Chapter 14

Bipartite Ramanujan Graphs

We will see how Marcus, Spielman, and Srivastava [MSS15] used the method of interlacing family of polynomials to prove the existence of bipartite Ramanujan graphs, using the 2-lift construction proposed by Bilu and Linial [BL06].

The expected characteristic polynomials in this problem are exactly the matching polynomials of graphs, and we will see some classical results of these polynomials.

14.1 Combinatorial Constructions of Ramanujan Graphs

Recall the definition of Ramanujan graphs from Chapter 7.

Definition 14.1 (Ramanujan Graphs). *Let $G = ([n], E)$ be a d-regular graph and $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ be the eigenvalues of its adjacency matrix. We say G is a Ramanujan graph if* $\max\{\alpha_2, |\alpha_n|\} \leq 2\sqrt{d-1}$.

We are interested in constructing an infinitely family of d-regular graphs that are all Ramanujan. This is best possible, as Alon-Boppana Theorem 7.10 proved that for any $\epsilon > 0$, every large engouh d-regular graph has $\max\{\alpha_2, |\alpha_n|\} \geq 2\sqrt{d-1} - \epsilon$. There is a meaning of the value $2\sqrt{d-1}$, which is the bound on the absolute value of the eigenvalues of the infinite d-regular tree, the best possible d-regular expander graphs in the combinatorial sense.

There are known constructions of Ramanujan graphs of constant degree from Cayley graphs. All known graphs are $(q+1)$-regular where q is a prime power. The analyses of these constructions use deep mathematical results and in particular some by Ramanujan (and hence the name). They are explicit in that the neighbors of a vertex can be computed in $O(\log n)$ time. See the survey by Hoory, Linial, and Wigderson [HLW06] for more details.

2-Lifts

It is of interest to find combinatorial constructions of Ramanujan graphs. Bilu-Linial [BL06] proposed a method to construct Ramanujan graphs using 2-lifts.

Definition 14.2 (2-Lift). *Let $G = ([n], E)$ be an undirected graph. A signing of the edges of G is a function $s : E(G) \to \{-1, +1\}$. The 2-lift $\hat{G}_s = (\hat{V}, \hat{E})$ of G associated with a signing s is defined as follows. The vertex set $\hat{V}$ of $\hat{G}_s$ is $\hat{V} = \{1, \ldots, n, 1', \ldots, n'\}$, where each vertex $i \in V(G)$ has two* *copies i and i' in $\hat{G}_s$. For each edge $ij \in E(G)$, if $s(ij) = 1$, then the edges ij and $i'j'$ are in $\hat{E}$; otherwise, if $s(ij) = -1$, then the edges ij' and $i'j$ are in $\hat{E}$.*

Bilu and Linial conjectured that if G is Ramanujan, then there is a 2-lift of G that is also Ramanujan. Note that if G is d-regular, then any 2-lift of G is also d-regular with the number of vertices doubled. So, if the conjecture is true, then it implies the existence of an infinite family of d-regular Ramanujan graphs for any degree d. Just start with the complete graph on $d+1$ vertices, which is Ramanujan, and keep doing a good 2-lift to double the graph size. Bilu and Linial used probabilistic method (Lovász local lemma) and the techniques in proving the converse of expander mixing lemma in Theorem 7.5 to prove that there exists a 2-lift with $\max\{\alpha_2, |\alpha_n|\} \lesssim \sqrt{d \log^3 d}$.

Spectrum of 2-Lift

There is a nice formulation to analyze the spectrum of a 2-lift of a graph.

Definition 14.3 (Signed Matrix). *Given a graph $G = ([n], E)$ and a signing $s : E(G) \to \{-1, +1\}$ of the edges, the signed matrix $A_s \in \mathbb{R}^{n \times n}$ is defined as follows. If $ij \in E$, then $(A_s)_{ij} = (A_s)_{ji} = s(ij)$, otherwise $(A_s)_{ij} = 0$.*

The proof of the following statement is left as a homework problem.

Problem 14.4 (Spectrum of 2-Lift). *Given a graph $G = ([n], E)$ and a signing $s : E(G) \to \{-1, +1\}$ of the edges, the spectrum of the adjacency matrix $A(\hat{G}_s)$ of the 2-lift $\hat{G}_s$ is equal to the disjoint union of the spectrum of the adjacency matrix $A(G)$ of G (called the old eigenvalues) and the spectrum of the signed matrix A_s (called the new eigenvalues). That is, the multiplicity of an eigenvalue α of $A(\hat{G}_s)$ is equal to the sum of the multiplicity of α of $A(G)$ and the multiplicity of α of A_s.*

With this statement, to prove that there is a Ramanujan 2-lift of a d-regular Ramanujan graph $G = (V, E)$, it is equivalent to proving that there is a signing $s : E(G) \to \{-1, +1\}$ so that the maximum absolute eigenvalue of the signed matrix A_s is at most $2\sqrt{d-1}$. Bilu and Linial made the following stronger conjecture which does not assume that G is a Ramanujan graph.

Conjecture 14.5 (Bili-Linial [BL06]). *For any d-regular graph $G = (V, E)$, there is a signing $s : E(G) \to \{-1, +1\}$ so that all eigenvalues of A_s have absolute value at most $2\sqrt{d-1}$.*

14.2 Bipartite Ramanujan Graphs from Interlacing Family

Marcus, Spielman, Srivastava [MSS15] proved Conjecture 14.5 for bipartite graphs.

Theorem 14.6 (Bili-Linial Conjecture for Bipartite Graphs [MSS15]). *Any d-regular bipartite graph $G = (V, E)$ has a signing $s : E(G) \to \{-1, +1\}$ so that the maximum eigenvalue of A_s is at most $2\sqrt{d-1}$.*

Note that for a bipartite graph, bounding the maximum eigenvalue is enough because the spectrum is symmetric (see Lemma 3.4), even for the signed matrix. This is the reason that Theorem 14.6 only holds for bipartite graph, because the new probabilistic method using interlacing polynomials can only bound the maximum eigenvalue (or one eigenvalue), but not the maximum eigenvalue and the minimum eigenvalue at the same time.

As the spectrum of a bipartite graph is symmetric, any d-regular bipartite graph has the maximum eigenvalue equal to d and the minimum eigenvalue equal to $-d$, which are called the trivial eigenvalues. A bipartite graph is Ramanujan if all its non-trivial eigenvalues are at most $2\sqrt{d-1}$.

Definition 14.7 (Bipartite Ramanujan Graphs). *Let $G = ([n], E)$ be a d-regular graph and $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ be the eigenvalues of its adjacency matrix. We say G is a bipartite Ramanujan graph if* $\max\{\alpha_2, |\alpha_{n-1}|\} \leq 2\sqrt{d-1}$.

A corollary of Theorem 14.6 and Problem 14.4 is that any bipartite Ramanujan graph G has a 2-lift that is Ramanujan. Note that a 2-lift of a bipartite graph is bipartite. So, starting from a complete bipartite graph with $2d$ vertices, which is Ramanujan (see Example 3.3), repeatedly applying a good 2-lift proves the following theorem.

Theorem 14.8 (Bipartite Ramanujan Graphs of Every Degree [MSS15]). *For every d, there is an infinite family of d-regular bipartite Ramanujan graphs.*

Probabilistic Method

Theorem 14.6 is proved by the method of interlacing family of polynomials developed in Chapter 12 and Chapter 13. Given a d-regular graph $G = (V, E)$ with $m := |E|$ edges, there are totally 2^m different signed matrices of G and we consider the uniform distribution on these 2^m signed matrices. The plan is to prove that there exists a signing $s : E(G) \to \{-1, +1\}$ such that

$$\lambda_{\max}\big(\det(xI - A_s)\big) \leq \lambda_{\max}\Big(\mathbb{E}_{s \in \{\pm 1\}^m}\big[\det(xI - A_s)\big]\Big) \leq 2\sqrt{d-1}. \tag{14.1}$$

The first inequality is a relatively straightforward application of Theorem 13.28 and we will prove it in this subsection. For the second inequality, it turns out that the expected characteristic polynomial is exactly the "matching polynomial" of the graph, a well-studied object in the literature, and the upper bound $2\sqrt{d-1}$ is already proved by Heilmann and Lieb in the 70s. We will compute the expected polynomial in the next subsection, and then review some classical results about matching polynomials in Section 14.3.

Theorem 14.9 (Probabilistic Method for Signed Matrices). *For any d-regular graph $G = (V, E)$, there exists a signing $s : E(G) \to \{\pm 1\}$ such that $\lambda_{\max}\big(\det(xI - A_s)\big) \leq \lambda_{\max}\big(\mathbb{E}_s\big[\det(xI - A_s)\big]\big)$, where the expectation is over the uniform distribution of all the signings of the edges.*

Proof. To apply Theorem 13.28, we would like to write A_s as a sum of independent random rank one symmetric matrices. Note that we can write $A_s = \sum_{e \in E} A_e$, where each A_e is a random variable with $(A_e)_{ij} = (A_e)_{ji} = s(ij)$ if $e = ij$ and all other entries zero. The issue is that A_e is a rank two matrix, not rank one. Instead, we consider the random variable $L_e = D_e + A_e$, where $(D_e)_{ii} = (D_e)_{jj} = 1$ if $e = ij$ with all other entries zero. So, each L_e is the signed Laplacian matrix of an edge, which is a rank one matrix. Denote $L_s = \sum_{e \in E} L_e$, which is a sum of independent random rank one symmetric matrices. Note that $L_s = dI + A_s$ as the graph is d-regular, and thus $\det(xI - A_s)$ can be written as $\det((x+d)I - L_s)$. Applying Theorem 13.28 on the random variables $\{L_e\}_{e \in E}$, there exists a signing $s : E(G) \to \{\pm 1\}$ such that

$$\lambda_{\max}\big(\det(yI - L_s)\big) \leq \lambda_{\max}\big(\mathbb{E}_s\big[\det(yI - L_s)\big]\big)$$

By doing a change of variable $y = x + d$, the same signing s satisfies the statement of the theorem. $\square$

Theorem 14.9 proves the first inequality in Equation 14.1.

Expected Characteristic Polynomial

Perhaps surprisingly, the expected characteristic polynomial is already known to be equal to the matching polynomial of a graph, a well-studied polynomial in Combinatorics.

Definition 14.10 (Matching Polynomials)**.** *Given an undirected graph $G = ([n], E)$, let m_i be the number of matchings in G with i edges with $m_0 = 1$, the matching polynomial of G is defined as*

$$\mu_G(x) := \sum_{i \geq 0} (-1)^i \cdot m_i \cdot x^{n-2i}.$$

The following identity is by Godsil and Gutman (see [God93]).

Theorem 14.11 (Expected Characteristic Polynomial is Matching Polynomial)**.** *Given an undirected graph $G = ([n], E)$, the expected characteristic polynomial of the signed matrices is*

$$\mathbb{E}_{s \in \{\pm 1\}^{|E|}} \big[\det(xI - A_s) \big] = \mu_G(x),$$

where the expectation is over the uniform distribution of all the signings of the edges.

Proof. Let $M_s = xI - A_s$. We expand the determinant of M_s as sum of permutations as in Fact 2.26, so that

$$\mathbb{E}_s \det(xI - A_s) = \mathbb{E}_s \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (M_s)_{i,\sigma(i)} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \mathbb{E}_s \left[\prod_{i=1}^{n} (M_s)_{i,\sigma(i)} \right],$$

where $\operatorname{sgn}(\sigma) = (-1)^{\operatorname{inv}(\sigma)}$ and $\operatorname{inv}(\sigma) := \big|\big\{(i,j) \mid i < j \text{ and } \sigma(i) > \sigma(j)\big\}\big|$ is the number of inversion pairs of the permutation σ. Since each edge is independent and $\mathbb{E}\left[(M_s)_{i,j}\right] = 0$ as each edge is equally likely to be ± 1, all the permutations with at least one variable with degree one vanished. Therefore, for the permutations remained, each edge $(M_s)_{ij}$ appears exactly twice and

$$\mathbb{E}_s \left[\prod_{i=1}^{n} (M_s)_{i,\sigma(i)} \right] = x^{n-2k} \cdot (M_s)^2_{i_l, j_l} = x^{n-2k}$$

for some k. So, each matching of size k will contribute $\operatorname{sgn}(\sigma)$ to the coefficient of x^{n-2k}. Check that each matching of size k has the same sign, with $\operatorname{sgn}(\sigma) = -1$ if k is odd and $\operatorname{sgn}(\sigma) = +1$ if k is even. We conclude that $\mathbb{E}_s \det(xI - A_s) = \sum_{k \geq 0} (-1)^k \cdot m_k \cdot x^{2n-k} = \mu_G(x)$. □

14.3 Matching Polynomials

Quite amazingly, the maximum root of the matching polynomial was studied by Heilmann and Lieb in 1972 and their result is exactly what is needed for bipartite Ramanujan graphs.

Theorem 14.12 (Heilmann-Lieb)**.** *For any undirected graph G of maximum degree d, the matching polynomial $\mu_G(x)$ is real-rooted with maximum root at most $2\sqrt{d-1}$.*

So, the results by Godsil-Gutman in Theorem 14.11 and Heilmann-Lieb in Theorem 14.12 combined to establish the second inequality in Equation 14.1, and this completes the proof of Theorem 14.6.

The original proof by Heilmann-Lieb uses recursion and induction. We present an approach by Godsil [God93] which consists of three steps:

1. The matching polynomial of a graph G of maximum degree d divides the matching polynomial of an associated tree T (called the path tree) of maximum degree d.

2. The matching polynomial of a tree T is equal to the characteristic polynomial $\det(xI - A_T)$ of its adjacency matrix A_T.

3. The maximum eigenvalue of the adjacency matrix A_T of a tree T of maximum degree d is at most $2\sqrt{d-1}$.

Since the characteristic polynomial of the adjacency matrix of a tree (and more generally a graph) is real-rooted, (1) and (2) imply that the matching polynomial of a graph is real-rooted. Therefore, the maximum root of the matching polynomial of G is

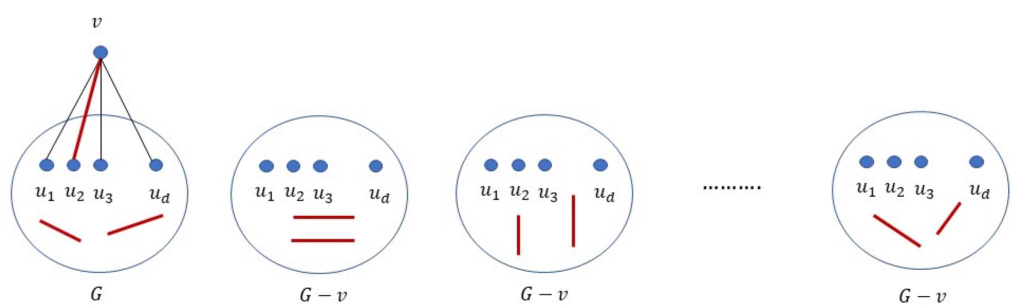
$$\lambda_{\max}\big(\mu_G(x)\big) \leq \lambda_{\max}\big(\mu_T(x)\big) = \lambda_{\max}\big(\det(xI - A_T)\big) \leq 2\sqrt{d-1},$$

where the first inequality is by (1), the equality is by (2), and the last inequality is by (3). Therefore, proving the three steps would complete the proof of Heilmann and Lieb's result in Theorem 14.12.

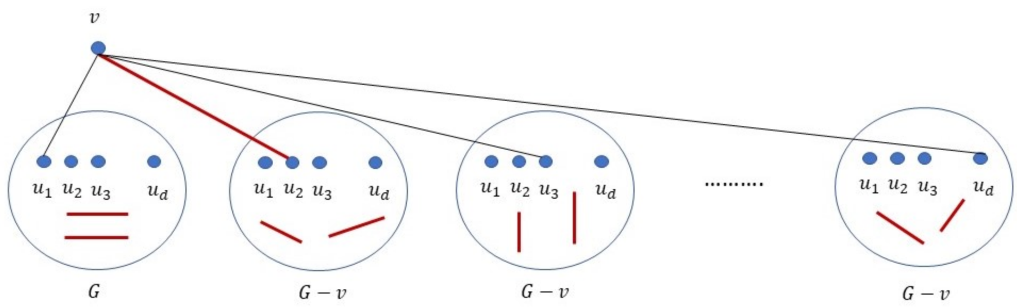
The third step is already done in Problem 3.10. The second step is left as an exercise, as its proof is similar to that in Theorem 14.11, showing that only permutations corresponding to matchings contribute to the characteristic polynomial.

First Step

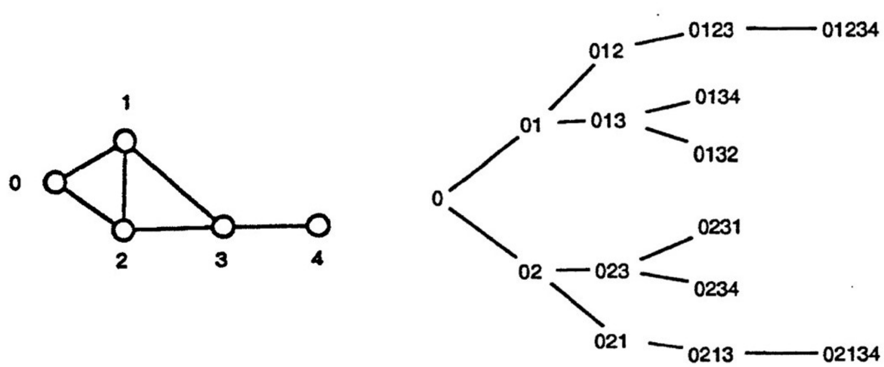
The proof of (1) in [God93] is not long but involves algebraic manipulations. Here we try to give a proof by pictures. Given a graph G, we start from an arbitrary vertex v. Let the degree of v be d, and the neighbors of v be $u_1, \ldots, u_d$. We add $d-1$ copies of $G-v$ to the graph and call the resulting graph H. Check that the matching polynomial of the disjoint union is equal to the product of the matching polynomials, i.e. $\mu_{G_1 \cup G_2}(x) = \mu_{G_1}(x) \cdot \mu_{G_2}(x)$ where $G_1 \cup G_2$ is the disjoint union of two graphs G_1 and G_2. Therefore, $\mu_H(x) = \mu_G(x) \cdot \big(\mu_{G-v}(x)\big)^{d-1}$, and so the matching polynomial of G divides the matching polynomial of H.



Consider the following graph H', where the edge vu_i in G is replaced by vu_i in the i-th copy of $G - v$ for $2 \leq i \leq d$ (see the picture below). The claim is that the matching polynomials of H and H' are the same. The reason is that there is a one-to-one correspondence between matchings in H and matchings in H', as v can only be matched to one vertex. See the pictures, where the red edges are the edges in a matching. Now, in H', there are no cycles involving v.



Applying the same operations (duplicate and branch, and remove isolated vertices) on u_1 in the first copy of $G - v$, on u_2 in the second copy of $G - v$, and so on, the resulting (big) graph will have no cycles and is a tree. The resulting tree is called the path tree of G, as there is a path in T for each path in G. See the following picture from [God93] for a complete example.



All these operations preserve the property that the matching polynomial of the old graph divides the matching polynomial of the new graph, and so by induction the matching polynomial of the original graph G divides the matching polynomial of the path tree, which has maximum degree at most d. This "proves" the first step. See Chapter 6 of [God93] for the formal proof.

14.4 Discussions and Problems

One obvious open question is whether this approach can be extended to construct a true Ramanujan graph (that satisfies $|\alpha_n| \leq 2\sqrt{d-1}$). There is a trick to get something close.

Problem 14.13 (Twice Ramanujan Graphs). *Show that the current approach can be used to construct a d-regular graph with* $\max\{\alpha_2, |\alpha_n|\} \leq 4\sqrt{d-1}$.

Another obvious open question is whether this approach can be made efficient algorithmically. Note that the natural attempt would not work, as it is NP-hard to compute the coefficients of matching polynomials. Marcus, Spielman, Srivastava [MSS18] gave another construction of bipartite Ramanujan graphs using interlacing families for permutations, and Cohen [Coh16] showed that their construction can be implemented in polynomial time.

The following is an exercise that completes the second step of Godsil's proof.

Exercise 14.14 (Matching Polynomial of a Tree). *Prove that the matching polynomial of a tree T is equal to the characteristic polynomial* $\det(xI - A_T)$ *of its adjacency matrix* A_T*.*

The following are some identities for matching polynomials, which can be used to give a formal proof that the matching polynomial of a graph G divides the matching polynomial of its path tree. They can be proved by some simple relations between the number of matchings in a graph and its subgraphs.

Problem 14.15 (Identities for Matching Polynomials [God93]).

1. $\mu_{G\cup H}(x) = \mu_G(x) \cdot \mu_H(x)$ *for disjoint* G *and* H*.*
2. $\mu_G(x) = \mu_{G\setminus e}(x) - \mu_{G\setminus uv}(x)$ *if* $e = uv$ *is an edge of* G*.*
3. $\mu_G(x) = x \cdot \mu_{G\setminus u}(x) - \sum_{i\sim u} \mu_{G\setminus ui}(x)$ *if* $u \in V(G)$*.*

14.5 References

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