CS 761 Randomized Algorithms **Last Updated:** July 19, 2018
Homework 4 **Due Date: July 27 (11pm), 2018**

You are allowed to discuss with others but not allowed to use any references except the course notes and the books "Probability and Computing" and "Randomized Algorithms". Please list your collaborators for each question. This will not affect your marks. In any case, you must write your own solutions.

There are totally 75 marks, and the full mark is 60.

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1. **Fractional Flow**

   (10 marks) Let $G = (V, E)$ be a directed graph and $s, t \in V$ and k is a positive integer. A fractional s-t flow solution with value k is an assignment of each edge e to a fractional value $x(e) \in [0, 1]$, satisfying that

$$\sum_{e \in \delta^{\text{out}}(s)} x(e) = \sum_{e \in \delta^{\text{in}}(t)} x(e) = k$$

   and

$$\sum_{e \in \delta^{\text{in}}(v)} x(e) = \sum_{e \in \delta^{\text{out}}(v)} x(e) \quad \forall v \in V - \{s, t\},$$

   where $\delta^{\text{in}}(v)$ denotes the set of directed edges with v as the head (incoming edges of v), and $\delta^{\text{out}}(v)$ denotes the set of directed edges with v as the tail (outgoing edges of v). Given a fractional s-t flow solution with value k, design a randomized algorithm to return an integral s-t flow with value k in $\tilde{O}(|E|)$ time, i.e. x satisfies the above constraints and moreover $x(e) \in \{0, 1\}$ for all $e \in E$.

2. **k-SAT**

   (10 marks) Generalize the randomized algorithm for 3-SAT to k-SAT. What is the expected time of the algorithm as a function of k?

   Remarks: (i) you can assume that the number of clauses is bounded by a polynomial in n where n is the number of variables; (ii) you will get most of the marks if you can beat 2^n for constant k.

3. **Cover Time**

   (10 marks) Prove that the cover time of a simple connected undirected regular graph is $O(n^2 \log n)$.

4. **Graph Partitioning by Random Walks**

   (15 marks) Let $G = (V, E)$ be an undirected d-regular graph and $S \subseteq V$ be a subset of vertices with $|S| \leq |V|/2$. Let $p_t = W^t p_0$ where $W = AD^{-1}$ is the random walk matrix (D is the diagonal degree matrix and A is the adjacency matrix) and p_0 is the initial distribution.

   (a) Prove that there is a vertex v in S such that if we start the random walk at v (i.e. $p_0 = \chi_v$), then $\sum_{i \in S} p_t(i) \geq 1 - t \cdot \phi(S)$ for any $t \geq 0$.

   (b) In the second part, we assume a stronger result that there is a vertex v in S such that if we start the random walk at v, then $\sum_{i \in S} p_t(i) \geq (1 - \phi(S))^t$ for any $t \geq 0$.

   Use this stronger result to argue that the random walk algorithm has the following performance guarantee when there is a small set S of small conductance. More precisely, say $|S| = |V|^{0.99}$, prove that we can use the random walk algorithm in L16 to find a set S' with conductance $O(\sqrt{\phi(S)})$ and $|S'| \leq |V|/2$.

5. **DNF counting**

   (15 marks) In this question, we generalize the DNF counting result in L17 to the setting where each variable is set to be true with probability p independently and we would like to approximate the probability that the DNF formula is satisfiable.

   For each clause C_i, let a_i and b_i be the number of unnegated and negated literals in C_i. We define $w_i = p^{a_i}(1-p)^{b_i}$ and $W = \sum_i w_i$.

   Consider the following algorithm. Choose a random clause C_i with probability of choosing C_i equal to w_i/W. Then, choose a random assignment σ satisfying C_i, i.e. each variable not appearing in C_i is set to be true with probability p independently. Let $N(\sigma)$ be the number of clauses satsified by the assignment σ. Return $P := W/N(\sigma)$ as our approximate answer.

   Prove that the expected value of P is exactly the probability that the DNF formula is satisfiable. Use this to obtain an (ϵ, δ)-approximation for the probability that the DNF formula is satisfiable and analyze its time complexity.

6. **Card Shuffling**

   (15 marks) We study the following Markov chain on shuffling n cards. In each step, we pick two random cards and exchange their positions.

   (a) Prove that this Markov chain will converge to the uniform distribution (of all permutations) from any initial permutation.

   (b) Prove that $\tau(\epsilon) \leq O(n^2/\epsilon)$, where $\tau(\epsilon)$ denotes the minimum time t such that $d_{\mathrm{TV}}(p^t - \pi) \leq \epsilon$.