Chapter 15

# Multivariate Barrier Method

Marcus, Spielman, and Srivastava [MSS15] proved Weaver's conjecture using the method of interlacing family of polynomials. A key component in their proof is a multivariate barrier method to bound the maximum root of the expected characteristic polynomial, which is an extension of the barrier method by Batson, Spielman and Srivastava for spectral sparsification in Chapter 10.

## 15.1 Weaver's Conjecture

It was observed that the linear-sized spectral sparsification result by Batson, Spielman, and Srivastava in Theorem 10.1 looks similar to the conjecture by Weaver, which is known to be equivalent to the Kadison-Singer problem (see [MSS15, MSS14]), whose positive resolution would have implications in several areas of mathematics.

**Conjecture 15.1** (Weaver's Conjecture)**.** *There exist positive constants $\alpha$ and $\epsilon$ so that the following holds. For every $m, n \in \mathbb{N}$ and every set of vectors $v_1, \ldots, v_m \in \mathbb{R}^n$ such that*

$$\sum_{i=1}^{m} v_i v_i^T = I_n \quad \textit{and} \quad \|v_i\|_2^2 \leq \alpha \ \textit{ for } \ 1 \leq i \leq n,$$

*there exists a partition of $\{1, \ldots, m\}$ into two sets $S_1$ and $S_2$ so that*

$$\lambda_{\max}\bigg(\sum_{i \in S_j} v_i v_i^T\bigg) \leq 1 - \epsilon \ \textit{ for } \ 1 \leq j \leq 2.$$

Note that since $\sum_{i \in S_1} v_i v_i^T + \sum_{i \in S_2} v_i v_i^T = I_n$, the conclusion in Weaver's conjecture is equivalent to

$$\epsilon I_n \preccurlyeq \sum_{i \in S_1} v_i v_i^T \preccurlyeq (1-\epsilon) I_n \quad \text{and} \quad \epsilon I_n \preccurlyeq \sum_{i \in S_2} v_i v_i^T \preccurlyeq (1-\epsilon) I_n,$$

and so the vectors in $S_1$ and in $S_2$ form spectral approximators of the identity matrix. In Theorem 10.1 by Batson, Spielman, and Srivastava, the task was to find scalars $s_1, \ldots, s_m$ with few nonzeros so that

$$(1-\epsilon) I_n \preccurlyeq \sum_{i=1}^{m} s_i v_i v_i^T \preccurlyeq (1+\epsilon) I_n \quad \iff \quad \frac{1}{2}(1-\epsilon) I_n \preccurlyeq \sum_{i=1}^{m} \frac{s_i}{2} v_i v_i^T \preccurlyeq \frac{1}{2}(1+\epsilon) I_n$$

So, if all the scalars $s_i/2$ are either zero or one, then Theorem 10.1 would have given a positive resolution to Weaver's conjecture. This is not always possible, however, since if there is a long vector (say $\|v_i\| = 1$), then setting $s_i/2$ to be zero or one would violate the minimum eigenvalue and the maximum eigenvalue bounds. This is why there is an additional condition $\|v_i\|_2^2 \leq \alpha$ in Weaver's conjecture, which says that long vectors are the only obstructions to finding such a partitioning.

### Graph Sparsification

In terms of graph sparsification, the question in Weaver's conjecture corresponds to finding an unweighted sparsifier. Recall the reduction in Lemma 9.11 and the discussions in Subsection 9.2, the length $\|v_i\|_2^2$ is equal to the effective resistance of the $i$-th edge in the graph. So, Weaver's conjecture in the graph setting states that if the maximum effective resistance of an edge is small enough, then there is a partitioning of the edges into two groups so that the subgraph formed by each group is a spectral approximator of the original graph. Some examples of graphs with small maximum effective resistance are expander graphs and edge-transitive graphs (such as hypercubes and Cayley graphs).

One could apply the matrix Chernoff bound in Theorem 9.13 to this problem, and it works for $\alpha \lesssim 1/\log^2 n$ with high probability, but this is not enough for Weaver's conjecture. The approach by Batson, Spielman, and Srivastava for spectral sparsification heavily depends on a careful choice of scalars and does not seem applicable for constructing unweighted sparsifiers. See [BST19] for a recent paper on constructing unweighted sparsifiers.

## 15.2 Probabilistic Formulation

Marcus, Spielman, and Srivastava formulated and proved a probabilistic statement that implies Weaver's conjecture.

**Theorem 15.2** (Marcus-Spielman-Srivastava [MSS15]). *Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be independent random vectors with finite support such that*

$$\mathbb{E}\left[\sum_{i=1}^{m} v_i v_i^T\right] = I_n \quad \textit{and} \quad \mathbb{E}\big[\|v_i\|_2^2\big] \leq \epsilon \textit{ for } 1 \leq i \leq m.$$

*Then*

$$\Pr\left[\lambda_{\max}\left(\sum_{i=1}^{m} v_i v_i^T\right) \leq \left(1+\sqrt{\epsilon}\right)^2\right] > 0.$$

### Reduction

Weaver's conjecture is about partitioning and Theorem 15.2 is about sum of random variables, but there is a simple reduction from the former to the latter.

For each vector $u_i \in \mathbb{R}^n$ in Weaver's problem, define a random vector $v_i \in \mathbb{R}^{2n}$ with two choices:

$$v_i = \sqrt{2}\begin{pmatrix} u_i \\ 0 \end{pmatrix} \text{ with probability } 1/2 \quad \text{and} \quad v_i = \sqrt{2}\begin{pmatrix} 0 \\ u_i \end{pmatrix} \text{ with probability } 1/2.$$

The first choice corresponds to putting $u_i$ in the first group, and the second choice corresponds to putting $u_i$ in the second group. Then, by the assumption that $\sum_{i=1}^m u_i u_i^T = I_n$,

$$\mathbb{E}\left[v_i v_i^T\right] = \begin{pmatrix} u_i u_i^T & 0 \\ 0 & u_i u_i^T \end{pmatrix} \quad \Longrightarrow \quad \sum_{i=1}^m \mathbb{E}\left[v_i v_i\right]^T = \sum_{i=1}^m \begin{pmatrix} u_i u_i^T & 0 \\ 0 & u_i u_i^T \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} = I_{2n}.$$

Similarly, by the assumption that $\|u_i\|_2^2 \leq \alpha$,

$$\mathbb{E}\left[\|v_i\|_2^2\right] = \mathbb{E}\left[v_i^T v_i\right] = 2\|u_i\|_2^2 \leq 2\alpha.$$

By Theorem 15.2, there exists a choice of $v_1, \ldots, v_m$ such that $\lambda_{\max}(\sum_{i=1}^m v_i v_i^T) \leq (1+\sqrt{2\alpha})^2$. As intended, we put vector $u_i$ into $S_1$ if we select the first choice for $v_i$, and put $u_i$ into $S_2$ otherwise. Then the conclusion from Theorem 15.2 implies that

$$\lambda_{\max}\left(\begin{pmatrix} 2\sum_{i\in S_1} u_i u_i^T & 0 \\ 0 & 2\sum_{i\in S_2} u_i u_i^T \end{pmatrix}\right) \leq (1+\sqrt{2\alpha})^2 \quad \Longrightarrow \quad \lambda_{\max}\left(\sum_{i\in S_j} v_i v_i^T\right) \leq \frac{1}{2}(1+\sqrt{2\alpha})^2$$

for $1 \leq j \leq 2$. So, when $\alpha$ is small enough (say $\alpha \leq \frac{1}{32}$), then $\frac{1}{2}(1+\sqrt{2\alpha})^2 < 1$ and thus Weaver's conjecture follows. We record the following corollary for future references.

**Corollary 15.3** (Solution to Weaver's Conjecture)**.** *Under the same setting in Conjecture 15.1, there exists a partition of $[m]$ into two sets $S_1$ and $S_2$ such that for $1 \leq j \leq 2$,*

$$\left(\frac{1}{2} - \sqrt{2\alpha} - 2\alpha\right) \cdot I_n \preccurlyeq \sum_{i\in S_j} v_i v_i^T \preccurlyeq \frac{1}{2}(1+\sqrt{2\alpha})^2 \cdot I_n.$$

Corollary 15.3 is quantitatively stronger than Weaver's formulation, as when $\alpha$ is small enough, we can bound how far is the solution from the ideal partitioning $\frac{1}{2}I$, which will be useful in applications.

## Proof Overview

The plan of the proof is to show that there exists a choice of the random variables $v_1, \ldots, v_m$ such that

$$\lambda_{\max}\left(\det\left(xI_n - \sum_{i=1}^m v_i v_i^T\right)\right) \leq \lambda_{\max}\left(\mathbb{E}_{v_1,\ldots,v_m}\left[\det\left(xI_n - \sum_{i=1}^m v_i v_i^T\right)\right]\right) \leq \left(1+\sqrt{\epsilon}\right)^2. \quad (15.1)$$

We have established in Theorem 13.28 that the set of all possible characteristic polynomials $\det(xI_n - \sum_{i=1}^m v_i v_i^T)$ forms an interlacing family, and the root polynomial can be set to be $\mathbb{E}_{v_1,\ldots,v_m}\left[\det\left(xI_n - \sum_{i=1}^m v_i v_i^T\right)\right]$ where the expectation is taken over the independent uniform random distributions on $v_1, \ldots, v_m$. Therefore, by the new probabilistic method for interlacing family in Theorem 12.12, we have already proved the first inequality, using the techniques from real stable polynomials described in Chapter 13.

The main goal of this chapter is to prove the second inequality in Equation 15.1, given the assumptions that $\mathbb{E}\left[\|v_i\|_2^2\right] \leq \epsilon$ for $1 \leq i \leq m$ and $\mathbb{E}\left[\sum_{i=1}^m v_i v_i^T\right] = I_n$. In Chapter 14, when we construct bipartite Ramanujan graphs, the expected characteristic polynomial turns out to be exactly the matching polynomial and there were known results bounding the maximum root. For Weaver's conjecture, bounding the maximum root of the expected polynomial is a major technical challenge (that took Marcus, Spielman, and Srivastava four years to solve).

Recall the multilinear formula in Theorem 13.20 that

$$\mathbb{E}_{v_1,\ldots,v_m}\left[\det\left(\lambda I - \sum_{i=1}^m v_i v_i^T\right)\right] = \prod_{i=1}^m \left(1-\partial_{x_i}\right)\det\left(\lambda I + \sum_{i=1}^m x_i \cdot \mathbb{E}\left[v_i v_i^T\right]\right)\Big|_{x_1=x_2=\cdots=x_m=0}.$$

This formula plays an important role in the first step, by showing that the expected characteristic polynomial is real-rooted to establish common interlacing for the new probabilistic method to work. Perhaps unexpectedly, the formula also plays an important role in the second step. Their idea is to first prove an upper bound of the "maximum root" of the multivariate polynomial $\det\big(\lambda I + \sum_{i=1}^m x_i \cdot \mathbb{E}\left[v_i v_i^T\right]\big)$, and then maintain a good upper bound after each $(1-\partial_{x_i})$ differential operator is applied. To establish the upper bound, they finally realized that the barrier method developed for linear-sized spectral sparsification in Chapter 10 can be extended to the multivariate setting in a syntatically similar way!

## 15.3 Multivariate Approach

To bound the maximum root of the univariate polynomial $\mathbb{E}\left[\det\left(\lambda I - \sum_{i=1}^m v_i v_i^T\right)\right]$, the approach taken is to bound the "maximum root" of the multivariate polynomial $\prod_{i=1}^m (1-\partial_{x_i})\det\big(\lambda I + \sum_{i=1}^m x_i \cdot \mathbb{E}\left[v_i v_i^T\right]\big)$, which will be defined in a moment.

First, we use the assumption and define some notations to slightly simplify the statement. Using the assumption that $\mathbb{E}\left[\sum_{i=1}^m v_i v_i^T\right] = I$, we rewrite the RHS of the multilinear formula as

$$\prod_{i=1}^m (1-\partial_{x_i})\det\left(\sum_{i=1}^m (\lambda+x_i)\cdot\mathbb{E}\left[v_i v_i^T\right]\right)\Big|_{x_1=\ldots=x_m=0} = \prod_{i=1}^m (1-\partial_{x_i})\det\left(\sum_{i=1}^m x_i\cdot\mathbb{E}\left[v_i v_i^T\right]\right)\Big|_{x_1=\ldots=x_m=\lambda}. \tag{15.2}$$

Denote the matrix $\mathbb{E}\left[v_i v_i^T\right] = B_i$ and note that $B_i \succcurlyeq 0$ for $1 \le i \le m$. Denote the polynomial after applying the differential operator $k$ times by

$$p_k(x_1,\ldots,x_m) := \prod_{i=1}^k (1-\partial_{x_i})\det\left(\sum_{i=1}^m x_i B_i\right),$$

so that $p_0(x_1,\ldots,x_m) = \det(\sum_{i=1}^m x_i B_i)$ and $p_m(x_1,\ldots,x_m) = \prod_{i=1}^m (1-\partial_{x_i})\det\big(\sum_{i=1}^m x_i B_i\big)$.

**Definition 15.4** (Above the Roots)**.** *Given a multivariate polynomial $p(x_1,\ldots,x_m)$, we say a point $y \in \mathbb{R}^m$ is "above the roots" of $p$ if $p(y+t) > 0$ for all $t = (t_1,\ldots,t_m) \in \mathbb{R}^m_{\ge 0}$.*

Our goal is to prove that the point $(1+\sqrt{\epsilon})^2 \cdot \vec{1}$ is above the roots of the multivariate polynomial $p_m(x_1,\ldots,x_m)$. Note that this implies that the maximum root of the univariate polynomial $p_m(\lambda,\ldots,\lambda) = \prod_{i=1}^m (1-\partial_{x_i})\det\big(\sum_{i=1}^m x_i B_i\big)|_{x_1=\ldots=x_m=\lambda}$ is at most $(1+\sqrt{\epsilon})^2$, and thus by the multilinear formula the maximum root of the univariate polynomial $\mathbb{E}_{v_1,\ldots,v_m}\big[\det\big(\lambda I - \sum_{i=1}^m v_i v_i^T\big)\big]$ is at most $(1+\sqrt{\epsilon})^2$.

Initially, since $\sum_{i=1}^m B_i = I_n$ by assumption, it follows that $p_0(t,t,\ldots,t) = \det(tI) > 0$ for any $t > 0$, and so the point $t\cdot\vec{1}$ is above the roots of $p_0$ for any $t > 0$. The strategy in [MSS15] is to prove inductively that $(\underbrace{t+\delta,\ldots,t+\delta}_{k \text{ coordinates}}, t\ldots,t)$ is above the roots of $p_k$ for some $\delta$ for all $1 \le k \le m$.

### Multivariate Barrier Functions

To execute the above inductive proof strategy, a similar approach as in Chapter 10 for spectral sparsification is used, to establish a "soft/comfortable" upper bound for the induction to go through.

Recall that in Definition 10.3, the potential function $\Phi^u(A) = \text{Tr}(uI - A)^{-1}$ is defined, and the invariant $\Phi^u(A) \leq \phi$ is maintained to guarantee that $u$ is well above the roots. Also recall from Remark 10.6 and Remark 11.6 that $\Phi^u(A) = p'_A(u)/p_A(u)$ where $p_A(x) = \det(xI - A)$ is the characteristic polynomial of $A$, and so the potential function has a natural interpretation in terms of polynomials. This univariate barrier function is generalized to the multivariate setting as follows.

**Definition 15.5** (Multivariate Barrier Functions)**.** *Given a real-stable polynomial $p \in \mathbb{R}[x_1, \ldots, x_m]$ and a point $y \in \mathbb{R}^m$ above the roots of $p$, for $1 \leq i \leq m$, the barrier function of $p$ in direction $i$ at $y$ is defined as*

$$\Phi_p^i(y) := \frac{\partial_{x_i} p(y)}{p(y)}.$$

*Equivalently, we can define*

$$\Phi_p^i(y) = \frac{q'_{y,i}(y_i)}{q_{y,i}(y_i)} = \sum_{j=1}^{d} \frac{1}{y_i - \lambda_j},$$

*where $q_{y,i}(t)$ is the univariate restriction $q_{y,i}(x) = p(y_1, \ldots, y_{i-1}, t, y_{i+1}, \ldots, y_m)$ where $\lambda_1, \ldots, \lambda_d$ are the roots of this univariate polynomial, which is real-rooted as substituting real numbers preserve real-stability by Proposition 13.13.*

For spectral sparsification, we maintain one potential function $\Phi^A(x)$ to show that $u \in \mathbb{R}$ is well above the roots by showing that $\Phi^A(u) \leq \phi$ for some small $\phi \in \mathbb{R}$. For Weaver's problem, we maintain $m$ potential functions $\Phi_p^1(x), \ldots, \Phi_p^m(x)$ to show that $y \in \mathbb{R}^m$ is well above the roots by showing that $\Phi_p^i(y) \leq \phi$ for $1 \leq i \leq m$ for some small $\phi \in \mathbb{R}$.

**Definition 15.6** (Induction Hypothesis)**.** *Let $x_0 = (t, \ldots, t) \in \mathbb{R}^m$ be the initial point above the roots of $p_0$ in Equation 15.2 for some $t > 0$, such that $\Phi_p^i(x_0) \leq \phi$ for some $\phi \in \mathbb{R}$ for $1 \leq i \leq m$. This is the base case. Let $x_k = (t+\delta, \ldots, t+\delta, t, \ldots, t)$ with the first $k$ coordinates being $t+\delta$. The induction hypothesis is to maintain that $\Phi_{p_k}^i(x_k) \leq \phi$ for $1 \leq i \leq m$, for $1 \leq k \leq m$ where $p_k$ is defined in Equation 15.2. The parameters $t, \phi, \delta$ will be chosen at the end.*

## 15.4 Bounding the Maximum Root

In this section, we do the calculations to carry out the induction as described in Definition 15.6.

### Monotonicity and Convexity

The following monotonicity and convexity properties are generalizations of Exercise 10.4 in the univariate case to the multivariate setting.

**Proposition 15.7** (Monotonicity and Convexity)**.** *Suppose $p \in \mathbb{R}[x_1, \ldots, x_m]$ is real-stable and $y$ is above the roots of $p$. Then, for all $i, j \in [m]$ and $\delta \geq 0$, the following two properties hold.*

1. *Monotonicity: $\Phi_p^i(y + \delta \cdot e_j) \leq \Phi_p^i(y)$ where $e_j$ is the $j$-th vector in the standard basis.*

2. *Convexity:* $\Phi_p^i(y + \delta \cdot e_j) - \Phi_p^i(y) \leq \delta \cdot \partial_{x_j} \Phi_p^i(y + \delta \cdot e_j)$.

The proof of the univariate case in Exercise 10.4 is easy, but the proof of the multivariate case in Proposition 15.7 is not. The proof in [MSS15] uses a deep result that any bivariate real-stable polynomial $p(x_1, x_2)$ can be written as $\pm\det(x_1 A + x_2 B + C)$ for some $A, B \succcurlyeq 0$ and some symmetric $C$. Then they do some explicit computations from this representation to prove monotonicity and convexity.

Tao [Tao13] gave a more elementary proof using complex analysis. We gave a proof sketch of Tao's proof in L15 in the previous offering of CS 860. We won't give a proof of this proposition in this offering, and refer the reader to [MSS15, Tao13]. One reason is that the arguments are different and independent from the rest of the proofs and also not self-contained, and another reason is that I don't understand the proofs well enough to provide any further explanations.

## Inductive Proof

As a warm up, we first see that when a point $y$ is well above the roots, then $y$ is still above the roots after the operation $1 - \partial_{x_j}$.

**Lemma 15.8** (Above the Roots after One Operation)**.** *Suppose that $p \in \mathbb{R}[x_1, \ldots, x_m]$ is real stable and $y \in \mathbb{R}^m$ is above the roots of $p$, with the additional property that $\Phi_p^i(y) < 1$ for $1 \leq i \leq m$. Then $y$ is still above the roots of $(1 - \partial_{x_j})p$ for $1 \leq j \leq m$.*

*Proof.* Let $z \in \mathbb{R}^m$ be a point above $y$ such that $z \geq y$. We would like to prove that $(1 - \partial_{x_j})p(z) \neq 0$ for any $1 \leq j \leq m$, and this would imply that $y$ is still above the roots of $(1 - \partial_{x_j})p$. By the monotonicity property in Proposition 15.7, $\Phi_p^j(z) \leq \Phi_p^j(y) < 1$ for $1 \leq j \leq m$. This implies that

$$1 > \Phi_p^j(z) = \frac{\partial_{x_j} p(z)}{p(z)} \quad \Longrightarrow \quad 0 \neq p(z) - \partial_{x_j} p(z) = (1 - \partial_{x_j})p(z).$$

□

The lemma shows that $y$ is still above the roots after one differential operation, but we cannot repeat this argument because the condition $\Phi_{(1-\partial_{x_i})p}^j(y)$ may no longer hold. To maintain the invariant, we will increase the upper bound in the corresponding coordinate to decrease the potential value. Assuming the monotonicity and convexity properties of the multivariate barrier functions in Proposition 15.7, the proof in the following inductive step is actually very similar to the univariate case as presented in Lemma 11.9 (and also in Problem 11.13). Basically, we can do exact calculation to compute the increase of the potential function after the $1 - \partial_{x_i}$ operation, and then use convexity to bound the decrease of the potential value by shifting up the barrier to $y + \delta \cdot e_j$.

**Lemma 15.9** (Maintaining the Potential Values)**.** *Suppose that $p \in \mathbb{R}[x_1, \ldots, x_m]$ is real stable and $y$ is above the roots of $p$. Suppose further that $\Phi_p^i(y) \leq 1 - \frac{1}{\delta}$ for $1 \leq i \leq m$ for some $\delta > 0$. Then, for $1 \leq i, j \leq m$,*

$$\Phi_{(1-\partial_{x_j})p}^i(y + \delta \cdot e_j) \leq \Phi_p^i(y),$$

*and $y + \delta \cdot e_j$ is still above the roots of $(1 - \partial_{x_j})p$.*

*Proof.* By Definition 15.5,

$$\Phi^i_{(1-\partial_{x_j})p} = \frac{\partial_{x_i}((1-\partial_{x_j})p)}{(1-\partial_{x_j})p} = \frac{\partial_{x_i}((1-\Phi^j_p)p)}{(1-\Phi^j_p)p} = \frac{(1-\Phi^j_p)\partial_{x_i}p}{(1-\Phi^j_p)p} + \frac{(\partial_{x_i}(1-\Phi^j_p))p}{(1-\Phi^j_p)p} = \Phi^i_p - \frac{\partial_{x_i}\Phi^j_p}{1-\Phi^j_p}.$$

Therefore,

$$\Phi^i_{(1-\partial_{x_j})p}(y+\delta e_j) = \Phi^i_p(y+\delta e_j) - \frac{\partial_{x_i}\Phi^j_p(y+\delta e_j)}{1-\Phi^j_p(y+\delta e_j)}.$$

To prove $\Phi^i_{(1-\partial_{x_j})p}(y+\delta e_j) \leq \Phi^i_p(y)$, it is equivalent to proving that

$$\Phi^i_p(y) - \Phi^i_p(y+\delta e_j) \geq -\frac{\partial_{x_i}\Phi^j_p(y+\delta e_j)}{1-\Phi^j_p(y+\delta e_j)}. \tag{15.3}$$

By convexity of the multivariate barrier function in Proposition 15.7, we have that

$$\Phi^i_p(y) - \Phi^i_p(y+\delta e_j) \geq -\delta \cdot \partial_{x_j}\Phi^i_p(y+\delta e_j).$$

So, Equation 15.3 holds if we could prove that

$$\delta \cdot \partial_{x_j}\Phi^i_p(y+\delta e_j) \leq \frac{\partial_{x_i}\Phi^j_p(y+\delta e_j)}{1-\Phi^j_p(y+\delta e_j)} \quad \iff \quad \delta \geq \frac{1}{1-\Phi^j_p(y+\delta e_j)}, \tag{15.4}$$

where the equivalence is by noting that $\partial_{x_j}\Phi^i_p = \partial_{x_i}\Phi^j_p$ and so the numerators are the same, and $\partial_{x_j}\Phi^i_p(y+\delta e_j) \leq 0$ as the barrier function is monotonically decreasing above the roots by Proposition 15.7 and so the inequality is reversed when we cancel the numerators. Our assumption implies that

$$\delta \geq \frac{1}{1-\Phi^j_p(y)} \geq \frac{1}{1-\Phi^j_p(y+\delta e_j)}$$

as desired, where the second inequality is again by monotonicity. We conclude that $\Phi^i_{(1-\partial_{x_j})p}(y+\delta \cdot e_j) \leq \Phi^i_p(y)$ for $1 \leq i,j \leq m$, and $y+\delta \cdot e_j$ is still above the roots of $(1-\partial_{x_j})p$ by Lemma 15.8. □

We will choose $\phi = 1 - \frac{1}{\delta}$ in the induction hypothesis in Definition 15.6.

## Choosing the Parameters

By Lemma 15.9, if we choose the initial $x_0 = (t,\dots,t)$ such that $\Phi^i_p(x_0) \leq 1 - \frac{1}{\delta}$ for $1 \leq i \leq m$ for some $\delta > 0$. Then, by induction, $x_m = (t+\delta,\dots,t+\delta)$ is above the roots of $p_m$. This would imply that $t+\delta$ is above the roots of the univariate polynomial $\mathbb{E}\left[\det\left(\lambda I - \sum_{i=1}^m v_i v_i^T\right)\right]$ by the multilinear formula. It remains to optimize $t$ and $\delta$ to prove the best upper bound.

Note that for the induction step to go through, we only used the property that the polynomial is real stable, and the general properties of monotonicity and convexity. We have not used the specific form of $p_k$ in Equation 15.2. Also, we have not used the crucial assumption that $\mathbb{E}\left[\|v_i\|_2^2\right] \leq \epsilon$. These are (only) needed in the following computation of the initial value.

Recall that $p_0(x_1,\dots,x_m) = \det(\sum_{i=1}^m x_i B_i)$ where $B_i = \mathbb{E}\left[v_i v_i^T\right] \succcurlyeq 0$. The assumptions of Theorem 15.2 are that $\sum_{i=1}^m B_i = \sum_{i=1}^m \mathbb{E}\left[v_i v_i^T\right] = I_n$ and $\mathbb{E}\left[\|v_i\|_2^2\right] = \mathrm{Tr}(B_i) \leq \epsilon$. The initial potential function is

$$\Phi^j_{p_0}(x) = \frac{\partial_{x_j}\det\left(\sum_{i=1}^m x_i B_i\right)}{\det\left(\sum_{i=1}^m x_i B_i\right)} = \frac{\det\left(\sum_{i=1}^m x_i B_i\right)\mathrm{Tr}\left(\left(\sum_{i=1}^m x_i B_i\right)^{-1} B_j\right)}{\det\left(\sum_{i=1}^m x_i B_i\right)} = \mathrm{Tr}\left(\left(\sum_{i=1}^m x_i B_i\right)^{-1} B_j\right),$$

where the second equality is by the Jacobi formula in Fact 2.39. Put in $x_0 = (t, \ldots, t)$, using the assumptions that $\sum_{i=1}^m B_i = I$ and $\mathrm{Tr}(B_i) \leq \epsilon$, the initial potential value is

$$\Phi^j_{p_0}(x_0) = \mathrm{Tr}\left((tI)^{-1} B_j\right) = \frac{1}{t}\,\mathrm{Tr}(B_j) \leq \frac{\epsilon}{t}.$$

If we set $t$ so that $\Phi^j_{p_0}(x_0) \leq \frac{\epsilon}{t} \leq 1 - \frac{1}{\delta}$, then by Lemma 15.9, we will get the final bound $t + \delta$. So, we should set $t$ so that $\frac{\epsilon}{t} = 1 - \frac{1}{\delta}$, and the final bound is

$$t + \frac{1}{1 - \frac{\epsilon}{t}}.$$

This is minimized when $t = \sqrt{\epsilon} + \epsilon$, and the final bound is $(1 + \sqrt{\epsilon})^2$. This completes the proof of Theorem 15.2 using the induction hypothesis in Definition 15.6.

## 15.5 Discussions

The major open problem is to design a polynomial time algorithm to find the solution in Theorem 15.2.

It is interesting to reflect on the journey to the solution to the Kadison-Singer problem. First, it started from the nice formulation of the spectral sparsification problem, as an intermediate step to design a fast Laplacian solver. Then, the reduction to the isotropy case, which helps to match the cut sparsification result by Benczur and Karger. Then, the barrier method was developed, starting with the heuristic argument from expected characteristic polynomial. Then, the interlacing property was observed, and the heuristic argument was developed as a new probabilistic method of interlacing family. The theory of real-stable polynomials was used to establish that the family in Theorem 15.2 is an interlacing family, with the crucial multilinear formula. Finally, the barrier method was understood as a way to bound roots, and it was extended to the multivariate setting through the multilinear formula to solve the problem. It is an amazing line of work with so many great ideas and techniques developed.

## 15.6 References


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