CS 466/666 Algorithm Design and Analysis **Last Updated:** May 15, 2020
Homework 1 **Due Date: June 1 (11pm), 2020**

You are allowed to discuss with others but not allowed to use any references except the course notes and the books "Probability and Computing" and "Randomized Algorithms". Please list your collaborators for *each* question. This will not affect your marks. In any case, you must write your own solutions.

There are totally 66 marks, and the full mark is 50. The extra marks will not be carried to other parts of the course. This homework is counted 10% of the course. Please read the course outline for the late submission policy.

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1. **Minimum Cuts**

   (10 marks) One advantage of Karger's random contraction algorithm for the minimum cut problem is that it can be used to output all minimum cuts. In this question, we assume Karger's algorithm as a black box, which can be used to output a minimum cut with probability at least $2/n(n-1)$ in time $O(n^2)$, where n is the number of vertices in the input graph. Explain how Karger's algorithm can be used to output all minimum cuts and analyze its running time to output all minimum cuts with success probability at least 0.9999.

2. **Minimum k-cut**

   (12 marks) Generalizing on the notion of a cut-set, we define a k-way cut-set in an undirected graph as a set of edges whose removal leaves the graph into k or more connected components. Show that the randomized contraction algorithm can be modified to find a minimum k-way cut-set in $n^{O(k)}$ time.

3. **Online Hiring**

   (10 marks) We need to hire a new staff. There are n applicants for this job. Assume that we will know how good they are (as a score) when we interview them, and the score for each applicant is different. So there is a unique candidate with the highest score, but we don't know that the applicant is the best when we interview him/her until we have interviewed all the applicants. The problem is that after we interview one applicant, we need to make an online decision to either give him/her an offer or forever lose the chance to hire that applicant. Suppose the applicants come in a random order (i.e. a uniformly random permutation), and we would like to come up with a strategy to hire the best applicant.

   Consider the following strategy. First, interview m applicants but reject them all. Then, after the m-th applicant, hire the first applicant you interview who is better than all of the previous applicants that you have interviewed.

   Let E be the event that you hire the best applicant. Let E_i be the event that the i-th applicant is the best and you hire him/her. Compute $\Pr(E_i)$ and show that

   $$\Pr(E) = \frac{m}{n} \sum_{j=m+1}^{n} \frac{1}{j-1}.$$

   Then, show that $\Pr(E) \geq \frac{m}{n}(\ln n - \ln m)$, and that $\Pr(E)$ can be arbitrarily close to $1/e$ for an appropriate choice of m when n tends to infinity.

4. **Random Permutations**

   (10 marks) A permutation $\pi : [n] \to [n]$ can be represented as a set of cycles as follows. Let there be one vertex for each number i for $1 \leq i \leq n$. If the permutation maps the number i to the number $\pi(i)$, then a directed arc is drawn from vertex i to vertex $\pi(i)$. This leads to a graph that is a set of disjoint directed cycles. Notice that some of the cycles could be self-loops. What is the expected number of cycles in a random permutation of n numbers?

5. **Quicksort**

   (12 marks) The expected runtime of the randomized quicksort algorithm is at most $2n \ln n$ steps where n is the number of elements to be sorted (see Section 2.5 of "Probability and Computing" for a proof). Prove that the probability that the actual runtime is more than say $100n \ln n$ is at most inverse polynomial in n. (Hint: Bound the probability that the recursion depth is large.)

6. **Chernoff Bound** (Bonus)

   (12 marks) Recall that X is a standard normal random variable if its probability density function is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

   (a) Compute the moment generating function $M_{X^2}(t) := E[e^{tX^2}]$. You may use the fact that $\int_{-\infty}^{\infty} f(x) = 1$ without providing a proof.

   (b) Use (a), or otherwise, to compute $E[X^4]$.

   (c) Let $X_1, \ldots, X_k$ be independent standard normal random variables, and $Y = \frac{1}{k} \sum_{i=1}^{k} X_i^2$. Derive a Chernoff bound to prove that $\Pr(Y > 1 + \epsilon) \leq e^{-k\epsilon^2/8}$ for $0 \leq \epsilon \leq 1$. You may use the Taylor expansion $\ln(1 - x) = -\sum_{i=1}^{\infty} x^i / i$ for $-1 \leq x \leq 1$.