You are allowed to discuss with others but not allowed to use any references except the course notes and the books "Probability and Computing" and "Randomized Algorithms". Please list your collaborators for each question. This will not affect your marks. In any case, you must write your own solutions.

The full mark is 50.

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1. **Graph Coloring**

   (10 marks) Let $G = (V, E)$ be an undirected graph and suppose each $v \in V$ is associated with a set $S(v)$ of $8r$ colors, where $r \geq 1$. Suppose further that for each $v \in V$ and $c \in S(v)$ there are at most r neighbors u of v such that c lies in $S(u)$. In class, we used local lemma to prove that there exists a proper coloring of G assigning to each vertex v a color from its class $S(v)$ such that, for any edge $(u, v) \in E$, the colors assigned to u and v are different. In this question, you are asked to provide a polynomial time randomized algorithm to find such a coloring. (It is okay if you need to increase $8r$ to say $32r$ for the analysis to work.)

2. **Discrepancy Minimization: Lower Bound**

   (10 marks) Prove that there exists a set system with n elements $[n]$ and n sets $S_1, \ldots, S_n$ such that any coloring $\chi : [n] \to \{-1, +1\}$ must have discrepancy $\max_j |\sum_{i \in S_j} \chi(i)| = \Omega(\sqrt{n})$.

3. **Discrepancy Minimization: Upper Bound**

   (10 marks) Let $H = (V, \mathcal{E})$ be a t-uniform hypergraph (i.e. each hyperedge is of size t) where $\deg_H(v) \leq t$ for all $v \in V$ (i.e. each vertex belongs to at most t hyperedges). Prove that there exists an assignment $\chi : V \to \{-1, +1\}$ so that for every hyperedge $e \in \mathcal{E}$ it holds that

$$\Big| \sum_{i \in e} \chi(i) \Big| \leq O(\sqrt{t \log t}).$$

4. **Hypercube**

   (10 marks) Let $U = \{1, 2, \ldots, 6\}^n$. Define the Hamming distance between $x, y \in U$ as $H(x, y) := |\{i \in [n] \mid x_i \neq y_i\}|$, and define $H(x, A) := \min\{H(x, y) \mid y \in A\}$ for any subset $A \subseteq U$. We would like to prove that if A is large, then most of the points in U will be very close to A. Concretely, consider any subset $A \subseteq U$ with $|A| \geq 6^{n-1}$, prove that for any $c > 0$,

$$\frac{\big|\{x \in U \mid H(x, A) \leq (c+2)\sqrt{n}\}\big|}{6^n} \geq 1 - e^{-c^2/2}.$$

5. **Method of Bounded Differences**

(10 marks) Let $f(X_1, \ldots, X_n)$ satisfy the c-Lipschitz condition so that, for any i and any values $x_1, \ldots, x_n$ and y_i,

$$|f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)| \leq c.$$

Let $Z_0 = \mathbb{E}[f(X_1, X_2, \ldots, X_n)]$ and $Z_i = \mathbb{E}[f(X_1, X_2, \ldots, X_n) \mid X_1, X_2, \ldots, X_i]$. Give an example to show that, if the X_i are not independent, then it is possible that $|Z_i - Z_{i-1}| > c$.