Chapter 1

Overview

The course has three parts. The first part is about eigenvalues, from classical to recent results in spectral graph theory. The second part is about polynomials, mostly on the method of interlacing polynomials and the theory of real-stable polynomials. The third part is about eigenvalues and polynomials, on high-dimensional expanders and log-concave polynomials.

1.1 First Part

The classical spectral graph theory relates (i) the second eigenvalue of the adjacency/Laplacian matrix and (ii) the graph expansion and (iii) the mixing time of random walks. We will start with the fundamental Cheeger's inequality, and then see its applications in analyzing mixing time and in constructing expander graphs.

Around 2010, there are a few interesting extensions/generalizations of Cheeger's inequality using other eigenvalues of the matrix. In previous offerings, we studied these generalizations in details. In this offering, we will just have an overview of these results. Instead, we will study a new Cheeger's inequality for vertex expansion from 2021.

Also around 2010, there are a few interesting results on a linear algebraic formulation of the graph sparsification problem. We will study a random sampling algorithm, and a deterministic algorithm using barrier functions. Then we will also study a related concept called spectral rounding, and see its applications in experimental design and network design.

To provide a more concrete idea, the graph sparsification problem can be formulated as the following pure linear algebraic problem. Given $v_1, \ldots, v_n \in \mathbb{R}^d$ such that $\sum_{i=1}^n v_i v_i^T = I_d$, find scalars $s_1, \ldots, s_n$ with few nonzeros such that $\sum_{i=1}^n s_i v_i v_i^T \approx I_d$. The deterministic result says that it is possible to have only $O(d)$ non-zeros scalars to achieve a constant factor approximation, implying that any undirected graph has a linear-sized sparsifier. It is striking that this linear algebraic formulation provides the best-known way to look at this combinatorial graph problem.

1.2 Second Part

The ideas and techniques developed in spectral sparsification turned out to be surprisingly powerful. It was observed that the deterministic spectral sparsification result is reminiscent to a major open problem in mathematics called the Kadison-Singer problem. This major problem is very remarkably solved in 2013 by a novel probabilistic method and a multivariate extension of the barrier method.

Interestingly, the new probabilistic method is based on viewing eigenvalues as roots of polynomials, and exploiting interlacing properties of these roots. Besides this method of interlacing family and the multivariate barrier method, the solution is also built on a beautiful theory for real-stable polynomials.

In the second part of the course, we will take this polynomial perspective and study the theory of real-stable polynomials to some extent. Then we will see how this is used in establishing interlacing properties for the new probabilistic method. And then we will see the multivariate barrier method and the solution to the Kadison-Singer problem.

Besides the Kadison-Singer problem, this method of interlacing family has several other interesting applications, including the construction of Ramanujan graphs and even the traveling salesman problem. To give one example, consider the following special case of the restricted invertibility problem. Given $v_1, \ldots, v_n \in \mathbb{R}^d$ such that $\sum_{i=1}^n v_i v_i^T = I_d$ and an integer k, the goal is to prove the existence of a subset S of k vectors with large minimum non-zero eigenvalue $\lambda_{\min}\big(\sum_{i \in S} v_i v_i^T\big)$. It turns out that the method of interlacing family allows us to reduce the problem to bounding the maximum root of the polynomial $\partial^n x^n (x-1)^n$!

1.3 Third Part

In the third part, we will study an active research topic about high-dimensional expanders. This new concept provides a local-to-global way to bound the second eigenvalue of the random walk matrix, and it leads to an elegant solution to a long standing open problem called the matriod expansion conjecture in 2019. Since then, lots of progress have been made in using this new approach to analyze mixing time of random walks.

Interestingly, this approach of bounding eigenvalues for random walks is also closely related to analytical properties of some associated polynomials. Consider the following natural algorithm for sampling a random spanning tree of an undirected graph $G = (V, E)$. Start with an arbitrary spanning tree T_0. In the i-th iteration, we add a random edge e to the tree and remove a random edge f on the cycle created and set $T_i := T_{i-1} + e - f$. And we simply repeat many iterations and hope that the tree will look random very soon. Amazingly, the analysis of this algorithm depends on the analytical properties of the following generating polynomials. Given an undirected graph $G = (V, E)$, we associate a variable x_e for each edge $e \in E$, and consider the generating polynomial of spanning trees $p(x) = \sum_{T \in \mathcal{T}} \prod_{e \in T} x_e$ where $\mathcal{T}$ denotes the set of all spanning trees of G. For spanning trees, we will see that this polynomial is completely log-concave and this implies that the above random sampling algorithm is fast.

This polynomial approach has been extended very nicely to prove optimal bounds on mixing time for several other problems, through the so-called log-Sobolev inequality. For these problems, a more general property called fractionally log-concavity is used to prove strong bounds on log-Sobolev inequality. This connection between polynomials and mixing time is very elegant.

High-dimensional expanders are also used in a recent breakthrough in constructing locally testable codes. It is an exciting and very active research area that has shown great promise.