CS 466 / 666 Algorithm Design and Analysis . Spring 2020

Lecture 19 : Spanning tree polytopes

We show that the spanning tree polytope is described by the subtour elimination LP.

The proof uses the uncrossing technique that underlies many results in combinatorial optimization.

Then we see an application in designing an additive approximation algorithm for bounded degree spanning trees.

The subtour elimination LP

The minimum spanning tree problem can be formulated by the following linear program.

$$\min \sum_e c_e x_e$$

$$x(E(S)) \leq |S|-1 \qquad \forall S \subset V \qquad \text{(subtour elimination constraints)}$$

$$x(E(V)) = |V|-1$$

$$x_e \geq 0$$

Recall that $E(S)$ denotes the set of edges with both endpoints in S, and $x(E(S))$ denotes $\sum_{e \in E(S)} x_e$.

We showed in L15 that there is a polynomial time seperation oracle to determine if a point x

satisfies all the constraints or not, and if not output a violating constraint.

This implies that the exponential-sized subtour LP can be solved in polynomial time by the ellipsoid method.

Integrality.

We would like to show that the LP is integral, again using the rank argument, but this LP has

exponentially many constraints and a simple argument like last time won't work.

Somewhat surprisingly, one can show that the rank of the subtour elimination constraints is small.

Theorem The number of linearly independent subtour elimination constraints is at most $|V|-1$.

With this theorem, then we can easily argue that the LP is integral as follows.

Let x be a basic optimal solution. First, we remove all edges with $x_e = 0$.

Then, by the theorem, there could be at most $|V|-1$ linearly independent tight constraints, and so

there could be at most $|V|-1$ non-zero edges left.

Note that every edge uv has $x_{uv} \leq 1$, as it is implied by the constraint $x(E(\{u,v\})) \leq |\{u,v\}|-1 = 1$.

Therefore, to satisfy the constraint $x(E(V)) = |V|-1$, we must have exactly $n-1$ edges with value one,

and hence a basic solution corresponds exactly to a spanning tree.

Basis of tight constraints

To prove the theorem, we need to understand more about the tight constraints.

We say $S \subseteq V$ is a tight set if $x(E(S)) = |S| - 1$.

Given $F \subseteq E$, we define χ_F as the characteristic vector in $\mathbb{R}^{|E|}$ with one for each edge in F and zero otherwise.

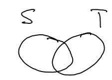
Note that $\chi_{E(S)}$ is the row vector in the constraint matrix for the constraint $x(E(S)) = |S| - 1$,

and we need to reason about their linear dependency.

Laminar basis

To prove that the rank is small, we are going to show that there exists a basis (i.e. a maximum set of linearly independent constraints) with a special structure, for which we can upper bound its cardinality.

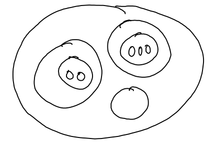
For this special structure, we need more definitions.



We say two sets $S, T \subseteq V$ are intersecting if $S-T$, $T-S$, and $S \cap T$ are all non-empty.

A family $\mathcal{L}$ of sets is a laminar family if there are no two intersecting subsets.

In other words, $\mathcal{L}$ is a laminar family if $S, T \in \mathcal{L}$ implies that either $S \subseteq T$, or $T \subseteq S$, or $S \cap T = \phi$.



A simple induction can prove the following statement.

Fact Let $\mathcal{L}$ be a laminar family on a ground set of n elements.

If $\mathcal{L}$ does not have singletons (i.e. sets of one element), then $|\mathcal{L}| \leq n-1$.

Laminarity is the special structure that we are looking for.

Note that in our LP there are no tight sets of size one (since those constraints are vacuous).

With the above fact, the above theorem would follow from the following theorem.

Theorem Let $\mathcal{F} = \{ S \mid x(E(S)) = |S| - 1 \}$ be the set of all tight sets.

Let $\operatorname{span}(\mathcal{F}) = \{ \chi_{E(S)} \mid S \in \mathcal{F} \}$ be the linear subspace in $\mathbb{R}^n$ spanned by the characteristic vectors in $\mathcal{F}$.

There exists a laminar family $\mathcal{L} \subseteq \mathcal{F}$ such that $\operatorname{span}(\mathcal{L}) = \operatorname{span}(\mathcal{F})$.

That is, there is a laminar family $\mathcal{L} \subseteq \mathcal{F}$ such that every characteristic vector $\chi_{E(S)}$ for $S \in \mathcal{F}$ can be written as a linear combination of the characteristic vectors $\chi_{E(T)}$ for $T \in \mathcal{L}$.

Or more simply, every row corresponds to a tight set in $\mathcal{F}$ can be written as a linear combination of the rows corresponding to tight sets in $\mathcal{L}$.

So, this laminar family $\mathcal{L}$ is a basis of the tight constraints.

Since such a laminar family (without singletons) can have at most $|V|-1$ sets, it follows that there are at most $|V|-1$ linearly independent tight constraints, and the previous theorem follows.

Uncrossing techniques

Our goal now is to construct a laminar basis, i.e. a basis with no intersecting sets.

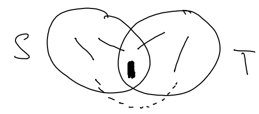
The uncrossing technique would allow us to "uncross" the intersecting sets.

The following claim makes use of the graph structure.

Claim For $S, T \subseteq V$, $\chi_{E(S)} + \chi_{E(T)} \leq \chi_{E(S\cap T)} + \chi_{E(S\cup T)}$.

Furthermore, equality holds if and only if $E(S-T, T-S) = \phi$, where $E(S-T, T-S)$ denotes the set of edges with one vertex in $S-T$ and another vertex in $T-S$.

proof (by picture)



- thin edges contribute 1 to both sides.
- thick edges contribute 2 to both sides.
- dotted edges contribute 0 to LHS but 1 to RHS.

Therefore, the inequality holds, and equality holds if and only if there are no dotted edges. □

The following lemma says that tight sets are closed under intersection and union, and furthermore there is linear dependency between them.

Informally, this will help us in replacing two intersecting tight sets by two non-intersecting tight sets.

Lemma If $S, T \subseteq \mathcal{F}$ and $S \cap T \neq 0$, then both $S\cap T$ and $S \cup T$ are in $\mathcal{F}$.

Furthermore, $\chi_{E(S)} + \chi_{E(T)} = \chi_{E(S\cap T)} + \chi_{E(S\cup T)}$.

Proof Since S, T are tight, we have

$$\begin{aligned} |S|-1+|T|-1 &= x(E(S)) + x(E(T)) \\ &\leq x(E(S\cap T)) + x(E(S\cup T)) \quad \text{(by the above claim)} \\ &\leq |S\cap T|-1 + |S\cup T|-1 \quad \text{(by LP constraints)} \\ &= |S|-1+|T|-1. \end{aligned}$$

So, equalities hold throughout.

The second inequality holds as equality implies that both $S\cap T$ and $S\cup T$ are tight sets.

The first inequality holds as equality implies the linear dependency by the above Claim as there are no edges with $x_e = 0$. □

The following proposition implies the second theorem.

Proposition If $\mathcal{L}$ is a maximal laminar subfamily of $\mathcal{F}$, then $\text{span}(\mathcal{L}) = \text{span}(\mathcal{F})$.

<u>proof</u> If $\mathcal{L}$ is a maximal laminar family, then every set $R \in \mathcal{F}$ but $R \notin \mathcal{L}$ must intersect some set in $\mathcal{F}$.

Let intersect $(\mathcal{L}, R)$ be the number of sets in $\mathcal{L}$ that intersect with R.

Suppose there exists a set $R \in \mathcal{F}$ but $R \notin \text{span}(\mathcal{L})$.

We consider such a R with minimum intersect $(\mathcal{L}, R)$ among all such sets.

Say R intersects $S \in \mathcal{L}$.

Consider $S \cap R$ and $S \cup R$. Since $S, R \in \mathcal{F}$, both $S \cap R$ and $S \cup R$ are in $\mathcal{F}$ by the lemma.

We will prove that both intersect $(\mathcal{L}, S \cap R) <$ intersect $(\mathcal{L}, R)$ and intersect$(\mathcal{L}, S \cup R) <$ intersect$(\mathcal{L}, R)$.

Assume this is true for now.

By the choice of R, we must have both $S \cap R \in \text{span}(\mathcal{L})$ and $S \cup R \in \text{span}(\mathcal{L})$.

But then $\chi_{E(S)} + \chi_{E(R)} = \chi_{E(S \cap R)} + \chi_{E(S \cup R)}$ by the lemma, and since $S, S \cap R, S \cup R$ are in $\text{span}(\mathcal{L})$, it implies that R is also in $\text{span}(\mathcal{L})$, a contradiction.

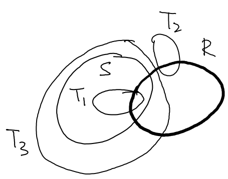
So, there is no R with $R \in \mathcal{F}$ but $R \notin \text{span}(\mathcal{L})$, and we are done.

To complete the proof, it remains to prove the following claim.

<u>Claim</u> If R intersects $S \in \mathcal{L}$, then both intersect $(\mathcal{L}, S \cap R)$, intersect $(\mathcal{L}, S \cup R) <$ intersect $(\mathcal{L}, R)$.

<u>proof</u> Consider a set $T \in \mathcal{L}$ and $T \neq S$.

Since $T \in \mathcal{L}$, it does not intersect with S.



So, whenever it intersects with $S \cap R$ (case 1) or $S \cup R$ (case 2 or case 3), it must intersect with R.

This implies intersect$(\mathcal{L}, S \cap R) \leq$ intersect $(\mathcal{L}, R)$ and intersect $(\mathcal{L}, S \cup R) \leq$ intersect$(\mathcal{L}, R)$.

Furthermore, both $S \cap R$ and $S \cup R$ do not intersect with S, but R does, and so this implies that we have strict inequalities. $\square$

The claim finishes the proof of the proposition. $\square$

Therefore, we have completed the proof that the subtour elimination LP is integral.

<u>Submodularity</u>

A function $f: 2^V \to \mathbb{R}$ that satisfies $f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$ $\forall S, T$ is called a submodular function.

One important example is the cut function : $x(\delta(S)) + x(\delta(T)) \geq x(\delta(S \cap T)) + x(\delta(S \cup T))$.

It is not an exaggeration to say that every polynomial time soluable combinatorial optimization problems has something to do with submodularity, and the uncrossing technique is used to handle it.

See the encyclopedia of Schrijver to illustrate this point.

Minimum bounded degree spanning trees

In this problem, we are given a degree upper bound B_v on each vertex, and our goal is to find a spanning tree with minimum cost that satisfies all the degree constraints.

This is NP-hard as the Hamiltonian path problem is a special case, and there is no reasonable approximation algorithms if we insist on satisfying all degree constraints.

But we can do something interesting if we allow small violations of the degree constraints.

Theorem: If there is a spanning tree with cost C that satisfies $\deg(v) \le B_v$ for all $v \in V$, then there is a polynomial time algorithm to find a spanning tree of cost $\le C$ and $\deg(v) \le B_v + 2$ $\forall v \in V$.

A corollary is that we can find a minimum spanning tree with max degree at most 2 more than the optimal.

Linear programming relaxation

$$\min \sum_{e \in E} c_e x_e$$

$$x(E(S)) \le |S| - 1 \qquad \forall S \subseteq V$$

$$x(E(V)) = |V| - 1$$

$$x(\delta(v)) \le B_v \qquad \forall v \in W \subseteq V \qquad (W \text{ is the set of degree constraints})$$

$$x_e \ge 0 \qquad \forall e \in E$$

Iterative algorithm

Initially $F = \phi$ (F is the partial solution so far)

While $|V| \ge 2$ do

① Compute a basic optimal solution to the LP. Remove all edges with $x_e = 0$.

② If there is a vertex u with only one edge uv incident on it, then $F \leftarrow F + \{uv\}$, $V \leftarrow V - \{v\}$, $B_u \leftarrow B_u - 1$.

③ If there is a vertex v with $\deg(v) \le 3$, remove the degree constraint on v, i.e. $W \leftarrow W - \{v\}$.

Return F.

Performance guarantee

We first assume that the iterative algorithm will terminate successfully, and we will prove later that it is always the case.

In step ②, note that $x_{uv} = 1$, because the constraints $x(E(V)) = |V| - 1$ and $x(E(V - \{v\})) \le |V| - 2$ actually implies the constraint $x(\delta(v)) \ge 1$, and since v is of degree 1, we have $x_{uv} = 1$.

Informally, the algorithm is trying to find a leaf to add to the spanning tree.

Therefore, we only choose those edges with $x_e = 1$, and by a simple inductive argument one can prove that the returned solution has cost at most the LP optimal value.

It remains to argue about the degree violation.

In step ③, we remove a degree constraint when there are at most three edges incident on it.

Note that at any time $B_v \geq 1$ $\forall v \in W$.

Therefore, in the worst case, we use all three edges and violate the degree constraint by two.

Analysis of basic solutions

We would like to prove that either there exists $x_e = 0$, or there is a degree one vertex, or there is a vertex with degree constraint and its degree is at most three.

Suppose to the contrary that none of these happen in a basic solution x.

Let W be the set of vertices with degree constraints

Then, $\deg(v) \geq 4$ for all $v \in W$ and $\deg(v) \geq 2$ for all $v \in V - W$.

This implies that $|E| \geq \frac{1}{2} \sum_v \deg(v) \geq \frac{1}{2}\left(\sum_{v \in W} 4 + \sum_{v \notin W} 2 \right) = \frac{1}{2}\left(4|W| + 2(|V| - |W|) \right) = |V| + |W|$.

On the other hand, how many tight constraints can we have?

There are at most $|V| - 1$ tight subtour elimination constraints by the theorem, and by definition there are at most $|W|$ tight degree constraints.

So, there are at most $|V| + |W| - 1$ tight constraints, fewer than the number of variables, contradicting to the fact that x is a basic solution. So, the algorithm must terminate.

Optimal algorithm

If we work harder (using laminarity, independence, and a better counting argument), then the degree violation can be decreased to at most one.

And the algorithm also becomes nicer.

While $W \neq \phi$

① Compute a basic optimal solution x. Remove all edges with $x_e = 0$.

② Remove the degree constraint on a vertex v if there are at most $B_v + 1$ edges incident on it.

Return an MST.

That is, the algorithm just keeps removing degree constraints one by one, and eventually just returns a minimum spanning tree.

See chapter 3 of "iterative methods in combinatorial optimization" if interested.

~~Open question~~: Design a "combinatorial" approximation algorithm with similar performance.

References

This "iterative rounding" approach is first introduced by Kamal Jain, who used it to solve a general network design problem and showed that any extreme point solution has an edge with value at least $\frac{1}{2}$, thereby obtaining a 2-approximation algorithm for the problem.

This approach works well when the problems being studied is a variant of some classical combinatorial optimization problems with integral LP formulations.