Chapter 5

Generalizations of Cheeger's Inequality

We give an overview of some recent generalizations of Cheeger's inequality using other eigenvalues. The presentation follows the chronological order.

5.1 Bipartiteness Ratio and Maximum Cut

Recall from Proposition 3.6 that a graph G is bipartite if and if the spectrum of its adjacency matrix is symmetric around the origin, and from Problem 3.7 that a connected graph G is bipartite if and only if $\alpha_1 = -\alpha_n$ where α_i is the i-th largest eigenvalue of the adjacency matrix. These results are for the spectrum of the adjacency matrix, and the following is a corresponding result for the spectrum of the normalized Laplacian matrix.

Exercise 5.1 (Spectral Characterization of Bipartiteness)**.** *Let $G = (V, E)$ be an undirected graph and λ_n be the largest eigenvalue of its normalized Laplacian matrix $\mathcal{L}(G)$. Then $\lambda_n = 2$ if and only if G has a bipartite component, i.e. a connected component that is a bipartite graph.*

Trevisan [Tre09] proved that λ_n is close to 2 if and only if G is close to having a bipartite component, in the same style as in Cheeger's inequality in Theorem 4.3. To state the result, we write the optimization formulation for $2 - \lambda_n$ and then motivate the corresponding definition of bipartiteness ratio of a subset of vertices S.

Exercise 5.2 (Optimization Formulation for $2 - \lambda_n$)**.** *Let $G = (V, E)$ be an undirected graph and λ_n be the largest eigenvalue of $\mathcal{L}(G)$. Then*

$$2 - \lambda_n = \min_{x \in \mathbb{R}^n} \frac{\sum_{ij \in E} \big(x(i) + x(j)\big)^2}{\sum_{i \in V} \deg(i) \cdot x(i)^2}.$$

Let $S \subseteq V(G)$ be a bipartite component with partition $S = (L, R)$ such that all the edges in S are between L and R. Then the vector $x \in \{-1, 0, 1\}^n$ where

$$x(i) = \begin{cases} +1 & \text{if } i \in L \\ -1 & \text{if } i \in R \\ 0 & \text{otherwise} \end{cases}.$$

is a solution to the above optimization problem with objective value 0. Using this association between a vector $x \in \{-1, 0, 1\}^n$ and a bipartition of a subset $S = (L, R)$, Trevisan considered the following definition of the bipartiteness ratio of $S = (L, R)$ where each edge within L and each edge within R contributes 2 in the numerator while an edge in $\delta(S)$ contributes 1 in the numerator.

Definition 5.3 (Bipartiteness Ratio)**.** *Let $G = (V, E)$ be an undirected graph with $V = [n]$. The bipartiteness ratio of a vector $x \in \{-1, 0, 1\}^n$ is defined as*

$$\beta(x) := \frac{\sum_{ij \in E} \left|x(i) + x(j)\right|}{\sum_{i \in V} \deg(i) \cdot |x(i)|}.$$

The bipartiteness ratio of a graph G is defined as

$$\beta(G) := \min_{x \in \{-1,0,1\}^n} \beta(x).$$

Trevisan proved the following analog of Cheeger's inequality for $2 - \lambda_n$ and $\beta(G)$.

Theorem 5.4 (Cheeger's Inequality for λ_n [Tre09])**.** *Let $G = (V, E)$ be an undirected graph and λ_n be the largest eigenvalue of $\mathcal{L}(G)$. Then*

$$\frac{1}{2}(2 - \lambda_n) \leq \beta(G) \leq \sqrt{2(2 - \lambda_n)}.$$

The easy direction is left as an exercise and the hard direction is left as a problem. Trevisan used the ideas in his proof of Cheeger's inequality as shown in section 4.3 to define $L_t = \{i \mid x(i) \geq \sqrt{t}\}$ and $R_t = \{i \mid x(i) \leq -\sqrt{t}\}$ and $S_t = (L_t, R_t)$ for a uniformly random t to prove the hard direction.

Maximum Cut

In the maximum cut problem, we are given an undirected graph $G = (V, E)$ and the task is to find a set $S \subseteq V$ that maximizes $|\delta(S)|$. This is a classical NP-complete problem. It is an exercise that there is always a subset $S \subseteq V$ with $|\delta(S)| \geq \frac{1}{2}|E|$, and this gives a trivial $\frac{1}{2}$-approximation algorithm for the problem. It was not known how to do better until Goemans and Williamson [GW95] introduced semidefinite programming into approximation algorithms and used it to design a 0.878-approximation algorithm for the maximum cut problem. Semidefinite programming was the only approach to do better than $\frac{1}{2}$-approximation for the maximum cut problem until Trevisan used Theorem 5.4 to design a spectral 0.531-approximation algorithm.

The power of the spectral method is that it gives a better upper bound on the optimal value than the trivial upper bound $|E|$. Suppose the maximum cut $(S, V - S)$ cuts at least $1 - \epsilon$ fraction of edges Then check that $\beta(G) \leq \epsilon$, and thus $2 - \lambda_n \leq 2\beta(G) \leq 2\epsilon$ by the easy direction of Theorem 5.4. So, if we compute that $2 - \lambda_n$ is large, then we know that the maximum cut only cuts at most $\frac{1}{2}\lambda_n$ fraction of edges, and thus the trivial approximation algorithm of cutting 50% of edges would be a $1/\lambda_n$-approximation algorithm. To summarize, when λ_n is bounded away from 2, then there is a better than $1/2$-approximation algorithm for the maximum cut problem, using the easy direction of Theorem 5.4.

On the other hand, when $2 - \lambda_n$ is small, by the hard direction of Theorem 5.4, we can find a subset $S = (L, R)$ with small $\beta(S)$, so that most edges with an endpoint in S are between L and R. So we commit on putting vertices in L on one side and vertices in R on the other side. Then we solve the maximum cut problem on $V - S$ recursively. Suppose the returned partition is $V - S = (L', R')$. Then we return the best of $(L' \cup L, R' \cup R)$ and $(L' \cup R, R' \cup L)$ as our solution, to ensure that at least 50% of edges in $\delta(S)$ are cut. To summarize, when λ_n is close to 2, we can find a set S and ensure that more than 50% of the edges with an endpoint in S will be cut in the returned solution, using the hard direction of Theorem 5.4.

With these ideas, it should be clear that they can be combined to give a better than $1/2$-approximation algorithm for the maximum cut problem. Soto [Sot15] improved the analysis and gave a spectral 0.614-approximation algorithm for the maximum cut problem.

5.2 Small-Set Expansion

A more refined notion of expansion is to study the expansion of sets of different size. We assume the graph is d-regular in this section.

Definition 5.5 (Expansion Profile)**.** *Let $G = (V, E)$ be a d-regular graph. For any $0 < \delta \leq 1/2$, define*

$$\phi_\delta(G) := \min_{S \subseteq V : |S| \leq \delta |V|} \phi(S)$$

to be the δ-small-set expansion of G. The curve $\phi_\delta(G)$ for $0 < \delta \leq 1/2$ is called the expansion profile of the graph G.

The problem of finding small sparse cuts is useful in applications such as community detection in a social network. Also, this problem is of much theoretical interest because of its close connection to the unique games conjecture [RS10]. The small-set expansion conjecture by Steurer and Raghavendra [RS10] states that for any $\epsilon > 0$, there exists $0 < \delta \leq 1/2$ such that it is NP-hard to distinguish between $\phi_\delta(G) \leq \epsilon$ and $\phi_\delta(G) \geq 1 - \epsilon$. This conjecture is still wide open, and if true this would imply optimal inapproximability results for many well-known problems, including Goemans-Williamson 0.878-approximation algorithm for the maximum cut problem!

Motivated by this connection, Arora, Barak and Steurer [ABS10] proved the following Cheeger's inequality for small-set expansion, which roughly says that if λ_k is small for a large enough k, then there is a set S with $|S| \approx |V|/k$ and $\phi(S) \approx \sqrt{\lambda_k}$.

Theorem 5.6 (Cheeger's Inequality for Small-Set Expansion [ABS10])**.** *Let $G = (V, E)$ be a d-regular graph and λ_k be the k-th smallest eigenvalue of $\mathcal{L}(G)$. For $k \geq n^{2\beta}$,*

$$\exists S \subseteq V \text{ with } |S| \lesssim n^{1-\beta} \text{ and } \phi(S) \lesssim \sqrt{\frac{\lambda_k}{\beta}}.$$

They used this theorem to design a sub-exponential time algorithm for the small-set expansion conjecture and the unique games conjecture, together with the ideas of subspace enumeration and graph decomposition. This is an influential paper as it opens up the line of research about higher eigenvalues of graphs.

Analytically Sparse Vectors from Random Walks

Using the threshold rounding in Lemma 4.8, if we could find a vector x with $|\operatorname{supp}(x)| \leq \delta|V|$ and $R_{\mathcal{L}}(x) \lesssim \lambda_k$, then we can find a set S with $|S| \leq \delta|V|$ and $\phi(S) \lesssim \sqrt{\lambda_k}$. This is the starting point.

The constraint $|\operatorname{supp}(x)| \leq \delta|V|$ is combinatorial and not easy to work with directly. Note that any vector satisfying this constraint satisfies the condition $\|x\|_1 \leq \sqrt{\delta|V|} \cdot \|x\|_2$ by Cauchy-Schwarz, an analytical condition more suitable for spectral analysis.

Definition 5.7 (Combinatorial and Analytical Sparse Vectors)**.** *Let $x \in \mathbb{R}^n$ be a vector and $\delta \in [0,1]$. We say x is δ-combinatorially sparse if $|\operatorname{supp}(x)| \leq \delta n$, and x is δ-analytically sparse if $\|x\|_1 \leq \sqrt{\delta n}\|x\|_2$.*

By a truncation argument similar to that in Problem 4.13, we can reduce the problem to finding a δ-analytically sparse vector with small Rayleigh quotient.

Problem 5.8 (Combinatorial Sparse Vector from Analytical Sparse Vector)**.** *Let $x \in \mathbb{R}^n_+$ be a non-negative vector that is δ-analytically sparse. Prove that there exists a non-negative vector $y \in \mathbb{R}^n_+$ that is 4δ-combinatorially sparse with $R_{\mathcal{L}}(y) \leq 2R_{\mathcal{L}}(x)$.*

The main idea in [ABS10] is to find such a vector from random walks, a topic that we will study in the next chapter. Let $W = \frac{1}{2}I + \frac{1}{2}\mathcal{A} = I - \frac{1}{2}\mathcal{L}$ be the lazy random walk matrix. Note that our assumption $\lambda_k \leq \lambda$ translates to $\alpha_k \geq 1 - \frac{\lambda}{2}$ where α_k denotes the k-th largest eigenvalue of W. The main argument using the spectrum is

$$\sum_{i=1}^{n} \|W^t \chi_i\|_2^2 = \sum_{i=1}^{n} \chi_i^T W^{2t} \chi_i = \operatorname{Tr}(W^{2t}) = \sum_{i=1}^{n} \alpha_i^{2t} \geq k\Big(1 - \frac{\lambda}{2}\Big)^{2t},$$

where the last equality is by Fact 2.35. Therefore, there exists $i \in [n]$ such that

$$\|W^t \chi_i\|_2^2 \geq \frac{k}{n}\Big(1 - \frac{\lambda}{2}\Big)^{2t},$$

and this gives an analytically sparse vector as $\|W^t \chi_i\|_1 = 1$ since it is a probability distribution.

On the other hand, by a relatively standard spectral argument using eigen-decomposition and the power mean inequality, one can prove that the Rayleigh quotient $R(W^t\chi_i)$ is small for every $i \in [n]$.

Problem 5.9 (Rayleigh Quotient of Random Walk Vector)**.** *Let $G = (V, E)$ be a graph with $V = [n]$, $\mathcal{L}$ be its normalized Laplacian matrix, and $W = I - \frac{1}{2}\mathcal{L}$ be its lazy random walk matrix. For any $i \in [n]$,*

$$R_{\mathcal{L}}(W^t \chi_i) \leq 2 - 2\|W^t \chi_i\|_2^{1/t}.$$

These two claims combine to give a vector $W^t\chi_i$ that has small Rayleigh quotient and is analytically sparse. More precisely, by setting $t = \frac{\ln k}{2\lambda}$ and doing some calculations, one can check that there exists i with $W^t\chi_i$ being $\frac{1}{\sqrt{k}}$-analytically sparse and $R(W^t\chi_i) \leq \frac{2\lambda \ln n}{\ln k}$. Then Theorem 5.6 follows when $k \geq n^{2\beta}$.

5.3 Higher-Order Cheeger Inequalities

Recall from Exercise 3.19 that $\lambda_k = 0$ if and only if G has at least k connected components. After seeing Cheeger's inequality in Theorem 4.3 and its analogy for λ_n in Theorem 5.4, we now expect that there is also a robust quantitative generalization of this fact.

Actually, the Cheeger inequality for small-set expansion in Theorem 5.6 can be seen as one such generalization, because when $\lambda_k = 0$ there exists a component of size at most n/k, and Theorem 5.6 proves that there exists a sparse cut of size roughly n/k when k is large enough.

In the following, we see another generalization that λ_k is small if and only if G has at least k disjoint subsets $S_1, \ldots, S_k$ each is close to a connected component.

Definition 5.10 (k-Way Edge Conductance)**.** *Let $G = (V, E)$ be a graph. The k-way edge conductance is defined as*

$$\phi_k(G) = \min_{S_1, S_2, \ldots, S_k \subseteq V} \max_{1 \le i \le k} \phi(S_i),$$

where the minimization is over pairwise disjoint subsets $S_1, \ldots, S_k$ of V.

The following higher-order Cheeger inequalities were obtained independently by two research groups.

Theorem 5.11 (Higher-Order Cheeger Inequalities [LOT14, LRTV12])**.** *Let $G = (V, E)$ be a graph and λ_k be the k-th smallest eigenvalue of its normalized Laplacian matrix. Then*

$$\frac{1}{2}\lambda_k \le \phi_k(G) \lesssim k^2 \cdot \sqrt{\lambda_k}.$$

Moreover,

$$\phi_k(G) \lesssim \sqrt{\log k \cdot \lambda_{2k}}.$$

Note that Theorem 5.11 guarantees that there are disjoint sparse cuts and hence also a sparse cut of size at most n/k, but Theorem 5.6 gives a stronger quantitative bound with no dependency on k which is crucial for the small-set expansion and the unique games conjectures. In short, Theorem 5.11 and Theorem 5.6 are incomparable, and it would be very interesting to obtain a common generalization of these two results. The following is another open question.

Question 5.12. *Is it true that $\phi_k(G) \lesssim \operatorname{polylog}(k) \cdot \sqrt{\lambda_k}$?*

Spectral Embedding

The high level plan is to find k disjoint supported vectors $x_1, \ldots, x_k$ such that each has small Rayleigh quotient $R_{\mathcal{L}}(x_i)$. Then we can apply the threshold rounding in Lemma 4.8 on each x_i to find $S_i \subseteq \operatorname{supp}(x_i)$ with small conductance $\phi(S_i) \lesssim \sqrt{R_{\mathcal{L}}(x_i)}$.

An interesting new idea in [LOT14, LRTV12] is to use the spectral embedding defined by the first k eigenvectors to find the k disjoint sparse cuts.

Definition 5.13 (Spectral Embedding)**.** *Let $G = (V, E)$ be a graph with $V = [n]$, $\lambda_1 \le \ldots \le \lambda_k$ be the k smallest eigenvalues of $\mathcal{L}(G)$, and $v_1, \ldots, v_k \in \mathbb{R}^n$ be corresponding orthonormal eigenvectors. Let $U \in \mathbb{R}^{n \times k}$ be the $n \times k$ matrix where the j-th column is v_j. The spectral embedding $u_i \in \mathbb{R}^k$ of a vertex i is defined as the i-th row of U.*

The spectral embedding is used in practice to find disjoint sparse cuts. A popular heuristic is to apply some well-known geometric clustering algorithms, in particular the k-means algorithm, to partition the point set in the spectral embedding into k groups/clusters, and use this partitioning to cut the graph into k sets. It is still an open problem to analyze this heuristic rigorously.

The proof in [LOT14] analyzed a slightly different algorithm that clusters the points based on directions. As $v_1, \ldots, v_k$ are orthonormal vectors, the matrix U in Definition 5.13 satisfies $U^T U = I_k$, and this implies that the spectral embedding satisfies the following isotropy condition.

Exercise 5.14 (Isotropy Condition)**.** *Let $u_1, \ldots, u_n \in \mathbb{R}^k$ be the spectral embedding of the vertices in Definition 5.13. For any $x \in \mathbb{R}^k$ with $\|x\|_2 = 1$, prove that*

$$\sum_{i=1}^{n} \langle x, u_i \rangle^2 = 1.$$

Note that $\sum_{i=1}^{n} \|u_i\|^2 = k$ as $U^T U = I_k$. The isotropy condition implies that the points $u_1, \ldots, u_n \in \mathbb{R}^k$ must be "well spread out" in different directions.

Definition 5.15 (Radial Projection Distance)**.** *Let $u_1, \ldots, u_n \in \mathbb{R}^k$ be the spectral embedding of the vertices in Definition 5.13. The radial projection distance between two vertices i and j is defined as*

$$d(i,j) = \left\| \frac{u_i}{\|u_i\|_2} - \frac{u_j}{\|u_j\|_2} \right\|$$

if $\|u_i\| > 0$ and $\|u_j\| > 0$. Otherwise, if $u_i = u_j = 0$ then $d(i,j) := 0$, else $d(i,j) := \infty$.

Problem 5.16 (Spreading Property)**.** *Let $G = (V, E)$ be a graph with $V = [n]$. Let $u_1, \ldots, u_n \in \mathbb{R}^k$ be the spectral embedding of the vertices in Definition 5.13. Let $S \subseteq V$ be such that $d(i,j) \leq \Delta$ for all $i, j \in S$. Then*

$$\sum_{i \in S} \|u_i\|^2 \leq \frac{1}{1 - \Delta^2}.$$

Informally, the spreading property implies that the points cannot be concentrated in less than k directions, as otherwise the spectral embedding cannot identify k clusters.

Suppose there are k clusters $S_1, \ldots, S_k$ such that $\sum_{i \in S_j} \|u_i\|^2 \approx 1$ for $1 \leq i \leq k$ and the pair-wise distance $d(S_i, S_j) := \min_{a \in S_i, b \in S_j} d(a,b)$ is large. Then [LOT14] uses an idea called smooth localization to find k disjoint supported vectors $x_1, \ldots, x_k \in \mathbb{R}^n$ each with small Rayleigh quotient.

To achieve this condition, [LOT14] also uses a random partitioning idea to partition $\mathbb{R}^k$ and removes all points close to boundaries so that the distances between different parts are lower bounded. For more details, see L04 in 2019 or the notes by Trevisan [Tre16] or the journal paper [LOT14].

Randomized Rounding Algorithm

The algorithm in [LRTV12] is elegant and simple to describe.

Algorithm 2 Randomized Rounding on Spectral Embedding [LRTV12]

Require: An undirected graph $G = (V, E)$ with $V = [n]$ and $m = |E|$, and a parameter k.

1: Compute the spectral embedding $u_1, \ldots, u_n \in \mathbb{R}^k$ in Definition 5.13.

2: Pick k independent Gaussian vectors $g_1, \ldots, g_k \in N(0,1)^n$. Construct disjointly supported vectors $h_1, \ldots, h_k \in \mathbb{R}^n$ as follows:

$$h_i(j) = \begin{cases} \langle u_j, g_i \rangle & \text{if } i = \operatorname{argmax}_{i \in [k]} \{\langle u_j, g_i \rangle\} \\ 0 & \text{otherwise} \end{cases}.$$

3: Apply the threshold rounding in Lemma 4.8 on each h_i to obtain a set $S_i \subseteq \operatorname{supp}(h_i)$ and $\phi(S_i) \leq \sqrt{2R_{\mathcal{L}}(h_i)}$.

4: **return** all S_i with $\phi(S_i) \lesssim \sqrt{\log k \cdot \lambda_k}$.

It is proved in [LRTV12] that this algorithm will return $\Omega(k)$ subsets with constant probability. The proof is by computing the expectation and the variance of the numerator and the denominator, using some properties of Gaussian random variables.

5.4 Improved Cheeger Inequalities

Recall that it is an open question to explain the empirical performance of the spectral partitioning algorithm. In practical instances for image segmentation and data clustering, it is reasonable to expect that there are only a few outstanding objects/clusters in the input image/dataset. One way to formalize this is to assume that the k-way conductance $\phi_k(G)$ of the input graph is large for a small constant k. By the higher-order Cheeger inequality in Theorem 5.11, this is qualitatively equivalent to λ_k is large for a small constant k, which is typically satisfied in practical instances of image segmentation. The following improved Cheeger's inequality shows that the spectral partitioning algorithm performs better in these inputs.

Theorem 5.17 (Improved Cheeger's Inequality [KLL+13]). *Let $G = (V, E)$ be a graph and λ_k be the k-th smallest eigenvalue of its normalized Laplacian matrix. For any $k \geq 2$,*

$$\frac{\lambda_2}{2} \leq \phi(G) \lesssim \frac{k\lambda_2}{\sqrt{\lambda_k}}.$$

The proof of Theorem 5.17 shows that the spectral partitioning algorithm achieves this guarantee. Note that when $\lambda_k = \Omega(1)$ for a small constant k, Theorem 5.17 implies that the spectral partitioning algorithm is a constant factor approximation algorithm for graph conductance. This provides some rigorous justification of the empirical success of the spectral partitioning algorithm.

Exercise 5.18 (Tight Example for Theorem 5.17). *Check that the improved Cheeger's inequality is tight up to a constant factor for the cycle examples.*

There are also related improved Cheeger's inequalities which work with $\phi_k(G)$ directly and with the robust vertex expansion of the graph [KLL17].

k-Step Functions

To see the main intuition in [KLL+13], consider the simpler scenario when λ_2 is small but λ_3 is big. Since λ_2 is small, the graph has a sparse cut $(S, V - S)$ by Cheeger's inequality in Theorem 4.3. As λ_3 is big, $\phi_3(G)$ is also big by the higher-order Cheeger's inequality in Theorem 5.11. This implies that the induced graph in each S and $V - S$ should be an expander graph, as otherwise there is a good way to cut them into smaller pieces which would contradict that $\phi_3(G)$ is big. Since the induced graphs in S and $V - S$ are expander graphs and $(S, V - S)$ is a sparse cut, we expect that the minimizer for the Rayleigh quotient in Lemma 4.4 should look like a binary solution and thus $\lambda_2 \approx \phi(G)$.

The proof of Theorem 5.17 has two main steps. The first step is to show that if λ_k is large for a small constant k, then any eigenvector of the second eigenvalue should look like a k-step function.

Definition 5.19 (k-Step Function). *Let $G = (V, E)$ be a graph with $V = [n]$. Given $y \in \mathbb{R}^n$ and $1 \leq k \leq n$, we say y is a k-step function if the number of distinct values in $\{y(i)\}_{i \in V}$ is at most k.*

Lemma 5.20 (Constructing k-Step Approximation). *Let $G = (V, E)$ be a d-regular graph with $V = [n]$. For any $x \in \mathbb{R}^n$, there is a $(2k + 1)$-step function y such that*

$$\|x - y\|_2^2 \leq \frac{4R_{\mathcal{L}}(x)}{d \cdot \lambda_k}.$$

The second step is to show that if the second eigenvector is close to a k-step function, then the spectral partitioning algorithm performs better.

Lemma 5.21 (Rounding k-Step Approximation)**.** *Let $G = (V, E)$ be a d-regular graph with $V = [n]$. Let $x \in \mathbb{R}^n$ and let $y \in \mathbb{R}^n$ be a $(2k+1)$-step function. The spectral partitioning algorithm applied on x outputs a set S with $|S| \leq n/2$ and*

$$\phi(S) \leq 4kR_{\mathcal{L}}(x) + 4\sqrt{2}k \cdot \sqrt{d} \cdot \|x - y\|_2 \cdot \sqrt{R_{\mathcal{L}}(x)}.$$

Note that Theorem 5.17 follows immediately from Lemma 5.20 and Lemma 5.21.

To prove Lemma 5.20, the idea is that if x is far from being a k-step function, then x must be "smooth/continuous", and it is possible to decompose x into k disjoint supported vectors $x_1, \ldots, x_k \in \mathbb{R}^n$ such that each has small Rayleigh quotient as shown in the following figure.

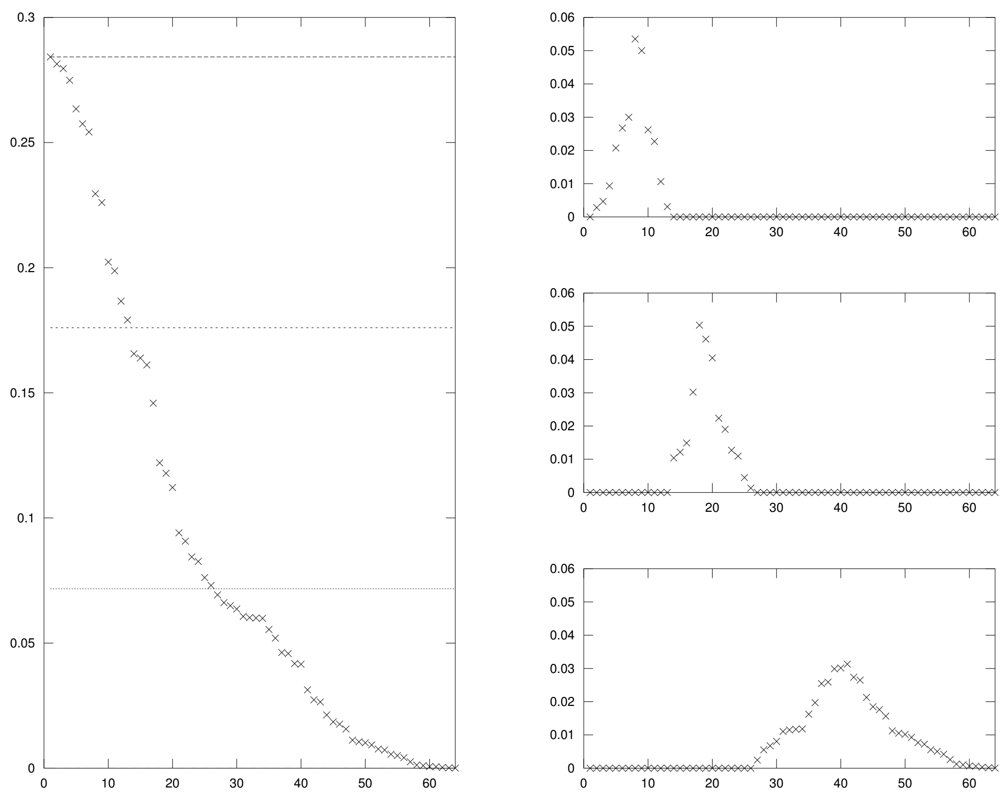

Figure 5.1: The figure on the left is the function x. We cut x into three disjointly supported vectors x_1, x_2, x_3 by setting $t_0 = 0$, $t_1 \approx 0.07$, $t_2 \approx 0.175$, and $t_3 = \max x(i)$. For each $1 \leq i \leq 3$, we define $x_i(j) = \min\{|x(j) - t_{i-1}|, |x(j) - t_i|\}$, if $t_{i-1} \leq x(j) \leq t_i$, and zero otherwise.

For Lemma 5.21, it is instructive to work out the special case when x is exactly a k-step function.

Problem 5.22 (Rounding k-Step Function)**.** *Prove Lemma 5.21 when x is a $(2k+1)$-step function.*

The general idea is to choose a random threshold t with probability proportional to the distance to the nearest step in y. See L05 in 2019 or [KLL+13] for details.

5.5 References

[ABS10] Sanjeev Arora, Boaz Barak, and David Steurer. Subexponential algorithms for unique games and related problems. In *51th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2010, October 23-26, 2010, Las Vegas, Nevada, USA*, pages 563–572. IEEE Computer Society, 2010. 35, 36

[GW95] Michel X. Goemans and David P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. ACM*, 42(6):1115–1145, 1995. 34

[KLL+13] Tsz Chiu Kwok, Lap Chi Lau, Yin Tat Lee, Shayan Oveis Gharan, and Luca Trevisan. Improved cheeger's inequality: analysis of spectral partitioning algorithms through higher order spectral gap. In *Symposium on Theory of Computing Conference, STOC'13, Palo Alto, CA, USA, June 1-4, 2013*, pages 11–20. ACM, 2013. 39, 40

[KLL17] Tsz Chiu Kwok, Lap Chi Lau, and Yin Tat Lee. Improved cheeger's inequality and analysis of local graph partitioning using vertex expansion and expansion profile. *SIAM J. Comput.*, 46(3):890–910, 2017. 39

[LOT14] James R. Lee, Shayan Oveis Gharan, and Luca Trevisan. Multiway spectral partitioning and higher-order cheeger inequalities. *J. ACM*, 61(6):37:1–37:30, 2014. 37, 38, 71, 72, 73

[LRTV12] Anand Louis, Prasad Raghavendra, Prasad Tetali, and Santosh S. Vempala. Many sparse cuts via higher eigenvalues. In *Proceedings of the 44th Symposium on Theory of Computing Conference, STOC 2012, New York, NY, USA, May 19 - 22, 2012*, pages 1131–1140. ACM, 2012. 37, 38

[RS10] Prasad Raghavendra and David Steurer. Graph expansion and the unique games conjecture. In *Proceedings of the 42nd ACM Symposium on Theory of Computing, STOC 2010, Cambridge, Massachusetts, USA, 5-8 June 2010*, pages 755–764. ACM, 2010. 35

[Sot15] José A. Soto. Improved analysis of a max-cut algorithm based on spectral partitioning. *SIAM J. Discret. Math.*, 29(1):259–268, 2015. 35

[Tre09] Luca Trevisan. Max cut and the smallest eigenvalue. In Michael Mitzenmacher, editor, *Proceedings of the 41st Annual ACM Symposium on Theory of Computing, STOC 2009, Bethesda, MD, USA, May 31 - June 2, 2009*, pages 263–272. ACM, 2009. 33, 34

[Tre16] Luca Trevisan. Lecture notes on graph partitioning, expanders and spectral methods. 2016. 28, 38