CS 466/666 Algorithm Design and Analysis — **Last Updated:** July 11, 2022
Homework 5 — **Due Date: July 25, 2022**

You are allowed to discuss with others but not allowed to use any references except the course notes and the books "Probability and Computing" and "Randomized Algorithms". Please list your collaborators for each question. This will not affect your marks. In any case, you must write your own solutions.

There are totally 62 marks, and the full mark is 40. This homework is counted 8% of the course. The extra marks will not be carried to other parts of the course. Please read the course outline for the late submission policy.

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1. **k-th Eigenvalue**

   (10 marks) In this question, we assume that the given undirected graph is d-regular. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of the normalized Laplacian matrix. Suppose $S_1, S_2, \ldots, S_k \subseteq V$ are pairwise disjoint sets with $\phi(S_i) \leq \phi$ for $1 \leq i \leq k$. Prove that $\lambda_k \leq 2\phi$.

   Hint: Use the following version of the Courant-Fischer theorem to upper bound λ_k. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \leq \lambda_n$ be the eigenvalues of the normalized Laplacian matrix $\mathcal{L}$. Then

   $$\lambda_k = \min_{S \subseteq \mathbb{R}^n : \dim(S) = k} \max_{x \in S, x \neq 0} \frac{x^T \mathcal{L} x}{x^T x}.$$

2. **Page Ranking**

   (12 marks) Suppose someone searches a keyword (say "car") and we would like to identify the webpages that are the most relevant for this keyword and the webpages that are the most reliable sources for this keyword (a page is a reliable source if it points to many of the most relevant pages). First we identify the pages with this keyword and ignore all other pages. Then we run the following ranking algorithm on the remaining pages. Each vertex corresponds to a remaining page, and there is a directed edge from page i to page j if there is a link from page i to page j. Call this directed graph $G = (V, E)$. For each vertex i, we have two values $s(i)$ and $r(i)$, where intendedly $r(i)$ represents how relevant is this page and $s(i)$ represents how reliable it is as a source (the larger the values the better). We start from some arbitrary initial values, say $s(i) = 1/|V|$ for all i, as we have no ideas at the beginning. At each step, we update s and r (where s and r are vectors of $s(i)$ and $r(i)$ values) as follows: First we update $r(i) = \sum_{j : ji \in E} s(j)$ for all i, as a page is more relevant if it is linked by many reliable sources. Then we update $s(i) = \sum_{j : ij \in E} r(j)$ for all i (using the just updated values $r(j)$), as a page is a more reliable source if it points to many relevant pages. To keep the values small, we let $R = \sum_{i=1}^{|V|} r(i)$ and $S = \sum_{i=1}^{|V|} s(i)$, and divide each $s(i)$ by S and divide each $r(i)$ by R. We repeat this step for many times to refine the values.

   Let $s, r \in \mathbb{R}^{|V|}$ be the vectors of the s and r values. Give a matrix formulation for computing s and r. Suppose G is weakly connected (when we ignore the direction of the edges, the underlying undirected graph is connected) and there is a self-loop at each vertex. Prove that there is a unique limiting s and a unique limiting r for any initial s as long as $s \geq 0$ and $s \neq 0$.

   Hint: You may use the Perror-Frobenius theorem which states that for any irreducible matrix, there is a unique positive eigenvalue with maximum absolute value and the entries of the corresponding eigenvector are all positive.

3. **Hitting Time**

   (10 marks) Consider a random walk on an undirected graph $G = (V, E)$ that starts at a vertex $v \in V$, and stops when it reaches s or t. Let $p(v)$ be the probability that if the random walk starts at v then it reaches s before t. Establish a connection between these probabilities and some parameters of an appropriate electrical flow problem.

4. **Cat and Mouse**

   (10 marks) A cat and a mouse each independently take a random walk on a connected, undirected, non-bipartite graph G. They start at the same time on different nodes, and each makes one transition at each time step. The cat catches the mouse if they are ever at the same node at some time step. Prove the best upper bound that you can on the expected time before the cat catches the mouse, where n is the number of vertices and m is the number of edges of G.

5. **Spanning Trees**

   Let $G = (V, E)$ be an undirected graph.

   (a) (10 marks) Let T be a uniform random spanning tree of G. Prove that $\Pr(e \in T) = R_{\text{eff}}(e)$ for any $e \in E$, where $R_{\text{eff}}(e)$ is the effective resistance of e when every edge has resistance 1.

   Hint: You may use the equation $\det(M + xx^T) = (1 + x^T M^{-1} x)\det(M)$, for any non-singular matrix $M \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$.

   (b) (Bonus 10 marks) Let T be a uniform random spanning tree of G. Prove that, for any two edges $e, f \in E$,

   $$\Pr(e \in T \mid f \in T) \leq \Pr(e \in T).$$

   In words, conditioned on the event $f \in T$, the probability of the event $e \in T$ could not increase, i.e., the events $e \in T$ and $f \in T$ are negatively correlated.