CS 466/666 Algorithm Design and Analysis — **Last Updated:** July 22, 2019
Homework 5 — **Due Date: August 2 (11pm), 2019**

You are allowed to discuss with others but not allowed to use any references except the course notes and the books "Probability and Computing" and "Randomized Algorithms". Please list your collaborators for each question. This will not affect your marks. In any case, you must write your own solutions.

There are totally 55 marks, and the full mark is 50. This homework is counted 8% of the course.

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1. **Bipartite Matching**

   (10 marks) Consider the following linear program for the bipartite perfect matching problem, where $x(e)$ is a variable for an edge e and $\delta(v)$ is the set of edges incident on v.

   $$\begin{aligned} \sum_{e \in \delta(v)} x(e) &= 1 \quad \forall v \in V \\ x(e) &\geq 0 \end{aligned}$$

   Suppose we are given a feasible fractional solution to this linear program. Show that we can obtain a perfect matching (i.e. an integral solution with $x(e) \in \{0,1\}$) in $\widetilde{O}(m)$ time where m is the number of edges in the graph.

   (Hint: Extend the approach for finding a perfect matching in a regular bipartite graph.)

2. **k-SAT**

   (10 marks) Generalize the randomized algorithm for 3-SAT to k-SAT. What is the expected time of the algorithm as a function of k?

   Remarks: (i) You can assume that the number of clauses is bounded by a polynomial in n where n is the number of variables; (ii) You will get full marks if you can beat 2^n for any constant k. You may get some bonus marks if your bound is close enough to state-of-the-art.

3. **Hitting probability**

   (10 marks) Consider a random walk on an undirected graph $G = (V, E)$ that starts at a vertex $v \in V$, and stops when it reaches s or t. Let $p(v)$ be the probability that if the random walk starts at v then it reaches s before t. Establish a connection between these probabilities and some parameters of an appropriate electrical flow problem.

4. **Cover Time**

   (10 marks) Prove that the cover time of a simple connected undirected regular graph is $O(n^2 \log n)$.

5. **Card Shuffling**

(15 marks) We study the following Markov chain on shuffling n cards. In each step, we pick two random cards and exchange their positions.

(a) Prove that this Markov chain will converge to the uniform distribution (of all permutations) from any initial permutation.

(b) Prove that $\tau(\epsilon) \le O(n^2/\epsilon)$, where $\tau(\epsilon)$ denotes the minimum time t such that $d_{\text{TV}}(p^t - \pi) \le \epsilon$.