Chapter 13

Real Stable Polynomials

To use the method of interlacing family of polynomials in Chapter 12, we need to check whether a set of polynomials $p_1, \ldots, p_m$ have a common interlacing, and this is reduced to checking whether all convex combinations $q = \sum_{i=1}^m \mu_i p_i$ are real-rooted polynomials by Theorem 12.8. In this chapter, we will see some characterizations of real-rooted polynomials. The main object that we will study is the class of real-stable polynomials, a multivariate generalization of the class of real-rooted polynomials. We will use the techniques in the theory of real-stable polynomials to prove that the family in Example 12.11 is an interlacing family.

13.1 Real-Rooted Polynomials

A polynomial is real-rooted if all of its roots are real numbers. One important example of real-rooted polynomials is the characteristic polynomial of a real symmetric matrix (more generally Hermitian matrix), as the roots are the eigenvalues of the matrix and they are real numbers by Theorem 2.5.

Besides computing all the roots of a polynomial, there is a general characterization for checking whether a given polynomial is real-rooted.

Theorem 13.1 (Hermite-Sylvester). *A polynomial $p(x) = \prod_{l=1}^n (x - \lambda_l)$ is a real-rooted polynomial if and only if the $n \times n$ matrix H with $H_{ij} = \sum_{l=1}^n \lambda_l^{i+j-2}$ is a positive semidefinite matrix.*

Given a polynomial in the coefficient form $p(x) = \sum_{i=0}^n c_i x^i$, note that the entries of H (which are moments of the roots) can be computed from the coefficients efficiently by Newton's identities, and thus Hermite-Sylvester's theorem provides a polynomial time algorithm to check whether a polynomial is real-rooted. We will not use this theorem to check whether a polynomial is real-rooted, and we leave the proof as an interesting problem to solve for the reader.

Another approach to show that a polynomial $p(x)$ is real-rooted is to start with a known real-rooted polynomial $q(x)$ (e.g. the characteristic polynomial of a real symmetric matrix) and show that $p(x)$ can be obtained from $q(x)$ by some real-rootedness preserving operations.

Exercise 13.2 (Real-Rootedness Preserving Operations). *Prove that the following operations are real-rootedness preserving operations:*

1. *(Scaling:) If $p(x)$ is real-rooted, then $p(cx)$ is real-rooted for any $c \in \mathbb{R}$.*

2. *(Inversion:) If $p(x) = \sum_{i=0}^{n} c_i \cdot x^i$ is a degree n real-rooted polynomial, then so is the polynomial $x^n \cdot p\big(\frac{1}{x}\big) = \sum_{i=0}^{n} c_{n-i} \cdot x^i$.*

3. *(Differentiation:) If $p(x)$ is a real-rooted polynomial, then so is $p'(x)$, the derivative of $p(x)$.*

We will use this approach to prove that a polynomial is real-rooted, but in the more general multivariate setting which we will define in the next section.

In the remainder of this section, we collect some nice properties of real-rooted polynomials. They will not be used for the method of interlacing family of polynomials, and we refer the reader to the course notes of Oveis Gharan [Ove20] for proofs.

Gauss-Lucas Theorem

The following theorem is a generalization of item (3) in Exercise 13.2. The proof is by considering p'/p and writing a root of p' but not p as a convex combination of the roots of p.

Problem 13.3 (Gauss-Lucas Theorem). *If $p \in \mathbb{C}[x]$ is a non-constant polynomial with complex coefficients, then all roots of p' are in the convex hull of the set of roots of p.*

Ravichandran used the techniques developed for the restricted invertibility problem in Section 12.4 to prove the following quantitative generalization of the Gauss-Lucas theorem, which bounds the area of the convex hull after many differentiations.

Theorem 13.4 (Quantitative Gauss-Lucas Theorem [Rav18]). *Let $p \in \mathbb{C}[x]$ be a degree n polynomial with complex coefficients. Then, for any $c \geq 1/2$,*

$$\big|\mathcal{K}\big(p^{\lceil cn \rceil}\big)\big| \leq 4(c - c^2)\big|\mathcal{K}(p)\big|,$$

where $\mathcal{K}(p)$ denotes the convex hull of the roots of p and $|S|$ denotes the area of the convex set S in the plane.

Note that there are examples where taking the $(\frac{n}{2} - 1)$-th derivative does not decrease the area yet. One such example is $(x+1)^{n/2}(x-1)^{n/2}$.

Generating Polynomials

Given a probability distribution μ over $[n]$, we define its generating polynomial as

$$p_\mu(x) = \sum_{i=1}^{n} \mu_i \cdot x^i.$$

The following is an interesting characterization of when such a generating polynomial is real-rooted.

Proposition 13.5 (Real-Rooted Generating Polynomials). *The generating polynomial $p_\mu(x)$ is real-rooted if and only if μ is the distribution of a sum of independent Bernoulli random variables.*

We will study in a later chapter about probability distributions with real-stable generating polynomials, and we may discuss the proof of Proposition 13.5 there.

One consequence of Proposition 13.5 is that we can use Chernoff bounds to bound the coefficient $a_i = \Pr[X = i]$ with i far away from the mean $\mathbb{E}[X] = \sum_{i=1}^{n} i \cdot a_i$. From this connection, we expect to see a Bell curve when we plot the numbers $a_1, \ldots, a_n$ of a real-rooted polynomial with non-negative coefficients. This intuition can be made precise by the notion in the next subsection.

Log-Concavity

The following is an analog of a log-concave function for a sequence.

Definition 13.6 (Log-Concave Sequence). *A sequence $a_0, \ldots, a_n$ of non-negative numbers is said to be log-concave if for all $0 < i < n$,*

$$a_{i-1} \cdot a_{i+1} \leq a_i^2 \qquad \iff \qquad \frac{1}{2}\big(\log(a_{i-1}) + \log(a_{i+1})\big) \leq \log(a_i).$$

A sequence $a_0, \ldots, a_n$ of non-negative numbers is said to be ultra log-concave if for all $0 < i < n$,

$$\frac{a_{i-1}}{\binom{n}{i-1}} \cdot \frac{a_{i+1}}{\binom{n}{i+1}} \leq \left(\frac{a_i}{\binom{n}{i}}\right)^2$$

We can use the operations in Exercise 13.2 to reduce a degree n real-rooted polynomial to a quadratic real-rooted polynomial involving only a_{i-1}, a_i, a_{i+1}, and then consider the discriminant of the resulting quadratic real-rooted polynomial to prove the following result.

Problem 13.7 (Newton Inequalities). *For any real-rooted polynomial $p(x) = \sum_{i=0}^n a_i \cdot x^i$ with non-negative coefficients, the sequence $a_0, \ldots, a_n$ is ultra log-concave.*

In the third part of the course, we will study log-concave polynomials and see that some sequences from combinatorial problems are log-concave (such as the number of matchings of size i).

13.2 Real Stable Polynomials

The class of real-stable polynomials is a multivariate generalization of real-rooted polynomials.

Definition 13.8 ($\mathcal{H}$-Stable Polynomials). *A multivariate polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$ is $\mathcal{H}$-stable if $p(x_1, \ldots, x_n) \neq 0$ whenever $(x_1, \ldots, x_n) \in \mathcal{H}^n$ where $\mathcal{H} = \{y \in \mathbb{C} \mid \Im(y) > 0\}$ is the upper-half of the complex plane.*

In the third part of the course, we may see some other stable polynomials where the root-free region is differently specified (e.g. sector-stable polynomials).

Definition 13.9 (Real Stable Polynomials). *A multivariate polynomial p is called real stable if p is $\mathcal{H}$-stable and all coefficients of p are real numbers.*

Some simple examples of real stable polynomials are $p(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n$ and $p(x_1, \ldots, x_n) = a_1 x_1 + \ldots + a_n x_n$ where $a_i > 0$ for $1 \leq i \leq n$. Some simple non-examples of real stable polynomials are $p(x_1, x_2) = x_1 - x_2$ and $p(x_1, x_2, x_3, x_4) = x_1 x_2 - x_3 x_4$.

Note that it is a generalization of real-rooted univariate polynomials, using that complex roots of a polynomial with real coefficients come in conjugate pairs.

Exercise 13.10 (Univariate Real-Stable Polynomials). *A univariate polynomial $p \in \mathbb{R}[x]$ is real stable if and only if it is real-rooted.*

Sometimes it is more convenient to check whether a multivariate polynomial is real-stable by checking whether certain derived univariate polynomials are real-rooted.

Exercise 13.11 (Univariate Restrictions). *A polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is real stable if and only if for any $b \in \mathbb{R}^n_+$ and $a \in \mathbb{R}^n$, the univariate polynomial $p(a + yb)$ in y is not identically equal to zero and is real-rooted.*

Using Exercise 13.11, one can draw some pictures to see that the polynomial $1 - xy$ is real-stable while the polynomial $1 + xy$ is not real-stable.

In this course, the source of all real-stable polynomials comes from determinants.

Proposition 13.12 (Source of Real-Stable Polynomials). *If $A_1, \ldots, A_m \in \mathbb{R}^{n \times n}$ are positive semidefinite matrices, then $p(x_0, x_1, \ldots, x_m) := \det(x_0 I + \sum_{i=1}^m x_i A_i)$ is a real stable polynomial.*

Proof. The plan is to show that if $\Im(x_i) > 0$ for all $0 \leq i \leq m$, then the matrix $x_0 I + \sum_{i=1}^m x_i A_i$ is of full rank, and hence $\det(x_0 I + \sum_{i=1}^m x_i A_i) \neq 0$, implying real stability.

Let $v \in \mathbb{C}^n$, and write $v = c + \imath d$ where $c \in \mathbb{R}^n$ is the real part and $d \in \mathbb{R}^n$ is the imaginary part of v. Let $X = x_0 I + \sum_{i=1}^m x_i A_i$, and write X as $\Re(X) + \imath \Im(X)$ where $\Re(X)$ is the real part and $\Im(X)$ is the imaginary part of X. Note that if $\Im(x_i) > 0$ for all $0 \leq i \leq m$, then $\Im(X) \succ 0$, as $A_i \succcurlyeq 0$ for $0 \leq i \leq m$ and $I \succ 0$.

We claim that $Xv = 0$ only if $v = 0$, and hence X is of full rank. To prove this, we show that $v^* X v = (c - \imath d)^T (\Re(X) + \imath \Im(X))(c + \imath d) = 0$ only if $c = d = 0$. Note that the imaginary part of $v^* X v$ is

$$\Im\Big[(c - \imath d)^T (\Re(X) + \imath \Im(X))(c + \imath d)\Big] = c^T \Im(X) c + d^T \Im(X) d,$$

and this is equal to zero only if $c = d = 0$, because $\Im(X) \succ 0$ when $\Im(x_i) > 0$ for $0 \leq i \leq m$. $\square$

One could also prove Proposition 13.12 using the univariate restrictions in Exercise 13.11; see Oveis Gharan's notes [Ove20].

Later, we will start from the multivariate real-stable polynomials from Proposition 13.12, and then apply the real-stability preserving operations in the next section to prove that a univariate polynomial is real-stable, and hence real-rooted by Exercise 13.10.

13.3 Real Stability Preserving Operations

There are several real-stability preserving operations, with some deep characterizations. We just present the proofs of two operations that we need in this course, and state others without proofs.

The following operation will be useful in reducing the number of variables in the multivariate polynomial.

Proposition 13.13 (Specialization). *Let $p(x_1, \ldots, x_m)$ be a non-zero real-stable polynomial. For any $c \in \mathbb{R}$, $p(c, x_2, \ldots, x_m)$ is a real-stable polynomial.*

Proof. It is clear that $p(c, x_2, \ldots, x_m)$ has real coefficients as p has real coefficients and $c \in \mathbb{R}$. For stability, consider the sequence of polynomials $p_k = p(c + \imath 2^{-k}, x_2, \ldots, x_m)$ for $k \geq 1$. Note that each p_k is a $\mathcal{H}$-stable polynomial (but may have complex coefficients) as p is $\mathcal{H}$-stable. The sequence $\{p_k\}_{k \geq 1}$ is converging uniformly to the polynomial $p(c, x_2, \ldots, x_m)$.

Suppose, by contradiction, that $p(c, x_2, \ldots, x_m)$ has a root $z_2, \ldots, z_m$ with $\Im(z_i) > 0$ for $2 \leq i \leq m$. By Hurwitz's Theorem 13.14, for any small enough $\epsilon > 0$ and for every large enough k (depending on ϵ), the polynomial p_k also has a root $y_2, \dots, y_m$ with $|y_i - z_i| < \epsilon$ for $2 \le i \le m$. By choosing ϵ small enough, we still have $\Im(y_i) > 0$ for $2 \le i \le m$, but this means that p_k has a root with all imaginary parts positive, contradicting the $\mathcal{H}$-stability of p_k. $\square$

Hurwitz's theorem is from complex analysis. The following statement is from Wikipedia.

Theorem 13.14 (Hurwitz's Theorem). *Let $\{f_k\}_{k\ge 1}$ be a sequence of holomorphic functions on a connected open set G that converge uniformly on compact subsets of G to a holomorphic function f which is not constantly zero on G. If f has a zero of order l at z_0, then for every small enough $\rho > 0$ and for sufficiently large $k \in \mathbb{N}$ (depending on ρ), f_k has precisely l zeros in the disk defined by $|z - z_0| < \rho$ including multiplicity. Furthermore, these zeroes converge to z_0 as $k \to \infty$.*

The other operation that we need is the differential operator that we have seen a couple of times already. The following proposition is for univariate polynomials.

Proposition 13.15 (Partial Differentiation). *If $p \in \mathbb{C}[x]$ is $\mathcal{H}$-stable, then $p + s \cdot p'$ is $\mathcal{H}$-stable for any $s \in \mathbb{R}$.*

Proof. Since $p(x)$ is stable, it can be written as $c\prod_{j=1}^{n}(x - w_j)$ with $\Im(w_j) \le 0$ for $1 \le j \le n$. Then

$$p(x) + s \cdot p'(x) = p(x)\left(1 + \sum_{j=1}^{n} \frac{s}{x - w_j}\right).$$

For z with $\Im(z) > 0$, $p(z) \neq 0$ as p is $\mathcal{H}$-stable. Furthermore, since $\Im(z) > 0$ and $\Im(w_j) \le 0$ for $1 \le j \le n$, it follows that $\Im\big(\frac{1}{z - w_j}\big) < 0$ for $1 \le j \le n$, and thus $1 + \sum_{j=1}^{n} \frac{s}{x - w_j} \neq 0$. This proves that $g(z) + s \cdot g'(z) \neq 0$ if $\Im(z) > 0$, establishing $\mathcal{H}$-stability. $\square$

This result can be generalized to multivariate polynomials easily by univariate restriction.

Corollary 13.16 (Partial Differentiation). *If $p \in \mathbb{R}[x_1, \dots, x_m]$ is real-stable, then $(1 + s \cdot \partial_{x_1})p$ is real-stable for any $s \in \mathbb{R}$.*

Proof. It is clear that $(1 + s \cdot \partial_{x_1})p$ has real coefficients if p has. For any $y_2, \dots, y_m$ with $\Im(y_i) > 0$ for $2 \le i \le m$, the polynomial $q(x_1) := p(x_1, y_2, \dots, y_m)$ is stable by definition. Proposition 13.15 proves that $(1 + s \cdot \partial_{x_1})q(x_1)$ is also stable. This implies that $(1 + s \cdot \partial_{x_1})p$ has no roots in which all of the variables have positive imaginary part, proving stability. $\square$

The following are some other operations that preserve real-stability, whose proofs are elementary.

Exercise 13.17 (Real-Stability Preserving Operations). *Let $p(x_1, x_2, \dots, x_m)$ and $q(x_1, \dots, x_m)$ be real-stable polynomials. Then*

1. *(Product:) $p \cdot q$ is real stable.*
2. *(Symmetrization:) $p(x_1, x_1, x_3, \dots, x_m)$ is real stable.*
3. *(External Field:) $p(c_1x_1, c_2x_2, \dots, c_mx_m)$ is real stable for any $c_1, \dots, c_m \ge 0$.*
4. *(Inversion:) $p\big(-\frac{1}{x_1}, x_2, \dots, x_m\big) \cdot x_1^{d_1}$ is real stable where d_1 is the degree of x_1 in p.*

5. *(Differentiation:)* $\partial_{x_1} p$ *is real stable.*

Borcea and Brändén characterized a class of differential operators that preserve real stability.

Theorem 13.18 (Borcea-Brändén Theorem). *For vectors $\alpha, \beta \in \mathbb{N}^m$, let $x^\alpha = x^{\alpha(1)} \cdots x^{\alpha(n)}$ and $\partial^\beta = \partial_{x_1}^{\beta(1)} \cdots \partial_{x_m}^{\beta(m)}$ and let $D = \sum_{\alpha,\beta \in \mathbb{N}^m} c_{\alpha,\beta} \cdot x^\alpha \cdot \partial^\beta$ be a differential operator with $c_{\alpha,\beta} \in \mathbb{R}$ for all $\alpha, \beta \in \mathbb{N}^m$. Then D is a stability preserving operator (i.e. it maps any real-stable polynomial to a real-stable polynomial) if and only if the polynomial $\sum_{\alpha,\beta \in \mathbb{N}^m} c_{\alpha,\beta} \cdot x^\alpha \cdot (-w)^\beta \in \mathbb{R}[x_1, \ldots, x_m, w_1, \ldots, w_m]$ on $2m$ variables is real-stable.*

For examples, $1 - \partial x_1 \partial x_2$ is stability preserving because $1 - (-w_1)(-w_2) = 1 - w_1 w_2$ is a real stable polynomial, and similarly $1 + x_1 \partial x_2$ is stability preserving. For non-examples, $1 + \partial_{x_1} \partial_{x_2}$ is not stability preserving as $1 + w_1 w_2$ is not a stable polynomial, and similarly $1 - \partial_{x_1} \partial_{x_2} \partial_{x_3}$ is not stability preserving.

Problem 13.19 (Real Stability Preserving Operators). *Use Theorem 13.18, or otherwise (both are possible), to prove the following results.*

1. *For any $1 \leq k \leq n$, the k-th elementary symmetric polynomial $\sum_{S \subseteq \binom{[n]}{k}} x^S$ is real stable.*

2. *Let MAP be the operator that only retains the multiaffine monomials of a given polynomial, e.g. $MAP(1 + x + 3x^3 y + 2xy) = 1 + x + 2xy$. Prove that MAP is stability preserving.*

See [Wag11] for a survey on real-stable polynomials, with a proof of Theorem 13.18.

13.4 Multilinear Formula, Mixed Characteristic Polynomials, and Interlacing Family

In this section, we use the tools from real stable polynomials to prove that a generalization of the family in Example 12.11 is an interlacing family, which will be a key component in constructing bipartite Ramanujan graphs and resolving the Kadison-Singer problem in the next two chapters.

Mixed Characteristic Polynomial and Multilinear Formula

We consider the setting where each A_i is a random symmetric rank-one matrix with finite support (e.g. A_i is aa^T with probability 0.6, bb^T with probability 0.3, cc^T with probability 0.1), and $A = \sum_{i=1}^m A_i$ is a sum of independent rank-one matrices. We are interested in proving that the set of all possible characteristic polynomials $\det(xI - \sum_{i=1}^m A_i)$ forms an interlacing family. The following identity of the expected characteristic polynomial is at the heart of the approach by Marcus, Spielman, and Srivastava.

Theorem 13.20 (Multilinear Formula). *If $A_1, A_2, \ldots, A_m$ are independent random symmetric rank-one matrices, then*

$$\mathbb{E}_{A_1,\ldots,A_m}\left[\det\left(\lambda I - \sum_{i=1}^m A_i\right)\right] = \prod_{i=1}^m (1 - \partial_{x_i}) \det\left(\lambda I + \sum_{i=1}^m x_i \cdot \mathbb{E}[A_i]\right)\Big|_{x_1 = x_2 = \cdots = x_m = 0}.$$

The right hand side of the multilinear formula is called the mixed characteristic polynomial of the expected matrices $\mathbb{E}[A_1], \ldots, \mathbb{E}[A_m]$, which are not of rank one in general.

Definition 13.21 (Mixed Characteristic Polynomial). *The mixed characteristic polynomial of $n \times n$ matrices $B_1, \ldots, B_m$ (not necessarily rank-one) is defined as*

$$\mu[B_1, \ldots, B_m](\lambda) = \prod_{i=1}^{m} (1 - \partial_{x_i}) \det\left(\lambda I + \sum_{i=1}^{m} x_i \cdot B_i\right)\bigg|_{x_1 = x_2 = \cdots = x_m = 0}$$

There are different proofs of Theorem 13.20. We first present the proof from [MSS15a] (suggested by James Lee), and then discuss a proof by Tao [Tao13] which shows more clearly why it is a multilinear formula. The original proof by Marcus, Spielman, and Srivastava used the Cauchy-Binet formula in Fact 2.30.

Inductive Proof: The base case is similar to the calculations in Section 10.1 and in Exercise 11.8, with the only difference that $\mathbb{E}[A_i]$ is not necessarily a scaled identity matrix.

Lemma 13.22 (Expected Rank-One Update). *For any square matrix M and a random vector v,*

$$\mathbb{E}_v\left[\det(M - vv^T)\right] = (1 - \partial_x) \det\left(M + x \cdot \mathbb{E}\left[vv^T\right]\right)\Big|_{x=0}.$$

Proof. First, we assume M is invertible. By the matrix determinantal formula in Fact 2.29,

$$\det(M - vv^T) = \det(M) \cdot \left(1 - v^T M^{-1} v\right) = \det(M)\left(1 - \operatorname{Tr}\left(M^{-1} vv^T\right)\right).$$

Taking expectation on both sides,

$$\mathbb{E}_v\left[\det(M - vv^T)\right] = \det(M) - \det(M) \operatorname{Tr}\left(M^{-1} \mathbb{E}\left[vv^T\right]\right).$$

On the other hand, by the Jacobi's formula in Fact 2.39,

$$\partial_x \det\left(M + x \cdot \mathbb{E}\left[vv^T\right]\right)\Big|_{x=0} = \det(M) \operatorname{Tr}\left(M^{-1} \mathbb{E}\left[vv^T\right]\right),$$

and so the lemma follows when M is invertible. When M is not invertible, we can choose a sequence of invertible matrices that approach M. Since the lemma holds for each matrix in the sequence and the two sides are polynomials in the entries of the matrix, a continuity argument implies that the lemma also holds for M as well. $\square$

Then Theorem 13.20 can be proved by applying Lemma 13.22 repeatedly.

Exercise 13.23 (Inductive Proof of Multilinear Formula). *Complete the proof of Theorem 13.20 by using Lemma 13.22 inductively and the assumption that $A_1, \ldots, A_m$ are independent random variables.*

Multilinear Proof: The proof presented by Tao [Tao13] also starts from the matrix determinantal formula, which shows that $\det(\lambda I - \sum_{i=1}^m A_i)$ is multilinear in terms of A_i, when each A_i is a rank one matrix. Then we can understand that the RHS of Theorem 13.20 is just a Taylor expansion of the LHS.

Lemma 13.24 (Taylor Expansion of Multilinear Polynomial)**.** *Let $p(x_1, \ldots, x_m)$ be a multilinear polynomial in $x_1, \ldots, x_m$. Then*

$$p(x_1, \ldots, x_m) = \prod_{i=1}^m (1 + x_i \partial_{y_i}) p(y_1, \ldots, y_m)\Big|_{y_1 = \ldots = y_m = 0}.$$

Proof. As p is a multilinear polynomial, it can be written as $p(x_1, \ldots, x_m) = \sum_{S \subseteq [m]} c_S \prod_{i \in S} x_i$, where c_S is the coefficient of the monomial $\prod_{i \in S} x_i$. Note that $c_S = \prod_{i \in S} \partial_{y_i} p(y_1, \ldots, y_m)\big|_{y_1 = \ldots = y_m = 0}$, as differentiation and substitution kill all the terms except c_S. Therefore,

$$p(x_1, \ldots, x_m) = \sum_{S \subseteq [m]} \Big(\prod_{i \in S} x_i\Big)\Big(\prod_{i \in S} \partial_{y_i} p(y_1, \ldots, y_m)\big|_{y=0}\Big) = \prod_{i=1}^m \Big(1 + x_i \partial_{y_i}\Big) p(y_1, \ldots, y_m)\Big|_{y=0}.$$

□

Putting in $p(x_1, \ldots, x_m) = \det(B + x_1 A_1 + \ldots + x_m A_m)$ in Lemma 13.24 gives the following corollary.

Corollary 13.25 (Determinant of Sum of Rank One Matrices)**.** *If $A_1, \ldots, A_m$ are symmetric rank-one matrices, then*

$$\det(B + x_1 A_1 + \ldots + x_m A_m) = \prod_{i=1}^m (1 + x_i \partial_{y_i}) \det(B + y_1 A_1 + \ldots + y_m A_m)\Big|_{y_1 = \ldots = y_m = 0}.$$

To prove Theorem 13.20, we set $B = \lambda I$ and $x_1 = \ldots = x_m = -1$ in Corollary 13.25. Then, we take the expectation on both sides using the sum of monomials form, and move the expectation inside the summation by linearity of expectation, and then move the expectation inside the products by independence of the random variables $A_1, \ldots, A_m$ to obtain the following result.

Exercise 13.26 (Expansion Proof of Multilinear Formula)**.** *Complete the proof of Theorem 13.20 by proving that*

$$\mathbb{E}_{A_1, \ldots, A_m}\left[\det\left(\lambda I - \sum_{i=1}^m A_i\right)\right] = \mu\big[\mathbb{E}[A_1], \ldots, \mathbb{E}[A_m]\big](\lambda),$$

the mixed characteristic polynomial of $\mathbb{E}[A_1], \ldots, \mathbb{E}[A_m]$ in Definition 13.21.

Interlacing Family of Independent Rank-One Matrices

With the multilinear formula in Theorem 13.20, we are now ready to prove that the set of all possible characteristic polynomials $\{\det(\lambda I - \sum_{i=1}^m A_i)\}$ form an interlacing family. The following lemma will be useful in showing that all conditional expectation polynomials are real-rooted.

Proposition 13.27 (Expected Characteristic Polynomial is Real-Rooted)**.** *The expected characteristic polynomial $\mathbb{E}_{A_1, \ldots, A_m}\big[\det\big(\lambda I - \sum_{i=1}^m A_i\big)\big]$ is real-rooted for any independent random symmetric rank-one matrices $A_1, \ldots, A_m$.*

Proof. We start from the RHS of the multilinear formula in Theorem 13.20. Since each A_i is a random symmetric rank-one matrix, the expected matrix $\mathbb{E}[A_i] = \sum_j p_j v_j v_j^T \succcurlyeq 0$ is a positive semidefinite matrix. So, by Proposition 13.12, the multivariate polynomial $\det(\lambda I + \sum_{i=1}^m x_i \cdot \mathbb{E}[A_i])$ is a real-stable polynomial. By the results in stability preserving operations in Corollary 13.16 and Proposition 13.13, applying the differential operator $1 - \partial_{x_i}$ and substituting real numbers preserve stability. Therefore, the LHS of the multilinear formula is a real-stable univariate polynomial in λ, and thus real-rooted by Exercise 13.10. ☐

The following interlacing family plays a major role in the sequence of papers by Marcus, Spielman and Srivastava [MSS15b, MSS15a, MSS21].

Theorem 13.28 (Interlacing Family of Independent Rank-One Matrices)**.** *Let $A_1, A_2, \ldots, A_m$ be random symmetric rank-one matrices, where each A_i has l_i possibilities $v_{i,1}v_{i,1}^T, \ldots, v_{i,l_i}v_{i,l_i}^T$. The set of all $\prod_{i=1}^m l_i$ polynomials in $\{\det(\lambda I - \sum_{i=1}^m v_{i,j_i}v_{i,j_i}^T)\}$ form an interlacing family, where each $j_i \in \{1, \ldots, l_i\}$ for $1 \leq i \leq m$. Furthermore, the root polynomial of the interlacing family can be $\mathbb{E}_{A_1,\ldots,A_m}\big[\det\big(\lambda I - \sum_{i=1}^m A_i\big)\big]$ for any independent distributions on $A_1, \ldots, A_m$.*

Proof. The tree has depth m, with the root at depth 0. At depth $0 \leq i \leq m-1$, each node has l_{i+1} children. Each leaf of the tree is labeled by a sequence $(j_1, j_2, \ldots, j_m)$, representing a path from the root to the tree, where $j_i \in [l_i]$ represents the j_i-th child of an internal node in the $(i-1)$-th level. The polynomial in the leaf node corresponding to $(j_1, j_2, \ldots, j_m)$ is $\det(\lambda I - \sum_{i=1}^m v_{i,j_i}v_{i,j_i}^T)$, a choice $v_{j_i}v_{j_i}^T$ for each A_i for $1 \leq i \leq m$.

Given the independent distributions on $A_1, \ldots, A_m$, the polynomial in an internal node $(j_1, j_2, \ldots, j_k)$ at depth k is defined as $\mathbb{E}_{A_{k+1},\ldots,A_m}\big[\det\big(\lambda I - \sum_{i=1}^k v_{i,j_i}v_{i,j_i}^T - \sum_{i=k+1}^m A_i\big)\big]$, the conditional expectation polynomial where $A_i = v_{j_i}v_{j_i}^T$ is fixed for $1 \leq i \leq k$. The root polynomial is then $\mathbb{E}_{A_1,\ldots,A_m}\big[\det\big(\lambda I - \sum_{i=1}^m A_i\big)\big]$.

We need to check that the two conditions of an interlacing family in Definition 12.10 are satisfied. The first condition is satisfied by construction, that the polynomial in each non-leaf node at depth k is a convex combination of the polynomials in its children, where the convex combination is based on the given probability distribution of A_k, which is independent of other random variables.

For the second condition, we need to prove that the polynomials in the children of a non-leaf node have a common interlacing. By Theorem 12.8, it suffices to prove that all convex combinations of the polynomials in the children of a non-leaf node are real-rooted. Consider an internal node $(j_1, \ldots, j_k)$ at depth k, with l_k children $(j_1, \ldots, j_k, 1), \ldots, (j_1, \ldots, j_k, l_k)$. Given any convex combination $\mu_1, \ldots, \mu_{l_k}$ with $\mu_a \geq 0$ for $1 \leq a \leq l_k$ and $\sum_{a=1}^{l_k} \mu_a = 1$, we need to prove that

$$\sum_{a=1}^{l_k} \mu_a \cdot \mathbb{E}_{A_{k+2},\ldots,A_m}\left[\det\left(\lambda I - \sum_{i=1}^k v_{i,j_i}v_{i,j_i}^T - v_{k+1,a}v_{k+1,a}^T - \sum_{i=k+2}^m A_i\right)\right]$$

is real-rooted. Observe that this is just the expected characteristic polynomial $\mathbb{E}_{B_1,\ldots,B_m}\det(\lambda I - \sum_{i=1}^m B_i)$ for a related set of independent random symmetric rank-one matrices, where $B_1, \ldots, B_k$ are just the (deterministic) random variables with $B_i = v_{i,j_i}v_{i,j_i}^T$ with probability one, B_{k+1} is the random variable with $B_{k+1} = v_{k+1,a}v_{k+1,a}^T$ with probability μ_a for $1 \leq a \leq l_{k+1}$, and $B_{k+2}, \ldots, B_m$ are just the same as the random variables $A_{k+2}, \ldots, A_m$. By Proposition 13.27, any such convex combination is real-rooted, and hence the children have a common interlacing by Theorem 12.8. We conclude that the polynomials in the leaves form an interlacing family. ☐

Note that this generalizes the family in Example 12.11, and thus completes the proof for the restricted invertibility result in Theorem 12.14.

We will use this interlacing family for constructing bipartite Ramanujan graphs and resolving the Kadison-Singer problem in the next two chapters.

13.5 References

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