CS 860 Spectral graph theory. Spring 2019, Waterloo.

Lecture 14: Bipartite Ramanujan graphs

We see how to use the method of interlacing family to prove the existence of bipartite Ramanujan graphs, using the 2-lift construction proposed by Bilu and Linial.

The expected characteristic polynomials in this setting are exactly the matching polynomial of graphs, and we will see some classical results of these polynomials.

Ramanujan graphs

Given a d-regular undirected graph G, let $d = \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ be the eigenvalues of its adjacency matrix.

We say G is Ramanujan if $\max\{\alpha_2, |\alpha_n|\} \leq 2\sqrt{d-1}$.

We are interested in constructing an infinite family of d-regular graphs that are all Ramanujan.

This is best possible, as Alon and Boppana proved that for any $\varepsilon > 0$, every large enough d-regular graph has $\max\{\alpha_2, |\alpha_n|\} \geq 2\sqrt{d-1} - \varepsilon$.

There is a meaning of the value $2\sqrt{d-1}$. It is a bound on the absolute value of the eigenvalues of the infinite d-regular tree, intuitively the best possible d-regular expander graph.

There are known constructions of Ramanujan graphs of constant degree from Cayley graphs.

All known graphs are $(q+1)$-regular where q is a prime power.

The proofs use deep mathematical results and in particular some by Ramanujan (and hence the name).

They are explicit in that the neighbors of a vertex can be computed in $O(\log n)$ time.

See the survey by Hoory-Linial-Wigderson for more details.

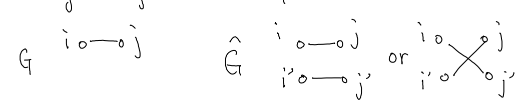
2-lift

It is of interest to find combinatorial constructions of Ramanujan graphs.

Bilu and Linial proposed a method to construct Ramanujan graphs using 2-lifts.

Given a graph $G = (V, E)$, a 2-lift of G is a graph $\hat{G} = (\hat{V}, \hat{E})$ where $\hat{V}$ is two copies of V, i.e. if $V = \{1, 2, \ldots, n\}$, then $V' = \{1, 2, \ldots, n, 1', 2', \ldots, n'\}$.

For each edge $ij \in E(G)$, there are two options to put corresponding edges in $\hat{G}$: either we put ij and $i'j'$, or put ij' and $i'j$.



So, given G, there are 2^m possible 2-lifts of G.

Bilu and Linial conjectured that if G is Ramanujan, then there is a 2-lift of G that is also Ramanujan.

Note that if G is d-regular, any 2-lift of G is also d-regular with double number of vertices.

So, if the conjecture is true, it implies the existence of an infinite family of d-regular Ramanujan graphs for any degree d. We just start with the complete graph on $d+1$ vertices, which is Ramanujan, and keep doing a good 2-lift to double the graph size.

Bilu and Linial used probabilistic method (Lovász local lemma) to prove that there is a 2-lift with $\max\{\alpha_2, |\alpha_n|\} \le O(\sqrt{d \log^3 d})$.

Bipartite Ramanujan graphs

Marcus, Spielman, Srivastava used the method of interlacing polynomials to prove a variant of Bilu-Linial conjecture.

Recall that the spectrum of the adjacency matrix of a bipartite graph is symmetric, so $\alpha_1 = d$ and $\alpha_n = -d$.

We say a bipartite graph is Ramanujan if $\max\{\alpha_2, |\alpha_{n-1}|\} \le 2\sqrt{d-1}$.

The following is the main theorem that we study today.

Theorem Given a bipartite Ramanujan graph G, there is a 2-lift of G that is Ramanujan.

Note that a 2-lift of a bipartite graph is bipartite. So, starting from a complete bipartite graph with $2d$ vertices, which is Ramanujan, it implies an infinite family of d-regular bipartite Ramanujan graph of any degree d.

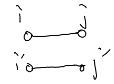
Spectrum of signed matrix

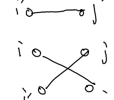
There is a nice formulation to analyze the spectrum of a 2-lift of a graph.

Let A be the adjacency matrix of G.

Let $\hat{G}$ be a 2-lift of G.

We encode the 2-lift $\hat{G}$ in a signed adjacency matrix A_S.

For $ij \in E(G)$, we set $(A_S)_{ij} = (A_S)_{ji} = 1$ if $ij \in E(\hat{G})$ and $i'j' \in E(\hat{G})$, i.e.  in $\hat{G}$;

otherwise, we set $(A_S)_{ij} = (A_S)_{ji} = -1$ if $ij' \in E(\hat{G})$ and $i'j \in E(\hat{G})$, i.e.  in $\hat{G}$

For $ij \notin E(G)$, we set $(A_S)_{ij} = (A_S)_{ji} = 0$.

Lemma The spectrum of the adjacency matrix of $\hat{G}$ is equal to the disjoint union of the spectrum of A_G (the old eigenvalues) and the spectrum of A_S (the new eigenvalues).

The proof of this lemma is left as a homework problem.

With this lemma, to prove that there is a Ramanujan 2-lift of a Ramanujan graph, it is equivalent to proving that there is a signing of a Ramanujan graph (an assignment of ± 1 to each edge) so that the maximum absolute eigenvalue (i.e. spectral radius) of A_S is at most $2\sqrt{d-1}$.

Bilu and Linial conjectured the stronger statement that any d-regular graph (not necessarily Ramanujan) has a signing that all eigenvalues of A_S have absolute value at most $2\sqrt{d-1}$.

Marcus, Spielman and Srivastava proved this conjecture for bipartite graphs.

Theorem Any bipartite graph has a signing such that the maximum eigenvalue of $A_S \leq 2\sqrt{d-1}$.

Note that for a bipartite graph, bounding the maximum eigenvalue is enough because the spectrum is symmetric.

This is the reason that the result only holds for bipartite graphs, because the new probabilistic method using interlacing polynomials can only bound the maximum eigenvalue (or one eigenvalue), but not the maximum eigenvalue and the minimum eigenvalue at the same time.

Outline

We will prove the theorem using the method of interlacing polynomials. The proof has two steps.

1. Prove that there exists a signing such that $\text{maxroot}(\det(xI - A_S)) \leq \text{maxroot}\left(\mathbb{E}_{s\in\{\pm1\}^m} \det(xI - A_S)\right)$.

   This is an application of the new probabilistic method in L13.

2. Prove that $\text{maxroot}\left(\mathbb{E}_{s\in\{\pm1\}^m} \det(xI - A_S)\right) \leq 2\sqrt{d-1}$.

   It turns out that the expected characteristic polynomial is exactly the "matching polynomial" of the graph, a well-studied object in the literature, and this bound is proved by Heilman and Lieb.

   We will review the existing results about matching polynomials.

Interlacing family

Our goal is to prove that $\|A_s\| \le \text{maxroot}\left(\mathbb{E}_{s\in\{\pm1\}^m} \det(\lambda I - A_s)\right)$ with positive probability.

We use the theorem of the new probabilistic method, which we restate below.

<u>Theorem</u> Let $v_1, v_2, \ldots, v_m \in \mathbb{R}^n$ be independent random vectors with finite support.

Then $\text{max-root}\left[\det(\lambda I - \sum_i v_i v_i^T)\right] \le \text{max-root}\left[\mathbb{E}\left(\det(\lambda I - \sum_i v_i v_i^T)\right)\right]$ with positive probability.

To apply the theorem, we would like to write A_s as a sum of rank one matrices.

Note that we can write $A_s = \sum_e A_e$ where $A_e = \begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$ or $A_e = \begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{array}$ is a random variable for each edge $e = ij$.

The problem is that A_e is a rank two matrix, not rank one.

Instead, we consider the random variable $L_e = \begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$ or $L_e = \begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$, the signed Laplacian of an edge, which is a rank one matrix.

Call $L_s = \sum_e L_e$, which is the sum of independent random rank-one PSD matrices.

Note that $A_s = -dI + \sum_e L_e = -dI + L_s$, and $\det(\lambda I - A_s)$ can be written as $\det((\lambda+d)I - L_s)$.

By letting $y = \lambda + d$ and apply the theorem, it follows that $\text{max-root}(\det(yI - L_s)) \le \text{max-root}\left(\mathbb{E}_s \det(yI - L_s)\right)$ with positive probability.

Therefore, there exists a signing s^* such that $\text{max-root}(\det(\lambda I - A_{s^*})) \le \text{max-root}\left(\mathbb{E}_s \det(\lambda I - A_s)\right)$.

<u>Theorem</u> There exists a signing $s^* \in \{\pm1\}^m$ such that $\|A_{s^*}\| \le \text{max-root}\left(\mathbb{E}_s \det(\lambda I - A_s)\right)$.

<u>Expected characteristic polynomials and matching polynomials</u>

Given a graph G, let m_i be the number of matchings in G with i edges (with $m_0 = 1$).

The matching polynomial is defined as $\mu_G(x) := \sum_{i \ge 0} x^{n-2i} (-1)^i m_i$.

Godsil and Gutman proved that the matching polynomial is exactly the expected characteristic polynomial.

<u>Theorem</u> $\mathbb{E}_{s\in\{\pm1\}^{|E|}} \det(xI - A_s) = \mu_G(x)$.

<u>proof</u> We expand the determinant as sum of permutations.

Let $B_s = xI - A_s = \begin{pmatrix} x & & & \\ & x & -1 & 1 \\ -1 & & x & \\ & 1 & & x \end{pmatrix}$.

Then $\mathbb{E}_s \det(xI - A_s) = \mathbb{E}_s \sum_{\sigma:\text{permutations}} \text{sgn}(\sigma) \prod_{i=1}^n (B_s)_{i,\sigma(i)} = \sum_\sigma \text{sgn}(\sigma)\, \mathbb{E}_s\left[\prod_{i=1}^n (B_s)_{i,\sigma(i)}\right]$

where $\text{sgn}(\sigma) = (-1)^{\text{inv}(\sigma)}$ and $\text{inv}(\sigma) := |\{(i,j) \mid i<j \text{ and } \sigma(i) > \sigma(j)\}|$ is the number of inversions of σ.

Since each edge is independent and $E[(B_S)_{ij}] = 0$ as each edge is equally likely to be ± 1, all the permutations with at least one variable with degree one vanished.

Therefore, the permutations remained can only be of the form $x^{n-2k} \prod_{\ell=1}^{k} (B_S)^2_{i_\ell j_\ell} = x^{n-2k}$, where each edge appears as a degree two term $i_1 - j_1 \quad i_2 - j_2 \quad \cdots \quad i_\ell - j_\ell$.

So, each matching of size k will contribute $\operatorname{sgn}(\sigma)$ to the coefficient of x^{n-2k}.

We claim that every matching of size k has the same sign, with $\operatorname{sgn}(\sigma) = -1$ if k is odd and $\operatorname{sgn}(\sigma) = 1$ if k is even.

The claim would imply that $\mathop{E}_{S} \det(xI - A_S) = \sum_{k \geq 0} (-1)^k m_k x^{2n-k} = \mu_G(x)$, proving the theorem.

It remains to check the claim. Given a permutation σ that corresponds to a matching of size k, we argue that the parity of the number of inversion is different from that of the identity permutation if k is odd and the same if k is even, and this would imply the claim.

To see this, let ij be a matched edge, so that $i < j$ and $\sigma(j) = i$ and $\sigma(i) = j$.

Consider the permutation that we just swap ij so that $\sigma(i) = i$ and $\sigma(j) = j$ with other positions unchanged.

Then, observe that for each $\ell \neq i, j$, the number of inversion pairs involving ℓ is either unchanged or decrease by two. and hence the same parity for those inversion pairs involving $\ell \neq i, j$.

However, the parity changes by one because of the pair ij (from ji to ij).

After k swaps, we get the identity permutation and the parity changes k times.

This implies that matchings of odd size have odd parity, and of even size have even parity.

This proves the claim and hence the proposition. $\square$

Matching polynomials

Heilmann and Lieb proved in 1972 that the matching polynomial is real-rooted and the maximum root is at most $2\sqrt{d-1}$ when the maximum degree of the graph is d.

As we discussed before, this combines with the previous theorems imply that there exists a signing s^* such that $\|A_{s^*}\| \leq \operatorname{maxroot}\left(\mathop{E}_{s} \det(\lambda I - A_s)\right) = \operatorname{maxroot}(\mu_G(x)) \leq 2\sqrt{d-1}$, completing the proof.

The original proof uses recursion and induction.

We present an approach by Godsil, which is more systematic and consists of three steps:

1. The matching polynomial of a graph of maximum degree d divides the matching polynomial of an associated tree (called the path tree) of maximum degree d.
2. The matching polynomial of a tree is equal to its characteristic polynomial.
3. The maximum eigenvalue of a tree of maximum degree d is at most $2\sqrt{d-1}$.

Since the characteristic polynomial is real-rooted, 1 + 2 implies that the matching polynomials are real-rooted, and 3 implies that the maxroot of the characteristic polynomial of the tree is at most $2\sqrt{d-1}$, implying Heilmann and Lieb's results.

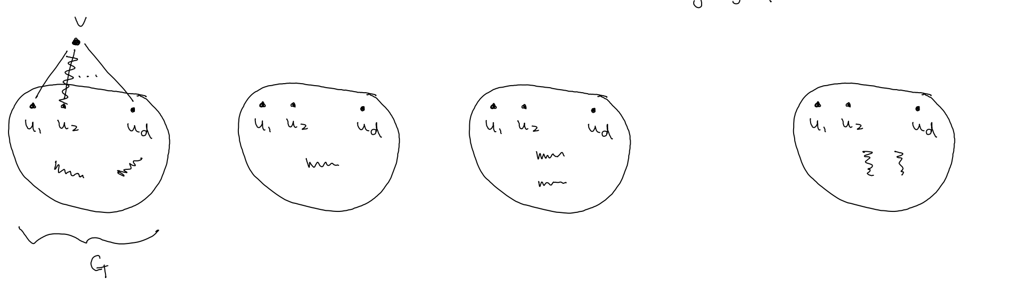
We have done 3 in HW1.

We leave 2 as an exercise. The proof is similar to that the expected characteristic polynomial is the matching polynomial, showing that only permutations correspond to matchings contribute to the char. poly.

We sketch the proof of 1.

Given a graph G, we look at an arbitrary vertex v.

We make $d-1$ copies of $G-v$ and call the resulting graph H.

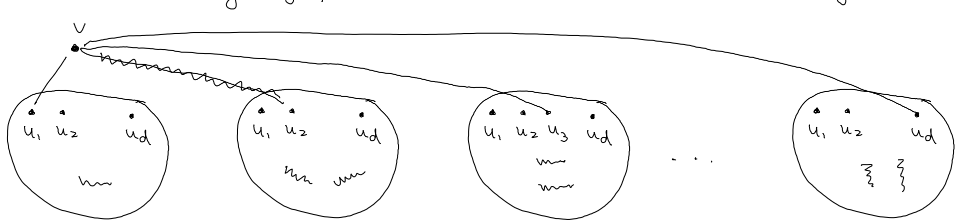


H is the disjoint union of these

Note that the matching polynomial of the disjoint union is equal to the product of the matching polynomials, i.e. $\mu_{G_1 \cup G_2}(x) = \mu_{G_1}(x)\,\mu_{G_2}(x)$ where $G_1 \cup G_2$ denotes the disjoint union of G_1 and G_2.

Therefore, $\mu_H(x) = \mu_G(x) \cdot \left(\mu_{G-v}(x)\right)^{d-1}$, and so the matching polynomial of G divides that of H.

Now, consider the following graph H', where vu_i in the first copy is replaced by vu_i in the i-th copy.



The claim is that the matching polynomials of H and H' are the same.

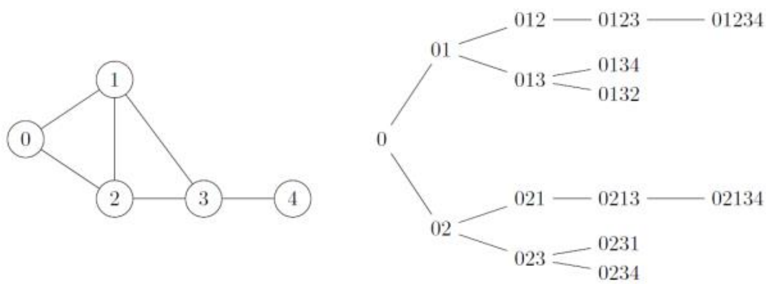
The reason is that there is a one-to-one correspondence between matchings in H and matchings in H',

as v can only be matched to one vertex (see the picture).

Now, in H', there are no cycles involving v.

Applying the same operations (duplicate and branch, and removing isolated components) on u_1 in the first copy, u_2 is the second copy and so on, the resulting (huge) graph will have no cycles and is a tree.

The resulting tree is called the path tree of G, as there is a path in T for each path in G.



picture from Shayan Oveis Gharan's notes

All these operations preserve the property that the matching polynomial of the small graph divides that of the bigger graph, and so eventually the matching polynomial of G divides the matching polynomial of the path tree, which has maximum degree at most d. This proves the first step.

The proof can be written much more formally, but I think it is easier to understand without symbols.

Open question An obvious open question is whether this approach can be extended to construct a true Ramanujan graph (i.e. $|\alpha_n| \leq 2\sqrt{d-1}$). We leave it as a homework problem to show that this approach works to construct a d-regular graph with $\max\{\alpha_1, |\alpha_n|\} \leq 4\sqrt{d-1}$.

Another obvious question is whether this approach can be made efficient algorithmically. Note that the obvious attempt would not work, as it is NP-hard to compute the coefficients of matching polynomials.

References

- Interlacing families I: Bipartite Ramanujan graphs of all degrees, by Marcus, Spielman, Srivastava.
- Algebraic combinatorics (chapter 5 and 6 for matching polynomials), by Godsil.