CS 798: Convexity and Optimization **Last Updated:** Jan 25, 2017
Homework 1 **Due Date: Feb 15, 2017 (in class)**

You are allowed to discuss with others but not allowed to use any references other than the course notes and the four primary references listed in the course page. If you discuss with others, please list your collaborators for each question. In any case, you must write your own solutions.

There are a total of 120 marks (not including the bonus), and the full mark is 100. This homework is counted 20% of the course.

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1. **Proving Inequality**

   (10 marks) Prove that for any $q > p > 0$ and positive real numbers $x_1, \ldots, x_n$, we have

   $$\Big(\frac{1}{n}\sum_{i=1}^{n} x_i^q\Big)^{1/q} \geq \Big(\frac{1}{n}\sum_{i=1}^{n} x_i^p\Big)^{1/p}.$$

   (Hint: Consider the function $f(x) = x^{q/p}$.)

   (**Bonus**: 5 marks) Does the inequality holds for any $q > p$? Give a proof or provide a counterexample. Note that when $p \to 0$, we have $(\frac{1}{n}\sum_{i=1}^{n} x_i^p)^{1/p} \to (\prod_{i=1}^{n} x_i)^{1/n}$, and so we define $(\frac{1}{n}\sum_{i=1}^{n} x_i^p)^{1/p} = (\prod_{i=1}^{n} x_i)^{1/n}$ when $p = 0$.

2. **Proving Convexity**

   (a) (15 marks) Prove that the function

   $$f(x) = \Big(\sum_{i=1}^{n} x_i^p\Big)^{1/p}$$

   with $\mathrm{dom} f = \mathbb{R}^n_{>0}$ is concave when $0 < p < 1$ and convex when $p \geq 1$. Conclude that $\|x\|_p := (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ is a norm when $p \geq 1$ but not a norm when $0 < p < 1$.

   (b) (10 marks) Prove that the function $f(X) = (\det X)^{1/n}$ is concave on dom $f = S^n_{++}$. You can use the fact that the geometric mean $f(x) = (\prod_{i=1}^{n} x_i)^{1/n}$ is a concave function.

3. **Sandwich Function**

   (10 marks) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $g : \mathbb{R}^n \to \mathbb{R}$ be a concave function with dom f = dom $g = \mathbb{R}^n$. Suppose $g(x) \leq f(x)$ for all x. Prove that there exists an affine function h such that $g(x) \leq h(x) \leq f(x)$ for all x.

4. **Polar Set**

   (10 marks) Suppose a polytope P is defined as $\{x \mid Ax \leq b\}$, where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m_{>0}$ and a_i denotes the i-th row of A. Prove that the polar set of P is the convex hull of $\{a_1/b_1, \ldots, a_m/b_m\}$.

5. **Quadratic Norm**

Given a positive definite matrix $Q \in S^n_{++}$, the quadratic norm in terms of Q is defined as $\|x\|_Q = \sqrt{x^TQx}$ for $x \in \mathbb{R}^n$.

(a) (5 marks) Prove that $\|x\|_Q$ is indeed a norm.

(b) (5 marks) Derive a simple formula for $\|x\|_{Q*}$, the dual norm of $\|x\|_Q$.

(c) (5 marks) Describe the polar set of an ellipsoid $\mathcal{E} := \{x \mid x^TQx \leq 1\}$. Draw a picture to illustrate the polar set of a 2D-ellipsoid with semi-axes v_1, v_2 and side lengths a_1, a_2.

6. **Cone Programming**

A second-order cone program (SOCP) is of the following form:

$$\begin{aligned} \min \quad & f^Tx \\ \text{s.t.} \quad & \|A_ix + b_i\|_2 \leq c_i^Tx + d_i, \quad 1 \leq i \leq m \\ & Fx = g, \end{aligned}$$

where $x \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n_i \times n}$, and $F \in \mathbb{R}^{p \times n}$. See [BV 4.4.2] for more about SOCP.

(a) (10 marks) Given two ellipsoids $\mathcal{E}_i := \{P_iu + q_i \mid \|u\|_2 \leq 1\}$ for $i = 1, 2$ where $P_i \in S^n_{++}$. We are interested in finding a hyperplane that strictly separates $\mathcal{E}_1$ and $\mathcal{E}_2$, i.e. find $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a^Tx + b > 0$ for $x \in \mathcal{E}_1$ and $a^Tx + b < 0$ for $x \in \mathcal{E}_2$. Formulate this problem as an SOCP feasibility problem.

(b) (10 marks) Show that any SOCP can be written as a semidefinite program.
(Hint: Use Schur complement; see [BV A.5.5].)

7. **Strong Duality**

(10 marks) Consider the following problem:

$$\begin{aligned} \min \quad & e^{-x} \\ \text{s.t.} \quad & x^2/y \leq 0, \end{aligned}$$

with domain $\mathcal{D} = \{(x, y) \mid y > 0\}$. Show that it is a convex optimization problem for which strong duality does not hold.

8. **John Ellipsoid**

(10 marks) Let $C \subseteq \mathbb{R}^n$ be a polyhedron described as $\{x \mid Ax \leq b\}$ and assume that its interior is non-empty. Prove that the maximum inscribed ellipsoid of C, expanded by a factor of n about its center, is an ellipsoid that contains C.

(Hint: One approach is to use the results in problem 4 and problem 5c, and the result in [BV 8.4] about minimum covering ellipsoid through a duality argument. You can use the fact that $\text{vol}((\mathcal{E} + c)^*) \geq \text{vol}(\mathcal{E}^*)$ where $\mathcal{E}$ is a symmetric ellipsoid and $\mathcal{E} + c$ is the translated ellipsoid with center c. You can also use other approaches such as convex programming, of course.)

9. **Dikin Ellipsoid**

(10 marks) Let P be a symmetric non-empty polytope with constraints $-b_i \leq a_i^Tx \leq b_i$ for $1 \leq i \leq m$. Prove that the Dikin ellipsoid $\mathcal{E}$ satisfies $\mathcal{E} \subseteq P \subseteq \sqrt{2m}\mathcal{E}$.