Chapter 20

Higher Order Random Walks

We study two related random walks on simplicial complexes, called the down-up walks and up-down walks. The main result is that they are fast mixing if the simplicial complex is a good local-spectral expander. A consequence is that the natural random walks on matroid bases is fast mixing, proving the long-standing matroid expansion conjecture.

20.1 Random Walks on Simplicial Complexes

Kaufman and Mass [KM17] defined two natrual random walks on faces of dimension k in a simplicial complex, the up-down walks that go through faces of dimension $k+1$ and the down-up walks that go through faces of dimension $k-1$. The most intuitive way to define these walks is to consider the following bipartite graphs.

Definition 20.1 (Bipartite Graph of a Layer). *Let (X, Π) be a pure d-dimensional simplicial complex. For any $-1 \leq k \leq d-1$, the bipartite graph $H_k = (X(k), X(k+1); E)$ has one vertex for each face in $X(k) \cup X(k+1)$, with an edge between a face $\alpha \in X(k)$ and a face $\beta \in X(k+1)$ if and only if $\alpha \subset \beta$ and the weight of this edge is $\frac{1}{k+2} \cdot \Pi_{k+1}(\beta)$.*

Up and Down Operators

We consider the random walk matrix of these bipartite graphs and define the important up and down operators, which correspond to one-step random walks on the bipartite graphs in Definition 20.1.

Definition 20.2 (Up and Down Operators). *Let (X, Π) be a pure d-dimensional simplicial complex. Let A_k be the adjacency matrix of H_k with $A_k(\alpha, \beta) = A_k(\beta, \alpha) = \frac{1}{k+2} \cdot \Pi_{k+1}(\beta)$ if $\alpha \subset \beta$ for any $\alpha \in X(k)$ and $\beta \in X(k+1)$ and zero otherwise. For each face $\alpha \in X(k)$, the weight degree of α is*

$$\deg(\alpha) := \sum_{\beta \in X(k+1): \beta \supset \alpha} A_k(\alpha, \beta) = \sum_{\beta \in X(k+1): \beta \supset \alpha} \frac{1}{k+2} \cdot \Pi_{k+1}(\beta) = \Pi_k(\alpha),$$

where the last equality is by Equation 19.1. For each face $\beta \in X(k+1)$, the weighted degree of β is

$$\deg(\beta) := \sum_{\alpha \in X(k): \alpha \subset \beta} A_k(\alpha, \beta) = \sum_{\alpha \in X(k): \alpha \subset \beta} \frac{1}{k+2} \cdot \Pi_{k+1}(\beta) = \Pi_{k+1}(\beta).$$

The random walk matrix W_k of H_k can thus be written as

$$W_k = \begin{pmatrix} 0 & D_{k+1} \\ U_k & 0 \end{pmatrix},$$

where D_{k+1} is a $X(k) \times X(k+1)$ matrix and U_k is a $X(k+1) \times X(k)$ matrix with

$$D_{k+1}(\alpha, \beta) = \frac{A_k(\alpha,\beta)}{\deg(\alpha)} = \frac{\Pi_{k+1}(\beta)}{(k+2)\Pi_k(\alpha)} \quad \text{and} \quad U_k(\beta,\alpha) = \frac{A_k(\alpha,\beta)}{\deg(\beta)} = \frac{1}{k+2}.$$

for $\alpha \in X(k)$ and $\beta \in X(k+1)$ satisfying $\alpha \subset \beta$. The matrix D_{k+1} is called the down operator from $X(k+1)$ to $X(k)$ and U_k is called the up operator from $X(k)$ to $X(k+1)$.

The following remark may clear up some potential confusion about the naming convention.

Remark 20.3 (Down Up Confusion)**.** *The name down operator comes from the perspective that D_{k+1} is an operator that maps a function $f : X(k+1) \to \mathbb{R}$ to a function $g = D_{k+1}f : X(k) \to \mathbb{R}$, and so the output is one dimension lower and it is called a down operator. In other words, the name comes from when we do right-multiplication on the matrix.*

When we do random walks, however, we do left-multiplication of the form $p^T W_k$. So D_{k+1} actually maps a distribution in $X(k)$ to a distribution in $X(k+1)$, and the output is one dimension higher. It is a bit confusing for us because we mostly think about random walks, but it won't be a big issue that we won't often talk about these down and up operators alone.

A useful property is the adjoint property of the up and down operators.

Exercise 20.4 (Adjoint Property)**.** *Let (X, Π) be a pure d-dimensional simplicial complex. Prove that for any $f : X(k) \to \mathbb{R}$ and $g : X(k+1) \to \mathbb{R}$,*

$$\langle U_k f, g\rangle_{\Pi_{k+1}} = \langle f, D_{k+1} g\rangle_{\Pi_k}.$$

Up-Down Walks and Down-Up Walks

The two random walks defined by Kaufman and Mass correspond to two-steps random walks on the bipartite graphs in Definition 20.1.

Definition 20.5 (Up-Down Walks and Down-Up Walks)**.** *Let (X, Π) be a pure d-dimensional simplicial complex. Let H_k be the bipartite graph in Definition 20.1 and W_k be the random walk matrix on H_k in Definition 20.2. Consider*

$$W_k^2 = \begin{pmatrix} D_{k+1}U_k & 0 \\ 0 & U_k D_{k+1} \end{pmatrix} =: \begin{pmatrix} P_k^{\triangle} & 0 \\ 0 & P_{k+1}^{\triangledown} \end{pmatrix},$$

where $P_k^{\triangle} \in \mathbb{R}^{X(k)\times X(k)}$ is called the up-down walk matrix and $P_{k+1}^{\triangledown} \in \mathbb{R}^{X(k+1)\times X(k+1)}$ is called the down-up walk matrix.

A simple but important property of $P_k^{\triangle}$ and $P_{k+1}^{\triangledown}$ is that they have the same spectrum. This will be used in an inductive proof to analyze the spectrum of $P_d^{\triangledown}$.

Exercise 20.6 (Same Spectrum of $P_k^{\triangle}$ and $P_{k+1}^{\triangledown}$)**.** *Prove that there is a one-to-one correspondence between the non-zero eigenvalues of $P_k^{\triangle}$ and $P_{k+1}^{\triangledown}$.*

It will be helpful to write out the entries of $P_k^{\triangle}$ and $P_{k+1}^{\triangledown}$ explicitly.

Exercise 20.7 (Entries of $P_k^{\triangle}$ and $P_{k+1}^{\triangledown}$)**.** *Let (X, Π) be a pure d-dimensional simplicial complex. For $\alpha, \alpha' \in X(k)$,*

$$P_k^{\triangle}(\alpha, \alpha') = \begin{cases} \frac{1}{k+2} & \text{if } \alpha = \alpha' \\ \frac{\Pi_{k+1}(\alpha \cup \alpha')}{(k+2)^2 \cdot \Pi_k(\alpha)} & \text{if } \alpha \cup \alpha' \in X(k+1) \\ 0 & \text{otherwise.} \end{cases}$$

For $\beta, \beta' \in X(k+1)$.

$$P_{k+1}^{\triangledown}(\beta, \beta') = \begin{cases} \sum_{\alpha \in X(k): \alpha \subset \beta} \frac{\Pi_{k+1}(\beta')}{(k+2)^2 \cdot \Pi_k(\alpha)} & \text{if } \beta = \beta' \\ \frac{\Pi_{k+1}(\beta')}{(k+2)^2 \cdot \Pi_k(\beta \cap \beta')} & \text{if } \beta \cap \beta' \in X(k) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $P_0^{\triangle}$ is just the standard lazy random walks on a graph. The non-lazy up-down walks turn out to be important in the analysis.

Definition 20.8 (Non-Lazy Up-Down Walks)**.** *Let (X, Π) be a pure d-dimensional simplicial complex. For $-1 \leq k \leq d-1$, the non-lazy up-down walk matrix $P_k^{\wedge} \in \mathbb{R}^{X(k) \times X(k)}$ is defined as*

$$P_k^{\wedge} := \frac{k+2}{k+1}\Big(P_k^{\triangle} - \frac{I}{k+2}\Big).$$

Explicitly, for $\alpha, \alpha' \in X(k)$,

$$P_k^{\wedge}(\alpha, \alpha') = \begin{cases} \frac{\Pi_{k+1}(\alpha \cup \alpha')}{(k+1)(k+2) \cdot \Pi_k(\alpha)} & \text{if } \alpha \cup \alpha' \in X(k+1) \\ 0 & \text{otherwise.} \end{cases}$$

For the notations, remember that the $\triangle$ in $P^{\triangle}$ represents that there could be self-loops, while the $\wedge$ in $P^{\wedge}$ represents that the two endpoints are different.

The stationary distributions of $P_k^{\triangle}, P_k^{\wedge}, P_k^{\triangledown}$ are all the same. This can be checked by direct calculations or check that the time reversible condition (i.e. $\pi_i P(i,j) = \pi_j P(j,i)$ for all i, j) is satisfied.

Exercise 20.9 (Stationary Distributions)**.** *The stationary distributions of $P_k^{\triangle}, P_k^{\wedge}, P_k^{\triangledown}$ are Π_k.*

This will allow us to use the inner product $\langle \cdot, \cdot \rangle_{\Pi_k}$ to bound the eigenvalues of $P_k^{\triangle}, P_k^{\wedge}, P_k^{\triangledown}$ using the Rayleigh quotients with Π_k in Definition 19.21.

Random Walks on Matroid Bases: To sample a uniform random basis of a matroid, we consider the matroid complex with the uniform distribution on the bases, and run the down-up walk $P_d^{\triangledown}$. Then, by Exercise 20.9, the stationary distribution is the uniform distribution. Note that the down-up walk $P_d^{\triangledown}$ is the natural algorithm that we start from an arbitrary basis B_0, and in each iteration $t \geq 0$ we drop a random element i of the current basis and then add a random element j so that $B_{t+1} := B_t - i + j$ is a basis, and repeat. Observe that the random spanning tree algorithm in

Chapter 6 is a special case. We know from Theorem 19.17 that a matroid complex is a 0-local-spectral expander. We will see in the next section that the up-down walks and the down-up walks of a good local-spectral expander mix quickly. Thus this provides a simple and efficient algorithm to sample a uniform matroid basis, answering a long-standing open question called the matroid expansion conjecture that we will explain at the end in the next section.

20.2 Kaufman-Oppenheim Theorem

Kaufman and Oppenheim [KO20] proved that if the simplicial complex is a good local-spectral expander, then the up-down walks and the down-up walks mix quickly.

Theorem 20.10 (Kaufman-Oppenheim Second Eigenvalue Bound [KO20])**.** *If (X, Π) is a γ-local-spectral expander, then for any $0 \leq k \leq d$ the second eigenvalue of $P_k^{\bigtriangledown}$ is*

$$\lambda_2(P_k^{\bigtriangledown}) \leq 1 - \frac{1}{k+1} + k\gamma.$$

We will present the proof of this theorem in the rest of this section, and discuss the matroid expansion conjecture at the end.

Garland's Method

An important step in the proof is to use Garland's method to decompose the down-up walk matrix and the up-down walk matrix into matrices of the links. We need a definition similar to, but different from, Definition 19.22.

Definition 20.11 (Restriction to Link)**.** *Given a simplicial complex (X, Π), a function $f : X(k) \to \mathbb{R}$ and a face $\tau \in X(k-1)$, the restriction of f to $X_\tau(0)$ is defined as $f_\tau : X_\tau(0) \to \mathbb{R}$ such that $f_\tau(x) = f(\tau \cup \{x\})$ for all $x \in X_\tau(0)$.*

The following lemma shows that the non-lazy up-down walk matrix $P_k^{\wedge}$ can be decomposed into the random walk matrix W_τ of the links for $\tau \in X(k-1)$. The reason that we consider the non-lazy up-down walk is that there are no self-loops in the random walk matrices of the links.

Lemma 20.12 (Decomposition of Non-Lazy Up-Down Walk Matrix)**.** *For any pure d-dimensional simplicial complex (X, Π) and any function $f : X(k) \to \mathbb{R}$,*

$$\langle f, P_k^{\wedge} f \rangle_{\Pi_k} = \mathbb{E}_{\tau \sim \Pi_{k-1}} \langle f_\tau, W_\tau f_\tau \rangle_{\Pi_0^\tau},$$

where W_τ is the random walk matrix of the link τ in Definition 19.12.

Proof. The main idea is to decompose $P_k^{\wedge}$ into transition matrices where each is about the transitions involving a particular link $\tau \in X(k-1)$. Let $P_\tau^{\wedge}$ be the $X(k) \times X(k)$ matrix with

$$P_\tau^{\wedge}(\sigma, \sigma') = \frac{\Pi_{k+1}(\alpha \cup \alpha')}{(k+1)(k+2) \cdot \Pi_k(\alpha)} \quad \text{if } \alpha \cap \alpha' = \tau \quad \text{and} \quad P_\tau^{\wedge}(\sigma, \sigma') = 0 \quad \text{otherwise.}$$

Note that $P_k^{\wedge} = \sum_{\tau \in X(k-1)} P_\tau^{\wedge}$ by Definition 20.8, as the transition between any two faces $\alpha, \alpha' \in X(k)$ involves a unique link $\tau = \alpha \cap \alpha' \in X(k-1)$. The observation is that this matrix $P_\tau^{\wedge}$ is almost the same as the random walk matrix W_τ of the link τ. Let $\alpha = \tau \cup \{x\}$ and $\alpha' = \tau \cup \{y\}$. Then, from Definition 19.12,

$$W_\tau(x,y) = \frac{\Pi(\tau \cup \{x,y\})}{(|\tau|+2)\cdot \Pi(\tau \cup \{x\})} = \frac{\Pi_{k+1}(\alpha \cup \alpha')}{(k+2)\cdot \Pi_k(\alpha)} \quad \Longrightarrow \quad P_\tau^\wedge(\sigma,\sigma') = \frac{1}{k+1} W_\tau(\sigma \setminus \tau, \sigma' \setminus \tau).$$

So, if we extend the small matrices W_τ appropriately to $\tilde{W}_\tau$ (i.e. put the (x,y)-entry of W_τ on the $(\tau \cup \{x\}, \tau \cup \{y\})$-entry of $\tilde{W}_\tau$ and set all other entries to be zero), then

$$P_k^\wedge = \frac{1}{k+1} \sum_{\tau \in X(k-1)} \tilde{W}_\tau,$$

and so it should be clear that a quadratic form involving $P_k^\wedge$ can be decomposed as a sum of quadratic forms involving W_τ as in the statement.

To write it concisely, we decompose the quadratic form directly (instead of decomposing the matrix $P_k^\wedge$). By writing the quadratic form as a sum of $|X(k)| \times |X(k)|$ terms,

$$\langle f, P_k^\wedge f \rangle_{\Pi_k} = \sum_{\sigma \in X(k), \sigma' \in X(k) : \sigma \cup \sigma' \in X(k+1)} \frac{\Pi_{k+1}(\sigma \cup \sigma')}{(k+1)(k+2)} \cdot f(\sigma) \cdot f(\sigma')$$

For each pair $\sigma, \sigma' \in X(k)$ with $\sigma \cup \sigma' \in X(k+1)$, their intersection $\tau := \sigma \cap \sigma' \in X(k-1)$. Let $x = \sigma \setminus \tau$ and $y = \sigma' \setminus \tau$. Then the corresponding term on the RHS is from the link X_τ with contribution

$$\begin{aligned}
& \Pi_{k-1}(\tau) \cdot \Pi_0^\tau(x) \cdot W_\tau(x,y) \cdot f_\tau(x) \cdot f_\tau(y) \\
= \; & \Pi_{k-1}(\tau) \cdot \frac{\Pi_k(\tau \cup \{x\})}{(|\tau|+1)\cdot \Pi_{k-1}(\tau)} \cdot \frac{\Pi_{k+1}(\tau \cup \{x,y\})}{(|\tau|+2)\cdot \Pi_k(\tau \cup \{x\})} \cdot f(\tau \cup \{x\}) \cdot f(\tau \cup \{y\}) \\
= \; & \frac{\Pi_{k+1}(\sigma \cup \sigma')}{(k+1)(k+2)} \cdot f(\sigma) \cdot f(\sigma').
\end{aligned}$$

The statement follows by noting that there is a one-to-one correspondence because each transition in $P_k^\wedge$ involves a unique link $\tau \in X(k-1)$. □

The next lemma shows that the down-up walk matrix can be decomposed as the down-up walk matrices of the links which are simple rank-one matrices.

Lemma 20.13 (Decomposition of Down-Up Walk Matrix). *For any pure d-dimensional simplicial complex (X, Π) and any function $f : X(k) \to \mathbb{R}$,*

$$\langle f, P_k^\triangledown f \rangle_{\Pi_k} = \mathbb{E}_{\tau \sim \Pi_{k-1}} \langle f_\tau, J_\tau f_\tau \rangle_{\Pi_0^\tau},$$

where $J_\tau := \vec{1}(\Pi_0^\tau)^T$ is a $X_\tau(0) \times X_\tau(0)$ rank-one matrix.

Proof. The proof is similar to that in Lemma 20.12. We decompose $P_k^\triangledown$ into transition matrices where each is about the transitions involving a particular link $\tau \in X(k-1)$. Let $P_\tau^\triangledown$ be the $X(k) \times X(k)$ matrix with

$$P_\tau^\triangledown(\sigma,\sigma') = \frac{\Pi_k(\sigma')}{(k+1)^2 \cdot \Pi_{k-1}(\tau)} \;\text{ if } \sigma \cap \sigma' \supseteq \tau \quad \text{and} \quad P_\tau^\triangledown(\sigma,\sigma') = 0 \;\text{ otherwise.}$$

Note that $P_k^{\bigtriangledown} = \sum_{\tau \in X(k-1)} P_\tau^{\bigtriangledown}$ by Exercise 20.7, where the summation in the self-loop probability for σ is split into the subsets $\tau \subset \sigma$ for $\tau \in X(k-1)$ where each takes a summand. Let $\sigma = \tau \cup \{x\}$ and $\sigma' = \tau \cup \{y\}$. By the definition of J_τ and Equation 19.2,

$$J_\tau(x,y) = \Pi_0^\tau(y) = \frac{\Pi(\tau \cup \{y\})}{(|\tau|+1) \cdot \Pi(\tau)} = \frac{\Pi_k(\alpha')}{(k+1) \cdot \Pi_{k-1}(\tau)} \quad \Longrightarrow \quad P_\tau^{\bigtriangledown}(\sigma, \sigma') = \frac{1}{k+1} J_\tau(\sigma \backslash \tau, \sigma' \backslash \tau).$$

Check that the remaining calculations are similar to that in Lemma 20.12, with the contribution from $\sigma, \sigma' \in X(k)$ involving at link $\tau \in X(k-1)$ is the same from LHS and RHS, being equal to

$$\frac{\Pi_k(\sigma) \cdot \Pi_k(\sigma')}{(k+1)^2 \cdot \Pi_{k-1}(\tau)} \cdot f(\sigma) \cdot f(\sigma').$$

□

Comparing Down-Up Walk and Non-Lazy Up-Down Walk

The decomposition of the down-up walk matrix in Lemma 20.13 shows that $P_k^{\bigtriangledown}$ can be written as the sum of rank-one matrices in the links each with second largest eigenvalue 0, while the decomposition of the non-lazy up-down walk matrix in Lemma 20.12 shows that $P_k^{\wedge}$ can be wrttien as the sum of random walk matrices in the links each with second largest eigenvalue at most γ for a γ-local-spectral expander. The main step in Kaufman-Oppenheim's theorem is to compare the spectrum of the down-up walk matrix $P_k^{\bigtriangledown}$ with the non-lazy up-down walk matrix $P_k^{\wedge}$, which intuitively can be understood as a term-by-term comparison between a complete graph and an expander graph.

Proposition 20.14 (Comparision of $P_k^{\bigtriangledown}$ and $P_k^{\wedge}$ [KO20, DDFH18])**.** *If (X, Π) is a γ-local-spectral expander, then*

$$P_k^{\wedge} - P_k^{\bigtriangledown} \preccurlyeq_{\Pi_k} \gamma I$$

for any $0 \leq k \leq d-1$, where $A \preccurlyeq_{\Pi} B$ donotes $\langle f, Af \rangle_\Pi \leq \langle f, Bf \rangle_\Pi$ for all f.

Proof. Using Lemma 20.12 and Lemma 20.13,

$$\big\langle f, (P_k^{\wedge} - P_k^{\bigtriangledown}) f \big\rangle_{\Pi_k} = \mathbb{E}_{\tau \sim \Pi_{k-1}} \big\langle f_\tau, (W_\tau - J_\tau) f_\tau \big\rangle_{\Pi_0^\tau}.$$

For each term, write $f_\tau = c\vec{1} + f_\tau^\perp$ where $\big\langle \vec{1}, f_\tau^\perp \big\rangle_{\Pi_0^\tau} = 0$ as in Equation 19.5. Then note that

$$\big\langle f_\tau, (W_\tau - J_\tau) f_\tau \big\rangle_{\Pi_0^\tau} = \big\langle c\vec{1} + f_\tau^\perp, (W_\tau - J_\tau)(c\vec{1} + f_\tau^\perp) \big\rangle_{\Pi_0^\tau} = \big\langle f_\tau^\perp, (W_\tau - J_\tau) f_\tau^\perp \big\rangle_{\Pi_0^\tau},$$

because $W_\tau \vec{1} = J_\tau \vec{1} = \vec{1}$ and also $(\Pi_0^\tau)^T W_\tau = (\Pi_0^\tau)^T J_\tau = (\Pi_0^\tau)^T$. Therefore,

$$\begin{aligned} \langle f, (P_k^{\wedge} - P_k^{\bigtriangledown}) f \rangle_{\Pi_k} &= \mathbb{E}_{\tau \sim \Pi_{k-1}} \big\langle f_\tau^\perp, (W_\tau - J_\tau) f_\tau^\perp \big\rangle_{\Pi_0^\tau} \\ &\leq \mathbb{E}_{\tau \sim \Pi_{k-1}} \gamma \big\langle f_\tau^\perp, f_\tau^\perp \big\rangle_{\Pi_0^\tau} \\ &\leq \mathbb{E}_{\tau \sim \Pi_{k-1}} \gamma \big\langle f_\tau, f_\tau \big\rangle_{\Pi_0^\tau} \\ &= \gamma \big\langle f, f \big\rangle_{\Pi_k}, \end{aligned}$$

where the first inequality is by the Rayleigh quotient characterization in Equation 19.4 and the assumption that (X, Π) is a γ-local-spectral expander, and the last equality is left as Exercise 20.15.

□

Exercise 20.15 (Decomposition of Identity)**.** *Prove that $\langle f, f \rangle_{\Pi_k} = \mathbb{E}_{\tau \sim \Pi_{k-1}} \langle f_\tau, f_\tau \rangle_{\Pi_0^\tau}$.*

Inductive Proof

Now we are ready to prove Kaufman-Oppenheim's Theorem 20.10. The proof is by an interesting induction, that we start from the spectrum of $P_0^{\bigtriangledown}$ and use Exercise 20.6 and Proposition 20.14 to reason about the spectrums of $P_k^{\bigtriangledown}$ and $P_k^{\bigtriangleup}$ and $P_k^{\wedge}$. We prepare with the following exercise which will be used in reasoning about the spectrums.

Exercise 20.16 (Bounding Spectrum by Quadratic Forms). *Let $A, B \in \mathbb{R}^{n \times n}$ be two self-adjoint matrices with respect to the inner product Π (see Equation 19.3). If $A \preccurlyeq_{\Pi} B$ as described in Proposition 20.14, then $\lambda_i(A) \leq \lambda_i(B)$ for all $1 \leq i \leq n$.*

Proof of Theorem 20.10. The proof is by induction on k. In the base case when $k = 0$, the matrix $P_0^{\bigtriangledown}$ is of rank one (see Exercise 20.7), and thus the second largest eigenvalue is at most 0, and the statement holds.

Now, assume the statement holds for k, and we would like to prove the inductive step. By Proposition 20.14, $P_k^{\wedge} \preccurlyeq_{\Pi_k} P_k^{\bigtriangledown} + \gamma I$. It follows from Exercise 20.16 and Exercise 20.9 that

$$\lambda_2(P_k^{\wedge}) \leq \lambda_2(P_k^{\bigtriangledown}) + \gamma \leq 1 - \frac{1}{k+1} + (k+1)\gamma,$$

where the second inequality is by the induction hypothesis on $P_k^{\bigtriangledown}$. Recall from Definition 20.8 that

$$P_k^{\wedge} = \frac{k+2}{k+1}\Big(P_k^{\bigtriangleup} - \frac{I}{k+2}\Big) \quad \Longrightarrow \quad P_k^{\bigtriangleup} = \frac{k+1}{k+2}P_k^{\wedge} + \frac{I}{k+2}.$$

Therefore, the second largest eigenvalue of $P_k^{\bigtriangleup}$ is

$$\lambda_2(P_k^{\bigtriangleup}) \leq \frac{k+1}{k+2}\Big(1 - \frac{1}{k+1} + (k+1)\gamma\Big) + \frac{1}{k+2} = 1 - \frac{1}{k+2} + \frac{(k+1)^2\gamma}{k+2} \leq 1 - \frac{1}{k+2} + (k+1)\gamma.$$

Finally, recall that $P_{k+1}^{\bigtriangledown}$ and $P_k^{\bigtriangleup}$ have the same spectrum by Exercise 20.6, and this completes the induction step. □

Combinatorial Interpretation: To summarize, one could visualize the proof as having a stack of bipartite graphs, one for each layer as in Definition 20.1. To reason about the spectrum of the top layer, we start from the down-up walk of the bottom layer. The key step using Garland's method is to observe that the (non-lazy) up-down walk in the layer above has a similar structure to the down-up walk in the layer below, by replacing each clique J_τ in a link τ in the down-up walk in Lemma 20.13 by an expander graph W_τ in the up-down walk in Lemma 20.12, whose expansion comes from the assumption of the local-spectral expander. So, if the down-up walk is an expander then the up-down walk is still an expander but with slightly weaker expansion (as we just replace a complete graph by an expander graph), and this is essentially the term-by-term comparison step of Kaufman and Oppenheim in Proposition 20.14. Finally, within the same layer, we use the simple but important property that the up-down walk and the down-up walk having the same spectrum to carry out the induction.

Matroid Expansion Conjecture

Recall from Theorem 19.17 that the matroid complex is a 0-local-spectral expander (using Oppenheim's trickling down Theorem 19.15). Then, by Kaufman-Oppenheim's Theorem 20.10, $\lambda_2(P_d^{\triangledown}) \leq 1 - \frac{1}{d+1}$ where $r := d+1$ is the rank of the matroid. By standard analysis of mixing time in Theorem 6.16, the ϵ-mixing time of the natural down-up walks is at most $O(r \log \frac{N}{\epsilon}) = O(r^2 \log \frac{n}{\epsilon})$ where N is the number of bases and n is the number of elements in the ground set.

Theorem 20.17 (Sampling Matroid Bases by Down-Up Walks [ALOV19])**.** *Given a matroid M with n elements and rank at most r, the mixing time of the down-up walk of the matroid complex is at most $O(r^2 \log \frac{n}{\epsilon})$.*

The matroid expansion conjecture by Mihail and Vazirani from 1989 states that the bases exchange graph has edge expansion at least one, which follows from Theorem 20.17 and Cheeger's inequality in Theorem 4.3.

Problem 20.18 (Matroid Expansion Conjecture)**.** *The bases exchange graph $G = (V, E)$ is the underlying unweighted graph of the down-up walk matrix of the matroid complex. Prove that the edge expansion of G is at least one, that is, $|\delta(S)|/|S| \geq 1$ for all $S \subseteq V$ with $|S| \leq |V|/2$.*

One may understand the resolution of the matroid expansion conjecture as using the right induction for the problem which may not be easy to come up with without the perspective of a simplicial complex and the concepts such as links. It would be great if someone could write a completely combinatorial proof (without using any linear algebra) of the matroid expansion conjecture using the combinatorial interpretation above.

Question 20.19 (Combinatorial Proof of Matroid Expansion Conjecture)**.** *Is there a purely combinatorial proof of the matroid expansion conjecture?*

20.3 References

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