CS 466/666 Algorithm Design and Analysis **Last Updated:** July 4, 2018
Homework 4 **Due Date: July 18 (11pm), 2018**

You are allowed to discuss with others but not allowed to use any references other than the course notes. Please list your collaborators for each question. This will not affect your marks. In any case, you must write your own solutions.

There are totally 55 marks, and the full mark is 50. This homework is counted 8% of the course.

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1. **Bipartite Graphs**

   (15 marks) Consider the adjacency matrix A of an undirected connected graph G. Let $\alpha_1 \geq \ldots \geq \alpha_n$ be the eigenvalues of A.

   (a) Show that
   $$\alpha_n = \min_{x \in \mathbb{R}^n} \frac{x^T A x}{x^T x}.$$

   (b) Prove that $\alpha_1 = -\alpha_n$ if and only if G is bipartite.
   Hint: You may use the Perror-Frobenius theorem which states that for any irreducible matrix, there is a unique positive eigenvalue with maximum absolute value and the entries of the corresponding eigenvector are all positive.

2. **k-th Eigenvalue**

   (10 marks) In this question, we assume that the given undirected graph is d-regular. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of the normalized Laplacian matrix. Suppose $S_1, S_2, \ldots, S_k \subseteq V$ are pairwise disjoint sets with $\phi(S_i) \leq \phi$. Prove that $\lambda_k \leq 2\phi$.

   Hint: Use the following version of the Courant-Fischer theorem to upper bound λ_k. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \leq \lambda_n$ be the eigenvalues of the normalized Laplacian matrix L. Then
   $$\lambda_k = \min_{S \subseteq \mathbb{R}^n : \dim(S) = k} \max_{x \in S} \frac{x^T L x}{x^T x}.$$

3. **Spanning Trees**

   (15 marks) Let $G = (V, E)$ be an undirected graph.

   (a) Let $V = \{1, \ldots, n\}$, $e = ij$, and b_e be the n-dimensional vector with $+1$ in the i-th entry and -1 in the j-th entry and 0 otherwise. Let B be an $n \times m$ matrix where the columns are b_e and m is the number of edges in G. Prove that the determinant of any $(n-1) \times (n-1)$ submatrix of B is ± 1 if and only if the $n-1$ edges corresponding to the columns form a spanning tree of G (otherwise the determinant is zero).

   (b) Let L be the Laplacian matrix of G and let L' be the matrix obtained from L by deleting the last row and last column. Use (a) to prove that $\det(L')$ is equal to the number of spanning trees in G.
   Hint: You can use the Cauchy-Binet formula (see wikipedia) to solve this problem.

4. **Page Ranking**

(15 marks) Suppose someone searches a keyword (like "car") and we would like to identify the webpages that are the most relevant for this keyword and the webpages that are the most reliable sources for this keyword (a page is a reliable source if it points to many of the most relevant pages). First we identify the pages with this keyword and ignore all other pages. Then we run the following ranking algorithm on the remaining pages. Each vertex corresponds to a remaining page, and there is a directed edge from page i to page j if there is a link from page i to page j. Call this directed graph $G = (V, E)$. For each vertex i, we have two values $s(i)$ and $r(i)$, where intentionally $r(i)$ represents how relevant is this page and $s(i)$ represents how reliable it is as a source (the larger the values the better). We start from some arbitrary initial values, say $s(i) = 1/|V|$ for all i, as we have no ideas at the beginning. At each step, we update s and r (where s and r are vectors of $s(i)$ and $r(i)$ values) as follows: First we update $r(i) = \sum_{j:ji \in E} s(j)$ for all i, as a page is more relevant if it is linked by many reliable sources. Then we update $s(i) = \sum_{j:ij \in E} r(j)$ for all i (using the just updated values $r(j)$), as a page is a more reliable source if it points to many relevant pages. To keep the values small, we let $R = \sum_{i=1}^{|V|} r(i)$ and $S = \sum_{i=1}^{|V|} s(i)$, and divide each $s(i)$ by S and divide each $r(i)$ by R. We repeat this step for many times to refine the values.

Let $s, r \in \mathbb{R}^{|V|}$ be the vectors of the s and r values. Give a matrix formulation for computing s and r. Suppose G is weakly connected (when we ignore the direction of the edges, the underlying undirected graph is connected) and there is a self-loop at each vertex. Prove that there is a unique limiting s and a unique limiting r for any initial s as long as $s \geq 0$ and $s \neq 0$.

As in question 1, you may use the Perror-Frobenius theorem which states that for any irreducible matrix, there is a unique positive eigenvalue with maximum absolute value and the entries of the corresponding eigenvector are all positive.