Chapter 19

High Dimensional Expanders

We begin the third part of the course about high dimensional expanders and log-concave polynomials. In this chapter, we study a definition of high dimensional expanders through local spectral expansion. We will introduce the necessary concepts and then prove a fundamental result by Oppenheim called the trickling down theorem. See [Ove20, HL21] for similar notes which are used in the preparation of this chapter.

High Dimensional Expanders

As we have seen in Chapter 7, expander graphs have nice combinatorial, probabilistic, and algebraic properties, which is an important reason that there is a rich theory with connections and applications in diverse areas. While it may be easy to generalize the definition of expander graphs to higher dimensions for some properties (e.g. combintorial expansion in hypergraphs), it is not easy to find a definition in higher dimension that generlizes all nice properties of expander graphs. There are various definitions of high dimensional expanders, some using concepts from algebraic topology; see [Lub18] for a survey with motivations and applications.

In this course, the main application of high dimensional expanders is in analyzing mixing time of Markov chains. We will study a more recent and elementary definition developed in [KM17, DK17, KO20], which was motivated by the study of random walks.

19.1 Simplicial Complexes

A simplicial complex is a high dimensional generalization of a graph.

Definition 19.1 (Simplicial Complex)**.** *A set system is a pair $X = (U, \mathcal{F})$ with U as the ground set and $\mathcal{F}$ is a set of subsets of U. A simplicial complex is a set system that is downward closed, such that if $\tau \in \mathcal{F}$ and $\sigma \subset \tau$, then $\sigma \in \mathcal{F}$.*

We follow the convention of using Greek letters $\sigma, \tau, \eta, \alpha, \beta$ for subsets in $\mathcal{F}$. The following are some basic definitions about simplicial complexes.

Definition 19.2 (Face, Dimension, Pure Simplicial Complex)**.** *Any subset $\sigma \in \mathcal{F}$ is called a face of the simplicial complex $X = (U, \mathcal{F})$. A face σ is of dimension k if its size is $|\sigma| = k + 1$, e.g. a 0-dimensional face is a singleton (a vertex), a 1-dimensional face is a pair (an edge), a 2-dimensional face is a triple, etc. Given a simplicial complex $X = (U, \mathcal{F})$, we use $X(k)$ to denote the set of* *faces of dimension k. A simplicial complex is d-dimensional if the maximum face size is $d+1$. A d-dimensional simplicial complex is pure if every maximal face is of size $d+1$.*

Simplicial complex is a very general definition. We can associate a simplicial complex to many classes of combinatorial objects.

Example 19.3 (Simplicial Complex from Spanning Trees). *Given a graph $G = (V, E)$, we can define a simplicial complex $X = (E, \mathcal{F})$ where the ground set in X is the edge set E of G. A subset of edges $E' \subseteq E$ is in $\mathcal{F}$ if and only if E' forms an acyclic subgraph in G. It should be clear that X is a pure simplicial complex. When G is connected, the maximal faces correspond to spanning tree, which are of size $|V| - 1$ and so X is $(|V| - 2)$-dimensional.*

More generally, every matroid naturally corresponds to a simplicial complex.

Example 19.4 (Simplicial Complex from Matroids). *A matroid $M = (U, \mathcal{I})$ is a set system where U is the ground set and $\mathcal{I}$ is the set of subsets of U which satisfies the following two properties:*

1. *$\mathcal{I}$ is downward close, i.e. $S \in \mathcal{I}$ and $T \subseteq S$ implies $T \in \mathcal{I}$.*
2. *If $S, T \in \mathcal{I}$ and $|T| > |S|$, then there exists $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$.*

So, by (1), $M = (U, \mathcal{I})$ is a simplicial complex. And, by (2), $M = (U, \mathcal{I})$ is a pure simplicial complex. The sets in $\mathcal{I}$ are usually called the independent sets, and the maximal sets are usually called bases. It is not difficult to check that the simplicial complex from spanning trees is a matroid.

A more general example is the class of linear matroids. Given a matrix $A \in \mathbb{F}^{m \times n}$, the linear matroid of A is defined as $M = ([n], \mathcal{I})$ where the ground set $[n]$ is the set of columns of A, and a subset S of columns is in $\mathcal{F}$ if and only if the columns in S are linearly independent. We leave it as an exercise to check that it is a matroid and includes the matroid from spanning trees as a special case.

There are many more simplicial complexes that one can define, e.g. simplicial complexes from cliques of graphs, simplicial complexes for graph coloring, etc. We may discuss some of these in later chapters.

Weighted Simplicial Complexes

We will consider pure simplicial complexes with weights on its faces. We follow the convention in [DDFH18] that the weights form a probability distribution on the faces of the same dimension.

Definition 19.5 (Weighted Simplicial Complexes). *A weighted pure simplicial complex (X, Π) is a pure simplicial complex with a probability distribution Π on the faces of maximal dimension.*

In applications of random sampling, the probability distribution in the maximal faces are usually the uniform distribution, so they are simply unweighted simplicial complexes, but the following definition of induced distributions will be important in our study. An alternative way to think about these induced distributions is to think of them as weighted degrees of a subset in the simplicial complex.

Definition 19.6 (Induced Distributions). *Given a d-dimensional weighted pure simplicial complex (X, Π), a probability distribution Π_k on $X(k)$ for $0 \leq k \leq d$ is defined inductively as follows. The base case is $\Pi_d = \Pi$. For $d-1 \geq k \geq 0$, the probability distribution $\Pi_k : X(k) \to \mathbb{R}$ is defined by considering the marginal distributions such that*

$$\Pi_k(\alpha) = \frac{1}{k+2} \sum_{\beta \in X(k+1) : \beta \supset \alpha} \Pi_{k+1}(\beta) \tag{19.1}$$

for each face $\alpha \in X(k)$. Equivalently, we can understand Π_k as the probability distribution of the following random process. Sample a random face $\beta \in X(d)$ using the probability distribution Π_d, and then sample a uniform random subset α of β in $X(k)$, so that

$$\Pi_k(\alpha) = \frac{1}{\binom{d+1}{k+1}} \sum_{\beta \in X(d) : \beta \supset \alpha} \Pi_d(\beta) = \frac{1}{\binom{d+1}{k+1}} \Pr_{\beta \sim \Pi_d}[\beta \supset \alpha].$$

We will often drop the subscript about the dimension of the face. Just keep in mind that each Π_k is a probability distribution.

19.2 Local Spectral Expanders

We first define links and graphs of simplicial complexes, and then define local spectral expanders.

Links

The following is the key definition that enables a local-to-global approach for simplicial complexes.

Definition 19.7 (Links). *Let $X = (U, \mathcal{F})$ be a simplicial complex. For a face $\alpha \in \mathcal{F}$, the link X_α is defined as*

$$X_\alpha := \{\beta \setminus \alpha \mid \beta \in \mathcal{F}, \beta \supset \alpha\}.$$

In words, X_α is defined by the faces τ that can be used to extend α such that $\alpha \cup \tau \in \mathcal{F}$.

If X is a pure d-dimensional simplicial complex and $\alpha \in X(k)$, then X_α is a pure $(d-|\alpha|)$-dimensional simplicial complex (where the empty set is a face of dimension -1). In the spanning tree complex $X = (E, \mathcal{I})$, given a subset of acyclic edges $F \in \mathcal{I}$, the link X_F is defined such that a subset of edges F' is a face in X_F if and only if $F \cup F'$ is an acyclic subgraph. In matroid terminology, the link X_F is obtained by "contracting" the elements in F. A general approach to study a simplicial complex is to decompose it into its links, as we will see later in this chapter.

The probability distributions $\Pi_0, \ldots, \Pi_d$ on X in Definition 19.6 can be used to define $\Pi_0^\alpha, \ldots, \Pi_{d-k-1}^\alpha$ on X_α using conditional probability, where $\Pi^\alpha(\tau) \propto \Pr_{\beta \sim \Pi_d}[\beta \supset \tau \mid \beta \supset \alpha]$

Definition 19.8 (Induced Distributions on Links). *Let (X, Π) be a d-dimensional weighted pure simplicial complex. For any face α and any $\tau \in X_\alpha$,*

$$\Pi^\alpha(\tau) := \Pr_{\beta \sim \Pi_{|\tau|+|\alpha|-1}}\left[\beta = \alpha \cup \tau \mid \beta \supset \alpha\right] = \frac{\Pi(\alpha \cup \tau)}{\sum_{\beta : |\beta| = |\tau| + |\alpha|, \beta \supset \alpha} \Pi(\beta)} = \frac{\Pi(\alpha \cup \tau)}{\binom{|\alpha \cup \tau|}{|\alpha|} \cdot \Pi(\alpha)}, \tag{19.2}$$

where the last equality follows from Definition 19.6.

Often, it is enough to understand that $\Pi^\alpha(\tau) \propto \Pi(\alpha \cup \tau)$, and just see the denominator in Definition 19.8 as a normalizing constant.

Exercise 19.9. *Verify that Π_k^α is a probability distribution for every $0 \le k \le d - |\alpha|$.*

Skeletons and Graphs

Definition 19.10 (k-Skeletons)**.** *Given $X = (U, \mathcal{F})$, the k-skeleton of X is the simplicial complex $X_k = (U, \mathcal{F}_k)$ where $\mathcal{F}_k$ is the set of faces of $\mathcal{F}$ with dimension at most k. When there are weights on the faces in $\mathcal{F}$, we use the same weight on the faces in $\mathcal{F}_k$.*

The special case of 1-skeleton will be of particular interest, which could be thought of as the underlying graph of the simplicial complex.

Definition 19.11 (Graph of Links)**.** *For a link X_α, the graph $G_\alpha = (X_\alpha(0), X_\alpha(1), \Pi_1^\alpha)$ is defined as the 1-skeleton of X_α. More explicitly, each singleton $\{v\}$ in X_α is a vertex v in G_α, each pair $\{u, v\}$ in X_α is an edge uv in G_α, and the weight of uv in G_α is equal to $\Pi_1^\alpha(\{u, v\})$.*

A simple observation is that if X is a pure d-dimensional simplicial complex and Π is the uniform distribution on $X(d)$, then for any $\alpha \in X(d-2)$ the weighting Π_1^α on the edges of G_α is uniform. We will use this simple observation later.

Random Walk Matrices

The definition of local spectral expanders will be based on the random walk matrices of the links.

Definition 19.12 (Random Walk Matrix of a Link)**.** *Given the graph $G_\alpha = (X_\alpha(0), X_\alpha(1), \Pi_1^\alpha)$ of a link X_α, let A_α be the adjacency matrix of G_α and let D_α be the diagonal degree matrix where*

$$D_\alpha(x, x) = \sum_{y \in X_\alpha(0)} A_\alpha(x, y) = \sum_{y \in X_\alpha(0)} \Pi_1^\alpha(\{x, y\}) = 2\Pi_0^\alpha(\{x\}).$$

Check that the last equality follows from Definition 19.6 and Definition 19.8. The random walk matrix W_α of G_α is defined as

$$W_\alpha := D_\alpha^{-1} A_\alpha \quad \textit{where} \quad W_\alpha(x, y) = \frac{\Pi_1^\alpha(\{x, y\})}{2\Pi_0^\alpha(\{x\})} = \frac{\Pi(\alpha \cup \{x, y\})}{(|\alpha| + 2) \cdot \Pi(\alpha \cup \{x\})} \textit{ for all } \{x, y\} \in X_\alpha(1).$$

Check that the last equality follows from Definition 19.8. Note that the distribution Π_0^α is the stationary distribution of W_α as

$$(\Pi_0^\alpha)^T W_\alpha = (\Pi_0^\alpha)^T D_\alpha^{-1} A_\alpha = \frac{1}{2}(\vec{1})^T A_\alpha = (\Pi_0^\alpha)^T.$$

Recall from Chapter 6 that the random walk matrix and the normalized adjacency matrix of a graph are similar matrices, and so the eigenvalues of the random walk matrices are real. The largest eigenvalue of W_α is one and the all-one vector is a corresponding eigenvector.

Local Spectral Expanders

Finally, we can state the definition of high dimensional expanders that we will use.

Definition 19.13 (Local Spectral Expanders [KM17, DK17, KO20]). *Let (X, Π) be a pure d-dimensional simplicial complex. We say (X, Π) is a γ-local-spectral expander if $\lambda_2(W_\alpha) \leq \gamma$ for all faces $\alpha \in X$, where $\lambda_2(W_\alpha)$ is the second largest eigenvalue of the random walk matrix W_α.*

More generally, given $\gamma_{-1}, \ldots, \gamma_{d-2}$, we say (X, Π) is a $(\gamma_{-1}, \ldots, \gamma_{d-2})$-local-spectral expander if $\lambda_2(W_\alpha) \leq \gamma_k$ for all faces $\alpha \in X(k)$ for all $-1 \leq k \leq d-2$.

The definitions in [KM17, DK17] require a lower bound on the minimum eigenvalue of W_α as well. The above definition is from [KO20] where Kaufman and Oppenheim realized that only upper bounding λ_2 is enough for fast mixing of higher order random walks that we will define in the next chapter.

We can understand the above definition as requiring the "local" random walks in each link graph are fast mixing. As the random walk matrix has the same spectrum as the normalized adjacency matrix, we can also understand that the above definition as requiring the "local" weighted graphs of the links have large edge conductance through Cheeger's inequality.

Example 19.14 (Complete Complex). *Consider the complete complex $X_d = (U, \mathcal{F}_d)$ where every subset $S \subseteq U$ with $|S| \leq d+1$ is in $\mathcal{F}_d$, equipped with the uniform distribution on the faces of dimension d. Then the graph of every link of dimension k is an unweighted complete graph with $d-k$ vertices, with second largest eigenvalue of the random walk matrix being $-1/(d-k-1)$.*

It is not surprising that a complete complex is a good high dimensional expander (if not, what is?). As in expander graphs in Chapter 7, the goal is usually to construct high dimensional expanders with few maximal faces. Unlike in the graph case, however, random simplicial complexes are *not* high dimensional expanders with high probability. It is difficult to construct sparse high-dimensional expanders, with only a few known algebraic constructions [Lub18]. This is a topic of great interest but is out of the scope of this course.

19.3 Oppenheim's Trickling Down Theorem

To show that a simplicial complex is a γ-local-spectral expander, we need to bound the second largest eigenvalue of the random walk matrix for every link up to dimension $d-2$. In applications where the goal is to do uniform sampling of the maximal faces, it is usually much easier to work with the random walk matrix of the "top" links of dimension $d-2$, because the graphs of these links are unweighted as we mentioned before. For "lower" links, just determining the edge weights may already involve some difficult counting problems. So, it would be very nice if we could bound the second largest eigenvalues of the lower links by the second largest eigenvalues of the top links, and Oppenheim's trickling down theorem [Opp18] provides such a general bound for any pure simplicial complex.

Theorem 19.15 (Oppenheim's Trickling Down Theorem [Opp18]). *Let (X, Π) be a pure d-dimensional weighted simplicial complex where Π satisfies Equation 19.1 and Equation 19.2. Suppose the graph $G_\emptyset = (X(0), X(1), \Pi_1)$ is connected and $\lambda_2(W_v) \leq \gamma$ for all $v \in X(0)$. Then*

$$\lambda_2(W_\emptyset) \leq \frac{\gamma}{1-\gamma}.$$

Note that the condition that the graph $G_\emptyset$ is connected is necessary, as the example of two disjoint cliques shows. Applying Theorem 19.15 inductively would give us the following bound.

Exercise 19.16 (Oppenheim's Bound [Opp18]). *Let (X, Π) be a pure d-dimensional weighted simplicial complex where Π satisfies Equation 19.1 and Equation 19.2. If G_α is connected for every $\alpha \in X(k)$ for every $k \leq d-2$, then for any $-1 \leq j \leq d-2$,*

$$\gamma_j \leq \frac{\gamma_{d-2}}{1-(d-2-j)\gamma_{d-2}}.$$

In general, when $\gamma_{d-2} > 0$, the bound deteriorates as we go to lower links. But if we could prove that $\gamma_{d-2} \leq 0$, then Oppenheim's bound in Exercise 19.16 would allow us to conclude that the simplicial complex is a 0-local-spectral expander, which is almost as strong as the complete complex in Example 19.14. An important example of 0-local-spectral expander is the matroid complex in Example 19.4.

Matroid Complex

The following result is proved in [ALOV19], as an important step in proving the matroid expansion conjecture that we will explain in the next chapter.

Theorem 19.17 (Matroid Complex is 0-Local-Spectral Expander [ALOV19]). *The simplicial complex of any matroid in Example 19.4 with the uniform distribution on the maximal faces is a 0-local-spectral expander.*

Proof. Let X be a pure d-dimensional simplicial complex from a matroid M. By Oppenheim's bound in Exercise 19.16, we just need to prove that the graph of every link is connected and the second largest eigenvalue of the random walk matrix of the links of diemnsion $d-2$ is at most 0.

The first claim that the graph of every link is connected follows from the second axiom of matroids stated in Example 19.4, and is left as a simple exercise.

For the second claim, we first consider the adjacency matrix A_α of a face α of dimension $d-2$. Since the probability distribution on the maximal faces is the uniform distribution, every non-zero entry of the adjacency matrix has the same weight. For bounding the spectrum, without loss of generality, we rescale the matrix such that $A_\alpha(i,j) = 1$ if $\alpha \cup \{i,j\}$ is a maximal face and $A_\alpha(i,j) = 0$ otherwise. We would like to argue that A_α has at most one positive eigenvalue, and this would imply that the normalized adjacency matrix $\mathcal{A}_\alpha$ has at most one positive eigenvalue by the Courant-Fischer Theorem 2.12, and this would imply that the random walk matrix W_α has at most one positive eigenvalue as W_α and $\mathcal{A}_\alpha$ are similar matrices.

To argue that A has at most one positive eigenvalue, let us start with the spanning tree complex in Example 19.3. In the spanning tree complex $X = (E, \mathcal{F})$ of a graph $G = ([n], E)$, the maximal faces are of size $n-1$ and thus of dimension $d := n-2$. Given a face $F \subseteq E$ of dimension $d-2$, with $|F| = n-3$, the subgraph formed by the edges in F has exactly three components left. Note that the edges remained in the link X_F are the edges with endpoints in different components. Two edges e, f in X_F form a face of size 2 if and only if $F \cup \{e, f\}$is a spanning tree if and only if e and f are not parallel edges if we contract the three components into single vertices. In other words, the edges in X_F can be partitioned into three equivalent classes E_1, E_2, E_3 such that two edges e, f form a face of size 2 in X_F if and only if they do not belong to the same subset. So, the adjacency matrix A_F can be written as $J - \chi_{E_1}\chi_{E_1}^T - \chi_{E_2}\chi_{E_2}^T - \chi_{E_3}\chi_{E_3}^T$, where J is the all-one matrix and

χ_{E_i} is the characteristic vector of E_i for $1 \leq i \leq 3$. Therefore, A_F is a rank-one matrix minus three positive semidefinite matrices. It follows from Courant-Fischer Theorem 2.12 or Cauchy interlacing Theorem 2.13 that A_F has at most one positive eigenvalue, and this concludes the proof for spanning tree complexes.

The same proof works for linear matroids, where two columns i, j form a face of size 2 if and only if they are parallel in the linear algebraic sense, and so again the columns can be partitioned into equivalence classes $E_1, E_2, \ldots, E_l$ (with l not necessarily equal to 3) so that $A = J - \sum_{i=1}^{l} \chi_{E_i} \chi_{E_i}^T$. In general, this holds for arbitrary matroids and is known as the matroid partition property and so the same proof works. □

The proof can be generalized so that the probability distribution on the maximal faces are product distributions.

Exercise 19.18 (Product Distributions). *Let $X = (E, \mathfrak{I})$ be a matroid complex. Suppose each element $e \in E$ has a weight w_e. Consider the probability distribution Π where each maximal face F has probability $\Pi(F)$ proportional to $\prod_{e \in F} w_e$. Prove that (X, Π) is still a 0-local-spectral expander.*

One may wonder what are other 0-local-spectral expanders. The following problem shows that they have very restrictive structures such that the graphs of the top links must be complete multipartite graphs.

Problem 19.19 (Complete Multi-Partite Graphs). *The adjacency matrix of a graph G has at most one positive eigenvalue if and only if G is a complete multi-partite graph.*

19.4 Garland's Method

The main goal in this section is to prove Oppenheim's Theorem 19.15. We will first prepare by introducing different inner products for the calculations in the proof. Then we will introduce the Garland's method which decomposes a structure of a simplicial complex to the corresponding structure of its links. Then we will present the proof of Oppenheim's theorem.

Inner Products and Rayleigh Quotients

Recall from Chapter 6 that the random walk matrix of a graph and the normalized adjacency matrix of a graph are similar matrices, and so the eigenvalues are real, but the eigenvectors may not be orthonormal using the standard inner product. It will be convenient to work with a different inner product so that the eigenvectors are orthonormal with respect to this inner product. Given a random walk matrix $W = D^{-1}A$, we have shown in Lemma 6.18 that the eigenvectors $u_1, \ldots, u_n \in \mathbb{R}^n$ satisfies $\langle u_i, u_j \rangle_D := \sum_{l=1}^{n} D(l,l) \cdot u_i(l) \cdot u_j(l) = 0$. For simplicial complexes, we will use the probability distribution Π_0 to define the inner product, which is equivalent to the degree distribution as shown in Definition 19.12.

Definition 19.20 (Inner Products using Π). *Given a simplicial complex (X, Π), for two functions $f, g : X(0) \to \mathbb{R}$, define*

$$\langle f, g \rangle_{\Pi_0} := \mathbb{E}_{i \sim X(0)} [f(i) g(i)] = \sum_{i \in X(0)} \Pi_0(i) f(i) g(i).$$

Similarly, given a link X_α and two functions $f, g : X_\alpha(0) \to \mathbb{R}$, define $\langle f, g\rangle_{\Pi_0^\alpha} := \mathbb{E}_{i \sim X_\alpha(0)}[f(i)g(i)]$. Note that W_α is self-adjoint with respect to this inner product, as

$$\big\langle f, W_\alpha g\big\rangle_{\Pi_0^\alpha} = \big\langle f, D_\alpha^{-1} A_\alpha g\big\rangle_{\Pi_0^\alpha} = \frac{1}{2}\big\langle f, A_\alpha g\big\rangle = \frac{1}{2}\big\langle A_\alpha f, g\big\rangle = \big\langle W_\alpha f, g\big\rangle_{\Pi_0^\alpha}. \tag{19.3}$$

Check that this implies that all eigenvalues of W_α are real with corresponding eigenvectors orthonormal with respect to this inner product.

We also define Rayleigh quotients using the inner product in Definition 19.20.

Definition 19.21 (Rayleigh Quotients using Π). *Given a simplicial complex (X, Π) and a link (X_α, Π^α), for a function $g : X_\alpha(0) \to \mathbb{R}$, the Rayleigh quotient of g is defined as*

$$\frac{\langle g, W_\alpha g\rangle_{\Pi_0^\alpha}}{\langle g, g\rangle_{\Pi_0^\alpha}}.$$

Check that there is a one-to-one correspondence between the Rayleigh quotients of W_α and the Rayleigh quotients of $\mathcal{A}_\alpha$ defined as $f^T \mathcal{A}_\alpha f / f^T f$, and in particular the second largest eigenvalue of W_α can be characterized as

$$\lambda_2(W_\alpha) = \max_{g : \langle g, \vec{1}\rangle_{\Pi_0^\alpha} = 0} \frac{\langle g, W_\alpha g\rangle_{\Pi_0^\alpha}}{\langle g, g\rangle_{\Pi_0^\alpha}}. \tag{19.4}$$

The advantage of working with W_α (instead of A_α or $\mathcal{A}_\alpha$) is that we know that the vector $\vec{1}/\|1\|_{\Pi_0^\alpha}$ is an eigenvector of W_α with eigenvalue 1 for every link α. Let $u_1, \ldots, u_n$ be the eigenvectors of W_α that are Π_0^α-orthonormal. Given any $y \in \mathbb{R}^n$, note that we can write $y = c_1 u_1 + \ldots + c_n u_n$ with $c_i = \langle y, u_i\rangle_{\Pi_0^\alpha}$, and in particular

$$c_1 = \langle y, u_1\rangle_{\Pi_0^\alpha} = \frac{\langle y, \vec{1}\rangle_{\Pi_0^\alpha}}{\|\vec{1}\|_{\Pi_0^\alpha}}. \tag{19.5}$$

Garland's Method

Our plan is to bound the second largest eigenvalue of W using the Rayleigh quotient formulation in Equation 19.4. Garland's method is a well-known technique in high dimensional expanders that decompose a term (Rayleigh quotient in this case) into the corresponding terms over the links, so that we can apply the properties (second largest eigenvalue in this case) in the links to bound the terms over the links in order to bound the original term. We first define a notation for the localization of a function into a link.

Definition 19.22 (Localization to Link). *Given a simplicial complex (X, Π), a function $f : X(k) \to \mathbb{R}$ and a face τ, the localization of f to $X_\tau(k)$ is defined as $f_\tau : X_\tau(k) \to \mathbb{R}$ such that $f_\tau(\sigma) = f(\sigma)$ for all $\sigma \in X_\tau(k)$.*

The following two lemmas show how to decompose the denominator and the numerator of Equation 19.4 respectively.

Lemma 19.23 (Decomposition of Denominator). *Given a simplicial complex (X, Π), for any two functions $f, g : X(0) \to \mathbb{R}$,*

$$\big\langle f, g\big\rangle_{\Pi_0} = \mathbb{E}_{v \sim \Pi_0}\Big[\big\langle f_v, g_v\big\rangle_{\Pi_0^v}\Big],$$

where f_v is the localization of f to $X_v(0)$ as defined in Definition 19.22.

Proof. The proof is by showing that the distribution Π_0 can be written as $\mathbb{E}_{v\sim\Pi_0}[\Pi_0^v]$ by using conditional probability. Note that $\langle f,g\rangle_{\Pi_0} = \sum_{w\in X(0)} \Pi_0(w)f(w)g(w)$ and

$$\mathbb{E}_{v\sim\Pi_0}\left[\langle f_v,g_v\rangle_{\Pi_0^v}\right] = \sum_{v\in X(0)} \Pi_0(v) \sum_{w\in X_v(0)} \Pi_0^v(w) f_v(w) g_v(w) = \sum_{w\in X(0)} \left(\sum_{v\in X(0)} \Pi_0(v)\tilde{\Pi}_0^v(w)\right) f(w)g(w)$$

where $\tilde{\Pi}_0^v$ is just an extension of Π_0^v (with zero entries) so that it has the same dimension as Π_0. We prove the statement by showing that $\Pi_0(w) = \sum_{v\in X(0)} \Pi_0(v)\cdot\tilde{\Pi}_0^v(w)$ as

$$\Pi_0(w) = \frac{1}{2}\sum_{v:\{w,v\}\in X(1)} \Pi_1(\{w,v\}) = \sum_{v:\{w,v\}\in X(1)} \Pi_0(v)\cdot\Pi_0^v(w) = \sum_{v\in X(0)} \Pi_0(v)\cdot\tilde{\Pi}_0^v(w).$$

where the first equality is by Equation 19.1 and the second equality is by Equation 19.2.

An alternative succinct way [Ove20] to write the above proof is

$$\mathbb{E}_{w\sim\Pi_0}[f(w)g(w)] = \mathbb{E}_{vw\in\Pi_1}\mathbb{E}_{w|\{v,w\}}[f(w)g(w)] = \mathbb{E}_{v\in\Pi_0}\mathbb{E}_{\{v,w\}|v}[f(w)g(w)] = \mathbb{E}_{v\sim\Pi_0}[\langle f_v,g_v\rangle_{\Pi_0^v}],$$

where in the first equality to sample a random vertex w we choose a random pair $\{v,w\}$ and then drop a random vertex with probability 1/2, in the second equality we use an equivalent process of first choosing a random vertex v then choose a random edge $\{v,w\}$ incident on it and then choose the other vertex w. □

Lemma 19.24 (Decomposition of Numerator)**.** *Given a simplicial complex (X,Π), for two functions $f,g : X(0)\to\mathbb{R}$,*

$$\langle f,Wg\rangle_{\Pi_0} = \mathbb{E}_{v\sim\Pi_0}\left[\langle f_v,W_vg_v\rangle_{\Pi_0^v}\right],$$

where W and W_v are the random walk matrices of the empty link $X_\emptyset$ and the link X_v respectively.

Proof. The proof is by showing that the adjacency matrix can be written as the expected matrix of the adjacency matrices of the links. We use Equation 19.3 to write the terms using the adjacency matrices so that $\langle f,Wg\rangle_{\Pi_0} = \frac{1}{2}\langle f,Ag\rangle$ and

$$\mathbb{E}_{v\sim\Pi_0}\left[\langle f_v,W_vg_v\rangle_{\Pi_0^v}\right] = \mathbb{E}_{v\sim\Pi_0}\left[\frac{1}{2}\langle f_v,A_vg_v\rangle\right] = \mathbb{E}_{v\sim\Pi_0}\left[\frac{1}{2}\langle f,\tilde{A}_vg\rangle\right] = \frac{1}{2}\left\langle f,(\mathbb{E}_{v\in\Pi_0}\tilde{A}_v)g\right\rangle$$

where $\tilde{A}_v$ is just the extended matrix of A_v (with zero rows and columns) so that it has the same dimension as A. We prove the statement by showing that $A = \mathbb{E}_{v\in\Pi_0}\tilde{A}_v$. Using conditional probability, for each entry (u,w),

$$\begin{aligned} A_{u,w} &= \Pi_1(\{u,w\}) = \frac{1}{3}\sum_{v:\{u,v,w\}\in X(2)} \Pi_2(\{u,v,w\}) = \sum_{v:\{u,v,w\}\in X(2)} \Pi_0(v)\cdot\Pi_1^v(\{u,w\}) \\ &= \sum_{v\in X(0)} \Pi_0(v)\cdot\Pi_1^v(\{u,w\}) = \sum_{v\in X(0)} \Pi_0(v)\cdot(\tilde{A}_v)_{u,w}, \end{aligned}$$

where the first line is by Equation 19.1 and Equation 19.2.

An alternative succinct way [Ove20] to write the above proof is to rewrite the first step as $\langle f,Wg\rangle_{\Pi_0} = \mathbb{E}_{\{v,w\}\sim\Pi_1}[f(v)g(w)]$ (exercise) and the remaining steps as

$$\begin{aligned} \mathbb{E}_{\{u,w\}\sim\Pi_1}[f(u)g(w)] &= \mathbb{E}_{\{u,v,w\}\sim\Pi_2}\mathbb{E}_{\{u,w\}|\{u,v,w\}}[f(u)g(w)] = \mathbb{E}_{v\sim\Pi_0}\mathbb{E}_{\{u,v,w\}|v}[f(u)g(w)] \\ &= \mathbb{E}_{v\sim\Pi_0}\mathbb{E}_{\{u,w\}\sim\Pi_1^v}[f(u)g(w)] = \mathbb{E}_{v\sim\Pi_0}[\langle f_v,W_vg_v\rangle_{\Pi_0^v}]. \end{aligned}$$

□

Proof of Oppenheim's Theorem

We are ready to prove Oppenheim Theorem 19.15. Let G be the graph of X and W be its random walk matrix. Since G is connected by assumption, the second largest eigenvalue of W is less than one. Let $\lambda < 1$ be the second largest eigenvalue of W and f be a corresponding eigenvector achieving the maximum of the Rayleigh quotient in Equation 19.4 with $\langle f, \vec{1}\rangle_{\Pi_0} = 0$. We assume without loss of generality that $\langle f, f\rangle_{\Pi_0} = 1$. Therefore, by Lemma 19.24,

$$\lambda = \langle f, Wf\rangle_{\Pi_0} = \mathbb{E}_{v\in\Pi_0}\Big[\big\langle f_v, W_v f_v\big\rangle_{\Pi_0^v}\Big]$$

As each W_v is a random walk matrix, the largest eigenvalue of each W_v is one with the corresponding eigenvector being $\vec{1}_v/\|\vec{1}_v\|_{\Pi_0^v} = \vec{1}_v$, where $\vec{1}_v$ is the localization of $\vec{1}$ into $X_v(0)$ as described in Definition 19.22. To bound $\langle f_v, W_v f_v\rangle_{\Pi_0^v}$, we use Equation 19.5 to decompose the vector f_v as

$$f_v = \langle f_v, \vec{1}_v\rangle_{\Pi_0^v} \cdot \vec{1}_v + f_v^{\perp} \quad \text{where} \quad \langle \vec{1}_v, f_v^{\perp}\rangle_{\Pi_0^v} = 0.$$

Expanding the quadratic form using $W_v\vec{1}_v = \vec{1}_v$, $\langle \vec{1}_v, f_v^{\perp}\rangle_{\Pi_0^v} = 0$ and the self-adjoint property in Equation 19.3, the cross terms are zero and we get

$$\langle f_v, W_v f_v\rangle_{\Pi_0^v} = \langle f_v, \vec{1}_v\rangle^2_{\Pi_0^v} + \langle f_v^{\perp}, W_v f_v^{\perp}\rangle_{\Pi_0^v} \leq \langle f_v, \vec{1}_v\rangle^2_{\Pi_0^v} + \gamma\langle f_v^{\perp}, f_v^{\perp}\rangle_{\Pi_0^v} = (1-\gamma)\langle f_v, \vec{1}_v\rangle^2_{\Pi_0^v} + \gamma\langle f_v, f_v\rangle_{\Pi_0^v}$$

where the inequality is by the characterization of the second largest eigenvalue in Equation 19.4 and the assumption that each link has second largest eigenvalue at most γ, and the last equality is by $\langle f_v, f_v\rangle_{\Pi_0^v} = \langle f_v, \vec{1}_v\rangle^2_{\Pi_0^v} + \langle f_v^{\perp}, f_v^{\perp}\rangle_{\Pi_0^v}$ by the same orthonormality argument. Therefore, using the decomposition of the denominator in Lemma 19.23 and $\langle f, f\rangle_{\Pi_0} = 1$,

$$\mathbb{E}_{v\in\Pi_0}\Big[\big\langle f_v, W_v f_v\big\rangle_{\Pi_0^v}\Big] \leq \mathbb{E}_{v\in\Pi_0}\Big[(1-\gamma)\langle f_v, \vec{1}_v\rangle^2_{\Pi_0^v} + \gamma\langle f_v, f_v\rangle_{\Pi_0^v}\Big] = \gamma + (1-\gamma)\cdot\mathbb{E}_{v\in\Pi_0}\Big[\langle f_v, \vec{1}_v\rangle^2_{\Pi_0^v}\Big].$$

Note that, by Equation 19.2 and Definition 19.12,

$$\langle f_v, \vec{1}_v\rangle_{\Pi_0^v} = \sum_{w\in X_v(0)} \Pi_0^v(w) f_v(w) = \sum_{w\in X_v(0)} \frac{\Pi_1(\{v,w\})}{2\Pi_0(v)}\cdot f_v(w) = \sum_{w\in X_v(0)} W(v,w)\cdot f_v(w) = (Wf)(v).$$

Hence, as f is an eigenvector of W with eigenvalue λ and $\langle f, f\rangle_{\Pi_0} = 1$,

$$\mathbb{E}_{v\in\Pi_0}\Big[\langle f_v, \vec{1}_v\rangle^2_{\Pi_0^v}\Big] = \mathbb{E}_{v\in\Pi_0}\big[(Wf)(v)^2\big] = \big\langle Wf, Wf\big\rangle_{\Pi_0} = \lambda^2\cdot\langle f, f\rangle_{\Pi_0} = \lambda^2.$$

To summarize,

$$\lambda = \mathbb{E}_{v\in\Pi_0}\Big[\big\langle f_v, W_v f_v\big\rangle_{\Pi_0^v}\Big] \leq \gamma + (1-\gamma)\cdot\mathbb{E}_{v\in\Pi_0}\Big[\langle f_v, \vec{1}_v\rangle^2_{\Pi_0^v}\Big] = \gamma + (1-\gamma)\lambda^2$$

Solving this quadratic inequality gives either $\lambda \geq 1$ or $\lambda \leq \gamma/(1-\gamma)$. Since G is connected and $\lambda < 1$, we conclude that $\lambda \leq \gamma/(1-\gamma)$ as stated in Theorem 19.15.

19.5 Problems

The following are two interesting problems.

Problem 19.25 (Spanning Tree Complex without Oppenheim)**.** *Use the results in Chapter 16 to prove directly that the spanning tree complex is a* 0*-local-spectral expander, without using Oppenheim's trickling down theorem.*

Problem 19.26 (Approximate Negative Correlation of Matroids)**.** *In Chapter 16, we have seen that the variables in a random spanning tree are negatively correlated, such that for any two edges* $e \neq f$*,*

$$\Pr_T[e \in T \mid f \in T] \leq \Pr_T[e \in T].$$

This is known to be not necessarily true for general matroids, but not all is lost. Use the result that any matroid complex is a 0*-local-spectral expander in Theorem 19.17 to prove that for any two elements* $i \neq j$ *in a matroid,*

$$\Pr_B[i \in B \mid j \in B] \leq 2 \Pr_B[i \in B],$$

where B *is a uniform random basis of the matroid.*

Question 19.27. *It is an open question what is the best constant that one could prove for the approximate negative correlation property of matroids in Problem 19.26. There are examples showing that the constant is at least* $^8/_7$*, and some conjectured that this is tight.*

19.6 References

[ALOV19] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials II: high-dimensional walks and an FPRAS for counting bases of a matroid. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, Phoenix, AZ, USA, June 23-26, 2019*, pages 1–12. ACM, 2019. 184

[DDFH18] Yotam Dikstein, Irit Dinur, Yuval Filmus, and Prahladh Harsha. Boolean function analysis on high-dimensional expanders. *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, 2018. 180

[DK17] Irit Dinur and Tali Kaufman. High dimensional expanders imply agreement expanders. In *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS*, pages 974–985. IEEE Computer Society, 2017. 179, 183

[HL21] Max Hopkins and Shachar Lovett. Course notes of "Expander graphs and high-dimensional expanders", lecture 6. 2021. https://cseweb.ucsd.edu/classes/sp21/cse291-g/CSE_291_Expanders_and_HDX_Course_Notes.pdf. 179

[KM17] Tali Kaufman and David Mass. High dimensional random walks and colorful expansion. In *8th Innovations in Theoretical Computer Science Conference, ITCS*, volume 67, pages 4:1–4:27, 2017. 179, 183

[KO20] Tali Kaufman and Izhar Oppenheim. High order random walks: Beyond spectral gap. *Comb.*, 40(2):245–281, 2020. 179, 183

[Lub18] Alexander Lubotzky. High dimensional expanders. In *Proceedings of the International Congress of Mathematicians (ICM 2018)*, pages 705–730, 2018. 179, 183

[Opp18] Izhar Oppenheim. Local spectral expansion approach to high dimensional expanders part I: descent of spectral gaps. *Discrete & Computational Geometry*, 59(2):293–330, 2018. 183, 184

[Ove20] Shayan Oveis Gharan. Course notes of "Polynomial paradigm in algorithm design", lecture 12. 2020. https://homes.cs.washington.edu/~shayan/courses/polynomials/. 128, 130, 153, 158, 171, 173, 175, 179, 187