You are allowed to discuss with others but not allowed to use any references except the course notes and the books "Probability and Computing" and "Randomized Algorithms". Please list your collaborators for each question. This will not affect your marks. In any case, you must write your own solutions.

The full mark is 50.

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1. **Bipartite Matching**

   (10 marks) Consider the following linear program for the bipartite perfect matching problem, where $x(e)$ is a variable for an edge e and $\delta(v)$ is the set of edges incident on v.

$$\begin{aligned} \sum_{e \in \delta(v)} x(e) &= 1 \quad \forall v \in V \\ x(e) &\geq 0 \end{aligned}$$

   Suppose we are given a feasible fractional solution to this linear program. Show that we can obtain a perfect matching (i.e. an integral solution with $x(e) \in \{0, 1\}$) in $\tilde{O}(m)$ time where m is the number of edges in the graph. (Hint: Extend the approach for regular bipartite perfect matching in L12.)

2. **Cover Time**

   (10 marks) Prove that the cover time of a simple connected undirected regular graph is $O(n^2 \log n)$.

3. **Graph Partitioning by Random Walks**

   (10 marks) Let $G = (V, E)$ be an undirected d-regular graph and $S \subseteq V$ be a subset of vertices with $|S| \leq |V|/2$. Let $p_t = W^t p_0$ where $W = \frac{1}{2}(I + \frac{A}{d})$ is the random walk matrix (where A is the adjacency matrix) and p_0 is the initial distribution.

   (a) Prove that there is a vertex v in S such that if we start the random walk at v (i.e. $p_0 = \chi_v$), then $\sum_{i \in S} p_t(i) \geq 1 - t \cdot \phi(S)$ for any $t \geq 0$.

   (b) In the second part, we assume a stronger result that there is a vertex v in S such that if we start the random walk at v, then $\sum_{i \in S} p_t(i) \geq (1 - \phi(S))^t$ for any $t \geq 0$.

   Use this stronger result to argue that the random walk algorithm has the following performance guarantee when there is a small set S of small conductance. More precisely, say $|S| = |V|^{0.99}$, prove that we can use the random walk algorithm in L14 to find a set S' with conductance $O(\sqrt{\phi(S)})$ and $|S'| \leq |V|/2$.

4. **Card Shuffling**

   (10 marks) We study the following Markov chain on shuffling n cards. In each step, we pick two random cards and exchange their positions.

   (a) Prove that this Markov chain will converge to the uniform distribution (of all permutations) from any initial permutation.

   (b) Prove that $\tau(\epsilon) \leq O(n^2/\epsilon)$, where $\tau(\epsilon)$ denotes the minimum time t such that $d_{\mathrm{TV}}(p^t - \pi) \leq \epsilon$.

5. **Algebraic Matching**

   (10 marks) Given a bipartite graph where each edge is red or blue and a parameter k, design an algorithm that determines if there is a perfect matching with exactly k red edges in $O(n^\omega)$ field operations with high probability, when the field size is $\Theta(\text{poly}(n))$.

   In this problem, you can assume that the determinant of a matrix where each entry is an element in $\mathbb{F}[x]$ of degree at most d can be computed in $O(dn^\omega)$ field operations, where $\mathbb{F}[x]$ is the set of single variate polynomials with coefficients in a finite field $\mathbb{F}$ and $\omega \approx 2.37$ is the matrix multiplication exponent. Note that the determinant of such a matrix would be a single variate polynomial, and you can assume that the list of coefficients of this polynomial will be returned to you in $O(dn^\omega)$ field operations.