CS 466/666 Algorithm Design and Analysis **Last Updated:** June 27, 2022
Homework 4 **Due Date: July 11, 2022**

You are allowed to discuss with others but not allowed to use any references except the course notes and the books "Probability and Computing" and "Randomized Algorithms". Please list your collaborators for each question. This will not affect your marks. In any case, you must write your own solutions.

There are totally 52 marks, and the full mark is 40. This homework is counted 8% of the course. The extra marks will not be carried to other parts of the course. Please read the course outline for the late submission policy.

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1. **Graph Coloring**

   (12 marks) Let $G = (V, E)$ be an undirected graph and suppose each $v \in V$ is associated with a set $S(v)$ of $32r$ colors, where $r \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most r neighbors u of v such that c lies in $S(u)$.

   Use local lemma to prove that there exists a proper coloring of G assigning to each vertex v a color from its class $S(v)$ such that, for any edge $(u, v) \in E$, the colors assigned to u and v are different.

   Furthermore, give a polynomial time randomized algorithm to find such a coloring.

2. **Balanced Coloring**

   (10 marks) Suppose we are given a ground set of n elements $U = \{u_1, u_2, \ldots, u_n\}$ and m subsets $S_1, \ldots, S_m \subseteq U$, where each S_i has exactly t elements and each element belongs to at most t subsets. Our goal is to find a 2-coloring of the ground set, say red and blue, so that each subset has roughly the same number of red and blue elements. Show that there exists a 2-coloring in which every subset S_i has at least $t/2 - c\sqrt{t \log t}$ red elements and at most $t/2 + c\sqrt{t \log t}$ red elements for some absolute constant c.

3. **Bipartite Matching**

   (10 marks) Consider the following linear program for the bipartite perfect matching problem, where $x(e)$ is a variable for an edge e and $\delta(v)$ is the set of edges incident on v.

$$\begin{aligned} \sum_{e \in \delta(v)} x(e) &= 1 \quad \forall v \in V \\ x(e) &\geq 0 \end{aligned}$$

   Suppose we are given a feasible fractional solution to this linear program. Show that we can obtain a perfect matching (i.e. an integral solution with $x(e) \in \{0, 1\}$) in $\tilde{O}(m)$ time where m is the number of edges in the graph.

4. **Bipartite Graphs**

(10 marks) Let G be a connected graph, and $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$ be the eigenvalues of the adjacecny matrix A of G. Prove that $\alpha_1 = -\alpha_n$ if and only if G is bipartite.

Hint: You may use the Perror-Frobenius theorem which states that for any irreducible matrix, there is a unique positive eigenvalue with maximum absolute value and the entries of the corresponding eigenvector are all positive.

5. **Spanning Trees**

(10 marks) Let $G = (V, E)$ be an undirected graph.

(a) Let $V = \{1, \ldots, n\}$, $e = ij$, and b_e be the n-dimensional vector with $+1$ in the i-th entry and -1 in the j-th entry and 0 otherwise. Let B be an $n \times m$ matrix where the columns are b_e and m is the number of edges in G. Prove that the determinant of any $(n-1) \times (n-1)$ submatrix of B is ± 1 if and only if the $n-1$ edges corresponding to the columns form a spanning tree of G (and otherwise the determinant is zero).

(b) Let L be the Laplacian matrix of G and let L' be the matrix obtained from L by deleting the last row and last column. Use (a) to prove that $\det(L')$ is equal to the number of spanning trees in G.

Hint: You can use the Cauchy-Binet formula (see wikipedia) to solve this problem.