CS 798: Convexity and Optimization **Last Updated:** Apr 10, 2017

Homework 3 **Due Date: Apr 17, 2017 (23:59 EST)**

You are allowed to discuss with others but not allowed to use any references other than the course notes and the four primary references listed in the course page. If you discuss with others, please list your collaborators for each question. In any case, you must write your own solutions.

The full mark is 60. This homework is counted 15% of the course.

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1. **Rounding Ellipsoid**

   (20 marks) Let $K \subseteq \mathbb{R}^n$ be a convex body with $rB_2^n \subseteq K \subseteq RB_2^n$ for some $0 < r < R$. We assume that there is a separation oracle for K: if $x \in K$ then the oracle will say "yes"; if $x \notin K$ then the oracle will return a separating hyperplane $v \in \mathbb{R}^n$ such that $\langle v, x\rangle > \langle v, y\rangle$ for all $y \in K$. Design a deterministic polynomial time algorithm using $O(n^3 \log(R/r))$ oracle calls to find an ellipsoid $\mathcal{E}$ such that

   $$c + \frac{1}{\sqrt{n}(n+1)}(\mathcal{E} - c) \subseteq K \subseteq \mathcal{E}$$

   where c is the center of the ellipsoid $\mathcal{E}$.

   (Hint: Let $\mathcal{E} = \{c + \Lambda V x \mid x \in \mathbb{R}^n, \|x\|_2 \leq 1\}$ where $v_1, \ldots, v_n \in \mathbb{R}^n$ are the columns of V that form an orthonormal basis (the semi-axes of $\mathcal{E}$) and $\Lambda \in \mathbb{R}^n$ is a diagonal matrix with positive diagonal entries $\lambda_1, \ldots, \lambda_n$ (the side lengths of $\mathcal{E}$) and c being the center of $\mathcal{E}$. Suppose $\mathcal{E}$ contains K. Check if the points $c \pm \frac{\lambda_i v_i}{n+1}$ are contained in K. Shrink the ellipsoid if some of these points are not in K.)

   Bonus (10 marks): Can you improve this result if K is symmetric around the origin?

2. **Matrix Factorization**

   (20 marks) Let A be an $m \times n$ matrix of rank r where each entry is in $\{-1, +1\}$. We know that A can be factorized as $A = U^T V$ where U is an $r \times m$ matrix with columns $u_1, \ldots, u_m$ and V is an $r \times n$ matrix with columns $v_1, \ldots, v_n$. Prove that there is such a factorization with $\|u_i\|_2 \leq \sqrt{r}$ for all $1 \leq i \leq m$ and $\|v_j\|_2 \leq \sqrt{r}$ for all $1 \leq j \leq n$.

   (Hint: Use John's theorem to find an appropriate linear transformation.)

3. **Log-concavity**

   (20 marks) Let $f : \mathcal{R} \to \mathcal{R}$ be a log-concave probability density function. By the general result that log-concavity is preserved by integration (follows from Prépoka-Leindler inequality), we know that the cumulative distribution function $F(x) = \int_{-\infty}^{x} f(y)dy$ is log-concave. Here we derive a simple proof of this statement in the special case when f is a differentiable log-concave probability density function. Let $f(x) = \exp(-h(x))$ where h is a differentiable convex function (which includes the Gaussian case).

   (a) Prove that f is log-concave if and only if $f''(x)f(x) \leq (f'(x))^2$ for all x.

   (b) Prove that

   $$\int_{-\infty}^{x} e^{-h(y)}dy \leq \frac{e^{-h(x)}}{-h'(x)},$$

   when $h'(x) < 0$ and use it to conclude that F is log-concave.