{\triangledown}) \le 1 - \frac{1}{d^{1+c}}.$$

21.2 Spectral Independence

Anari, Liu and Oveis Gharan [ALO20] defined a notion called spectral independence, which is a nice probabilistic formulation of Theorem 21.2 without using the language of high-dimensional expanders.

The following correlation matrix is a natural matrix that records the pairwise correlation of the elements. As we will see, this matrix is closely related to the random walk matrix of the empty link of a corresponding simplicial complex of the probability distribution.

Definition 21.5 (Correlation Matrix). *Let $\mu : \{0,1\}^n \to \mathbb{R}$ be a probability distribution on subsets of $[n]$. The correlation matrix of μ is a $2n \times 2n$ matrix Ψ, whose rows and columns are indexed by $[n] \times \{0,1\}$, with*

$$\Psi\big((i,a_i),(j,a_j)\big) = \Pr_{Z\sim\mu}\big[Z(j)=a_j \mid Z(i)=a_i\big] - \Pr_{Z\sim\mu}\big[Z(j)=a_j\big]$$

for $i \neq j$ and $a_i, a_j \in \{0,1\}$, and $\Psi\big((i,a_i),(j,a_j)\big) = 0$ if $i=j$.

Remark 21.6. *The correlation matrix in [ALO20] is defined slightly differently, with*

$$\Psi\big((i,a_i),(j,a_j)\big) = \Pr_{Z\sim\mu}\big[Z(j)=a_j \mid Z(i)=a_i\big] - \Pr_{Z\sim\mu}\big[Z(j)=a_j \mid Z(i)=1-a_i\big].$$

The above definition of the correlation matrices is from [AASV21, CGSV21].

The following conditional correlation matrices are the correlation matrices given a partial assignment. As we will see, they are closely related to the random walk matrices of the links of a corresponding simplicial complex of the probability distribution.

Definition 21.7 (Conditional Correlation Matrices). *Let $\mu : \{0,1\}^n \to \mathbb{R}$ be a probability distribution on subsets of $[n]$. Let $S \subseteq [n]$ be a subset of size k and let $a_S \in \{0,1\}^k$ be a binary string of length k with an entry for each element $i \in S$. Let $Z(S) = a_S$ be the event that $Z(i) = a_S(i)$ for all $i \in S$ when $Z \sim \mu$. The conditional correlation matrix Ψ_{a_S} is a $2(n-k) \times 2(n-k)$ matrix, whose rows and columns are indexed by $([n] \setminus S) \times \{0,1\}$, with*

$$\Psi_{a_S}\big((i,a_i),(j,a_j)\big) = \Pr_{Z\sim\mu}\big[Z(j)=a_j \mid Z(i)=a_i, Z(S)=a_S\big] - \Pr_{Z\sim\mu}\big[Z(j)=a_j \mid Z(S)=a_S\big]$$

for $i \neq j$ and $a_i, a_j \in \{0,1\}$, and $\Psi\big((i,a_i),(j,a_j)\big) = 0$ if $i=j$.

The following definition of spectral independence is closely related to local-spectral expansion of a corresponding simplicial complex of the probability distribution.

Definition 21.8 (Spectral Independence). *A probability distribution $\mu : \{0,1\}^n \to \mathbb{R}$ on subsets of $[n]$ is called η-spectrally independent if for any $S \subseteq [n]$ with $|S| \leq n-2$ and partial assignment $a_S \in \{0,1\}^{|S|}$,*

$$\lambda_{\max}\big(\Psi_{a_S}\big) \leq \eta.$$

Let's see some examples before we go on. First, it is easy to see that if μ is an independent product distribution (i.e. there exist $\lambda_1, \ldots, \lambda_n$ such that $\mu(S) \propto \sum_{i\in S} \lambda_i$), then μ is 0-spectrally independent. This suggests that spectral independence is an algebraic way to quantity the independence of a probability distribution.

A more interesting example is the class of negatively correlated distributions that we studied in Chapter 16.

Problem 21.9 (Spectral Independence of Strongly Rayleigh Distributions)**.** *Let $\mu : \{0,1\}^n$ be a homogeneous distribution such that for all $i \neq j$,*

$$\Pr_{Z \sim \mu}\big[Z(i) = 1 \mid Z(j) = 1\big] \leq \Pr_{Z \sim \mu}\big[Z(i) = 1\big].$$

Prove that $\lambda_{\max}(\Psi) \leq 1$ where Ψ is the correlation matrix of μ. Conclude that a homogeneous strongly Rayleigh distribution as defined in Definition 16.1 is 1-spectrally independent.

Glauber Dynamics

A natural random walk on a probability distribution $\mu : \{0,1\}^n \to \mathbb{R}$ is called Glauber dynamics.

Definition 21.10 (Glauber Dynamics)**.** *Let $\mu : \{0,1\}^n \to \mathbb{R}$ be a probability distribution on subsets of $[n]$. Start with an arbitrary subset $S_0 \in \operatorname{supp}(\mu)$. At each iteration $t \geq 1$, we choose a uniformly random element $i \in [n]$ and set*

$$S_t := \begin{cases} S_{t-1} \setminus \{i\} & \text{with probability } \frac{\mu(S_{t-1}\setminus\{i\})}{\mu(S_{t-1}\setminus\{i\})+\mu(S_{t-1}\cup\{i\})} \\ S_{t-1} \cup \{i\} & \text{otherwise.} \end{cases}$$

Check that this Markov chain has stationary distribution μ.

The main result of this formulation is to bound the spectral gap of the transition matrix of the Glauber dynamics by the spectral independence of the probability distribution.

Theorem 21.11 (Spectral Gap via Spectral Independence [ALO20])**.** *Let $\mu : \{0,1\}^n \to \mathbb{R}$ be a probability distribution that is η-spectrally independent. The random walk matrix of the Glauber dynamics of μ has spectral gap at least*

$$\frac{1}{n}\prod_{i=0}^{n-2}\Big(1 - \frac{\eta}{n-i-1}\Big).$$

Simplicial Complex for Glauber Dynamics

The proof of Theorem 21.11 is by (1) defining a simplicial complex X^μ for μ, (2) showing that the down-up walk $P^{\triangledown}_{n-1}$ of X^μ is exactly the Glauber dynamics in Definition 21.10, (3) seeing that the conditional correlation matrices of μ are basically the matrices $W_\tau - J_\tau$ of the links of X^μ in the proofs of Proposition 20.14 and Proposition 21.1, and (4) seeing that the spectral gap bound in Theorem 21.11 for Glauber dynamics follows from that in Theorem 21.2 for down-up walks.

Definition 21.12 (Simplicial Complex of Assignments)**.** *Let $\mu : \{0,1\}^n \to \mathbb{R}$ be a probability distribution on subsets of $[n]$. The simplicial complex (X^μ, Π) is defined with ground set $[n] \times \{0,1\}$, with a maximal face $\zeta := \big((1, Z(1)), (2, Z(2)), \ldots, (n, Z(n))\big)$ of dimension $n-1$ with $\Pi(\zeta) := \mu(Z)$ for each $Z \in \operatorname{supp}(\mu)$. In words, each maximal face of X^μ corresponds to an assignment of the n binary variables with non-zero probability in μ.*

Note that there is a one-to-one correspondence between a face ζ_{a_S} of X^μ and a partial assignment $a_S \in \{0,1\}^{|S|}$ on a subset $S \subseteq [n]$ of binary variables. Hence we denote the links of X^μ by $X^\mu_{a_S}$ for $S \subseteq [n]$ and for $a_S \in \{0,1\}^{|S|}$.

Step (2) is left as an exercise.

Exercise 21.13 (Glauber Dynamics and Down-Up Walks). *Verify that the down-up walk matrix $P_{n-1}^{\bigtriangledown}$ on X^{μ} is exactly the transition matrix of Glauber dynamics on μ in Definition 21.10.*

Step (3) is to see that the correspondence between conditional correlation matrices of μ and random walk matrices of links of X^{μ}.

Lemma 21.14 (Correlation Matrices and Random Walk Matrices of Links). *Let $\mu : \{0,1\}^n \to \mathbb{R}$ be a probability distribution on subsets of $[n]$ and X^{μ} be the simplicial complex in Definition 21.12. For a partial assignment $a_S \in \{0,1\}^{|S|}$ on a subset $S \subseteq [n]$,*

$$W_{a_S} - J_{a_S} \preccurlyeq_{\Pi_0^{a_S}} \frac{1}{n-|S|-1}\big(J_{a_S} + \Psi_{a_S}\big),$$

where Ψ_{a_S} is the conditional correlation matrix in Definition 21.7, W_{a_S} is the random walk matrix of the link $X_{a_S}^{\mu}$ in Definition 19.12, and $J_{a_S} := \vec{1}(\Pi_0^{a_S})^T$ is defined as in Lemma 20.13.

Proof. We first check that each off-diagonal entry of LHS and RHS matches. Let a_S be a partial assignment and a_i, a_j be two bits for $i, j \notin S$. Then $a_{S\cup\{i\}}$ is used to denote the partial assignment on $S\cup\{i\}$ which extends a_S with the i-th variable being assigned a_i, and $a_{S\cup\{i,j\}}$ is defined analogously. By Definition 19.12 and Equation 19.1,

$$\begin{aligned} W_{a_S}\big((i,a_i),(j,a_j)\big) &= \frac{\Pi(a_{S\cup\{i,j\}})}{(|S|+2)\cdot\Pi(a_{S\cup\{i\}})} \\ &= \frac{\binom{n}{|S|+2}^{-1}\cdot \Pr_{Z\sim\mu}\big[Z\big(S\cup\{i,j\}\big) = a_{S\cup\{i,j\}}\big]}{(|S|+2)\cdot\binom{n}{|S|+1}^{-1}\cdot \Pr_{Z\sim\mu}\big[Z\big(S\cup\{i\}\big) = a_{S\cup\{i\}}\big]} \\ &= \frac{1}{n-|S|-1}\cdot \Pr_{Z\sim\mu}\big[Z(j)=a_j \mid Z(i)=a_i, Z(S)=a_S\big]. \end{aligned}$$

Similarly, by Equation 19.2,

$$J_{a_S}\big((i,a_i),(j,a_j)\big) = \Pi_0^{a_S}((j,a_j)) = \frac{\Pi(a_{S\cup\{j\}})}{(|S|+1)\cdot\Pi(a_S)} = \frac{1}{n-|S|}\cdot \Pr_{Z\sim\mu}\big[Z(j)=a_j \mid Z(S)=a_S\big].$$

This shows that the non-diagonal entries of Ψ_{a_S} and $(n-|S|-1)\cdot W_{a_S} - (n-|S|)\cdot J_{a_S}$ are the same. Rearranging and noting that the diagonal entries on RHS are bigger than that on LHS proves the statement. □

We are ready to prove step (4) and thus Theorem 21.11.

Proof of Theorem 21.11. The plan is to use the assumption that μ is η-spectrally independent (instead of local-spectral expansion of X^{μ}) to prove Proposition 21.1 with γ_{k-1} replaced by $\frac{\eta}{n-k-1}$, and then the theorem follows by plugging in $\gamma_{k-1} = \frac{\eta}{n-k-1}$ into Theorem 21.2 to obtain

$$\lambda_2(P_{n-1}^{\bigtriangledown}) \le 1 - \frac{1}{n}\prod_{j=-1}^{n-3}(1-\gamma_j) = 1 - \frac{1}{n}\prod_{j=0}^{n-2}\Big(1 - \frac{\eta}{n-k-1}\Big).$$

To see Proposition 21.1 holds with $\gamma_{k-1} = \frac{\eta}{n-k-1}$, we use Lemma 21.14 in Equation 21.1 with $X_\tau = X_{a_S}$ so that

$$\begin{aligned}\langle f, (P_k^\wedge - P_k^\triangledown) f\rangle_{\Pi_k} &= \mathbb{E}_{\tau\sim\Pi_{k-1}}\langle f_\tau^\perp, (W_\tau - J_\tau) f_\tau^\perp\rangle_{\Pi_0^\tau} \\ &\leq \mathbb{E}_{\tau\sim\Pi_{k-1}}\frac{1}{n-k-1}\langle f_\tau^\perp, (J_\tau + \Psi_\tau) f_\tau^\perp\rangle_{\Pi_0^\tau} \\ &\leq \mathbb{E}_{\tau\sim\Pi_{k-1}}\frac{\eta}{n-k-1}\langle f_\tau^\perp, f_\tau^\perp\rangle_{\Pi_0^\tau},\end{aligned}$$

where the last inequality uses the assumption that $\lambda_{\max}(\Psi_\tau) \leq \eta$ and $\langle f_\tau^\perp, J_\tau f_\tau\rangle = 0$. Then the rest of the proof of Proposition 21.1 is the same with γ_{k-1} replaced by $\frac{\eta}{n-k-1}$. □

To summarize, Theorem 21.11 can be seen as finding the corresponding simplicial complex so that Glauber dynamics is the same as down-up walks, and then interpreting the matrix $W_\tau - J_\tau$ in the proof of Proposition 21.1 as correlation matrices to define spectral independence. Spectral independence is a nice formulation so that probabilists do not need to know about high-dimensional expanders to use the result, and indeed this notion has led to many recent developments and we will discuss some in the next section.

21.3 Applications

In this section, we just briefly discuss some of the recent developments and point to the relevant references.

Sampling Independent Sets from Hardcore Distributions

The first major application of the spectral independence formulation is to prove fast mixing for sampling independent sets from the hardcore distribution.

Definition 21.15 (Hardcore Distributions)**.** *Given a graph $G = (V, E)$ and a parameter $\lambda > 0$, define the hardcore distribution $\mu_\lambda : \{0,1\}^{|V|} \to \mathbb{R}$ as $\mu_\lambda(S) = \lambda^{|S|}/Z_G(\lambda)$ for each independent set $S \subseteq V$, where*

$$Z_G(\lambda) := \sum_{S\subseteq V: S \text{ is an independent set}} \lambda^{|S|}$$

is the normalization constant called the partition function.

Estimating the partition function is a well-studied problem in statistical physics. Given a graph of maximum degree Δ, there is a critical threshold $\lambda(\Delta) = (\Delta-1)^{\Delta-1}/(\Delta-2)^\Delta$ called the "tree uniqueness threshold", where $\lambda < \lambda(\Delta)$ corresponds to the regime where the "influence" of a vertex u on another vertex v in the infinite Δ-regular tree decays exponentially fast in the distance between u and v.

The tree uniqueness threshold is about a mathematical property, but very interestingly this is also about computational complexity. A seminal work of Weitz showed that for any $\lambda < \lambda(\Delta)$, there is a deterministic fully polynomial time approximate scheme to estimate $Z_G(\lambda)$. Another seminal work of Sly proved that for any $\lambda > \lambda(\Delta)$, there is no such scheme to estimate $Z_G(\lambda)$ unless NP = RP. Both proofs connect explicitly the mathematical property to the computational complexity.

It was conjectured that the simple Glauber dynamics in Definition 21.10 for the hardcore distributions mixes in polynomial time whenever $\lambda < \lambda(\Delta)$. Anari, Liu and Oveis Gharan [ALO20] introduced spectral independence and used this notion to resolve the conjecture positively. Their proof uses the self-avoiding walk tree defined by Weitz to write a recurrence to bound the maximum row sum of the correlation matrices Ψ in Definition 21.5 to bound their maximum eigenvalue $\lambda_{\max}(\Psi)$ to apply Theorem 21.11 to conclude fast mixing. The proof is interesting and nontrivial which extends previous techniques in "correlation decay". One advantage of this randomized algorithm is that the dependency on Δ in the running time is much better than that of Weitz.

Sampling Graph Coloring from Glauber Dynamics

The work on spectral independence [ALO20] inspired many recent developments. One natural generalization is to sample from distributions $\mu : [k]^n \to \mathbb{R}$ where the variables are of larger arity. This class of distributions includes the problem of sampling a random graph coloring of a graph. The long standing major open problem for sampling graph coloring is that the simple Glauber dynamics as in Definition 21.10 mixes rapidly as long as the number of colors k is at least $\Delta + 2$ where Δ is the maximum degree of the input graph. Note that the Glauber dynamics may not be irreducible when $k \leq \Delta + 1$. The best known result is by Vigoda that the Glauber dynamic mixes in polynomial time as long as $k \geq 11\Delta/6$, so there is a very large gap between the upper bound and the lower bound.

The random graph coloring problem is very well-studied, where previous results are mostly based on the coupling techniques to prove fast mixing. Using spectral independence with "correlation decay" arguments, the previous results can be recovered [CGSV21, CLV21] with improved running time. Some of these results also rely on log-Sobolev inequalities and entropy techniques that we will study in later chapters. The main goal in this line of work is to use these new ideas originally from high-dimensional expanders to make progress on the long standing open problem about mixing time of Glauber dynamics for graph coloring.

Problem 21.16 (Simplicial Complex for Graph Coloring)**.** *Define a simplicial complex for graph coloring so that the down-up walk matrix corresponds exactly to the Glauber dynamics. Define the corresponding notion of spectral independence and compare to those defined in [CGSV21, CLV21].*

Coupling and Spectral Independence

A general question is how does the spectral independence method relate to other methods for proving fast mixing such as the most popular coupling techniques. Recent work in [Liu21, BCC$^+$22] show that certain types of coupling proofs imply spectral independence as well, suggesting the spectral independence method could be a unifying method in analyzing mixing times of Markov chains.

21.4 References

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