CS 466 / 666 : Algorithm Design and Analysis . Spring 2018 . Waterloo.

Lecture 14: Electrical networks

We present some basic results about electrical flows and effective resistances.

Then we show a connection to analyzing commute time and cover time of random walks in undirected graphs.

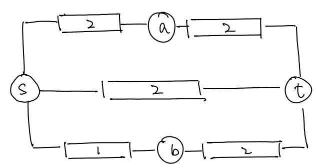
Electrical flows

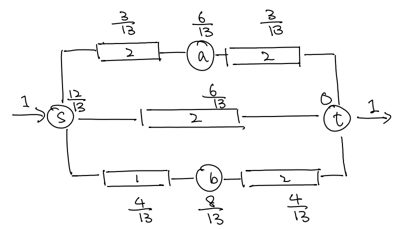
An electrical network is an undirected graph, where each edge e is a resistor with resistance r_e.

The electrical flows in these networks are governed by two rules:

- Kirchhoff's law : the sum of incoming currents is equal to the sum of outgoing currents.
- Ohm's law : there exists a voltage vector $\phi: V \to \mathbb{R}$ such that $\phi_u - \phi_v = f_{uv} r_{uv}$, where f_{uv} is the electrical flow across the edge uv, which is positive in the forward direction and negative in the backward direction.

For example, consider this network :





If one ampere is injected into s and one ampere is removed from t, then the voltages at the nodes and the currents on the resistors are shown in the figure on the right.

The first question is : how to compute the electrical flow of an electrical network ?

We will show that this can be done by solving a system of linear equations.

Before we show that, we first set up some notation for the matrix formulation of the problem.

Notation

Let $G = (V,E)$ be the underlying undirected graph of the electrical network, with $n := |V|$ and $m := |E|$.

Let $\phi \in \mathbb{R}^n$ be the vector of potentials at vertices.

Let $f(u,v)$ be the current flowing from vertex u to vertex v on an edge uv.

This is a directed quantity, and we define $f(v,u) = -f(u,v)$.

So, we interpret positive $f(u,v)$ as the flow going forward from u to, v and negative going backward

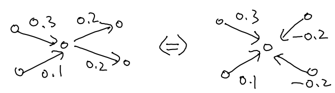
Let $f \in \mathbb{R}^m$ be the vector of currents flowing over the edges, where each edge $e=(u,v)$ only appears once as $f(u,v)$ where $u<v$ (assuming there is an ordering of the vertices).

Let $w_e = 1/r_e$ be the "conductance" of edge e.

Matrix formulation

The Ohm's law enforces that $f(u,v) = \dfrac{\phi(u)-\phi(v)}{r_{uv}} = w_{uv}(\phi(u)-\phi(v))$.

To see the Kirchhoff's law, first consider a vertex which is not a source or a sink, then the total incoming flow should be equal to the total outgoing flow.



Since $f(uv) = -f(vu)$, this is equivalent to that the total outgoing flow is equal to zero, i.e. $\sum_{u: vu \in E} f(vu) = 0$.

More generally, let b_u be the unit of currents injecting into u, i.e. b_u is positive for a source, negative for a sink, and zero for other vertices.

Then, the Kirchhoff's law enforces that $\sum_{u: vu \in E} f(vu) = b_v$.

Combining with Ohm's law, this gives $b_v = \sum_{u: vu \in E} f(vu) = \sum_{u: vu \in E} w_{uv}(\phi(v) - \phi(u)) = \deg_w(v)\phi(v) - \sum_{u: vu \in E} w_{uv}\phi(u)$,

where $\deg_w(v) = \sum_{u: uv \in E} w_{uv}$ is the weighted degree of vertex v.

Notice that when $w_{uv}=1$ for all edges uv, this can be written as $\vec{b} = L\vec{\phi}$ where L is the Laplacian matrix.

More generally, let L be the weighted Laplacian matrix where $L_{u,v} = \begin{cases} -w_{uv} & \text{if } u \neq v \\ \deg_w(u) & \text{if } u = v \end{cases}$,

$\vec{\phi} \in \mathbb{R}^n$ is the voltage vector and $\vec{b} \in \mathbb{R}^n$ is the "demand" vector.

Then, satisfying Ohm's law and Kirchhoff's law on every edge and every vertex can be compactly written as $L\vec{\phi} = \vec{b}$.

Computing voltages

To compute $\vec{\phi}$, we just need to solve a Laplacian system of linear equations.

This can certainly be solved in $O(n^3)$ time using Gaussian elimination, but it is now known that this can be solved in near-linear time. Hopefully we will have time to discuss this in some detail.

Pseudo-inverse of a matrix and the set of all solutions to $Lx=b$.

Notice that L is not of full rank, and so it is not invertible.

But if we assume without loss of generality that the graph is connected, then we know from L11 that nullspace$(L) = \vec{1}$. So, it is still possible to characterize the set of all solutions.

Let $x \in \mathbb{R}^n$ and we write $x = \sum_{i=1}^{n} c_i v_i$ where v_i is an orthonormal basis of eigenvectors of L with $v_1 = \vec{1}$.

Then $Lx = \sum_{i=1}^{n} c_i \lambda_i v_i = \sum_{i=2}^{n} c_i \lambda_i v_i$ as $\lambda_1 = 0$, and so Lx is always perpendicular to $\vec{1}$.

So, for $Lx = b$ to have a solution, it is necessary that b is perpendicular to $\vec{1}$.

Observe that this is always satisfied in the electrical flow problem, as $\sum_{v \in V} b_v = 0$ because the total currents injected to the electrical network is equal to the total currents removed from the network.

On the other hand, it is sufficient for $b \perp \vec{1}$ so that $Lx = b$ has a solution.

To see this, if $b \perp 1$, then b can be written as $b = \sum_{i=2}^{n} a_i v_i$ for some $a_2, a_3, \ldots, a_n \in \mathbb{R}$.

Now, note that $x = \sum_{i=2}^{n} \frac{a_i}{\lambda_i} v_i$ satisfies $Lx = L\left(\sum_{i=2}^{n} \frac{a_i}{\lambda_i} v_i\right) = \sum_{i=2}^{n} \frac{a_i}{\lambda_i} L v_i = \sum_{i=2}^{n} \frac{a_i}{\lambda_i} (\lambda_i v_i) = \sum_{i=2}^{n} a_i v_i = b$.

The linear transformation to map $b = \sum_{i=2}^{n} a_i v_i$ to $x = \sum_{i=2}^{n} \frac{a_i}{\lambda_i} v_i$ can be written as follows.

Define the <u>pseudo-inverse</u> L^{+} of L as $L^{+} = \sum_{i=2}^{n} \frac{1}{\lambda_i} v_i v_i^T$.

Then $L^{+} b = \left(\sum_{i=2}^{n} \frac{1}{\lambda_i} v_i v_i^T\right)\left(\sum_{i=2}^{n} a_i v_i\right) = \sum_{i=2}^{n} \frac{a_i}{\lambda_i} v_i$, by the orthonormality of $v_1, \ldots, v_n$.

In general, the pseudo-inverse of a real symmetric matrix L is defined as $\sum_{i:\lambda_i \neq 0} \frac{1}{\lambda_i} v_i v_i^T$, where λ_i are the eigenvalues of L and the v_i form an orthonormal basis of eigenvectors.

L^{+} maps any vector b in the range of L to the unique vector x such that $Lx = b$ and $x \perp$ kernel(L).

And the set of all solutions satisfying $Lx = b$ is $\{ L^{+} b + y \mid y \in \text{kernel}(L) \}$.

In the electrical flow problem on a connected graph, L^{+} maps any vector $b \perp \vec{1}$ to the unique vector x such that $Lx = b$ and $x \perp \vec{1}$, and the set of all solutions for $Lx = b$ is $\{ L^{+} b + c\vec{1} \mid c \in \mathbb{R} \}$.

So, the set of all solutions to $Lx = b$ is just a "shift" of $L^{+} b$.

In particular, this means that there is a unique solution to $Lx = b$ with $x_t = 0$, for instance.

<u>Computing electrical flows</u>

Once we have computed the voltages, then it is easy to compute the flows, i.e. $f(u,v) = w_{uv}(\phi(u) - \phi(v))$.

Let us write down a matrix formulation for our discussion later.

Let B be an $n \times m$ matrix where each column be associated to an edge $e = ij$, where b_e has $+1$ in the i-th entry and -1 in the j-th entry and zero otherwise.

Let W be the $m \times m$ diagonal matrix where $W_{e,e} = w_e$, where w_e is the conductance of the edge.

Then, it is easy to check that $\vec{f} = W B^T \vec{\phi}$.

Note that $L = \sum_{e \in E} w_e b_e b_e^T = B W B^T$.

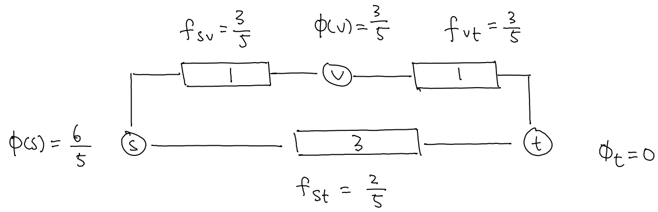
So, $\vec{b} = L\vec{\phi} = BWB^T\phi = B\vec{f}$, which can also be checked directly from the definition.

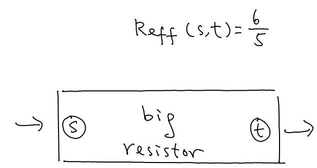
Effective resistance

This is an interesting concept that comes up in different places.

The effective resistance between vertices s and t is defined as $\phi(s) - \phi(t)$ when one unit of electrical flow is sent from s to t.

For example,





We can think of it as the resistance between s and t when the whole network as a single resistor.

We denote the effective resistance by $R_{eff}(s,t)$.

To compute $R_{eff}(s,t)$, first we compute the voltage vector of an unit electrical flow from s to t.

By the matrix formulation, this is the solution to $L\vec{\phi} = b_{st}$, where b_{st} is the vector with $+1$ in the s-th entry and -1 in the t-th entry.

So, $\vec{\phi} = L^{+} b_{st}$, and $R_{eff}(s,t) = \phi(s) - \phi(t) = b_{st}^T L^{+} b_{st}$.

Once we have L^{+}, we can compute $R_{eff}(u,v)$ for all u,v easily.

Energy

This is an important concept about electrical flows.

Recall from physics that the energy dissipated in a resistor network with electrical flow $f(u,v)$ $\forall u,v$ is defined as $\mathcal{E}(\vec{f}) := \sum_{uv \in E} f^2(u,v)\, r_{uv}$.

Note that $\mathcal{E}(\vec{f}) = \sum_{uv \in E} f^2(u,v)\, r_{uv} = \sum_{uv \in E} (\phi(u) - \phi(v))^2 / r_{uv} = \sum_{uv \in E} w_{uv} (\phi(u) - \phi(v))^2 = \phi^T \left(\sum_{uv \in E} w_{uv} b_{uv} b_{uv}^T \right) \phi = \phi^T L \phi$.

Intuitively, if we think of the whole network as one resistor of resistance $R_{eff}(s,t)$ from s to t, then $R_{eff}(s,t) = (\phi(s) - \phi(t)) / f(s,t) = \mathcal{E}(\vec{f})$ if f is a one-unit electrical flow from s to t.

This can be proved formally as $R_{eff}(s,t) = b_{st}^T L^{+} b_{st} = (L\phi)^T L^{+} (L\phi) = \phi^T L L^{+} L \phi = \phi^T L \phi = \mathcal{E}(\vec{f})$, as it is easy to verify that $L L^{+} L = L$.

To summarize, the effective resistance between s and t is the energy of the one-unit s-t electrical flow.

Energy minimization

It turns out that the electrical flow from s to t is the one-unit flow that minimizes the energy.

Let $\vec{g}$ be a one-unit s-t (standard) flow, satisfying the flow conservation constraint at every vertex.

By the above definition, the energy of $\vec{g}$ is $\mathcal{E}(\vec{g}) = \sum_{e \in E} r_e \cdot g(e)^2$.

Theorem (Thompson's principle) $\quad R_{\text{eff}}(s,t) \leq \mathcal{E}(\vec{g})$ for any one-unit s,t-flow $\vec{g}$.

Proof Let $\vec{f}$ be the one unit s-t electrical flow, and $\vec{\phi}$ be the corresponding voltage vector.

Consider $\vec{c} = \vec{g} - \vec{f}$.

As both $\vec{f}$ and $\vec{g}$ satisfy flow conservation constraints, it can be checked that $B\vec{f} = B\vec{g} = b_{st}$,

as the v-th entry of $B\vec{g}$ is $\sum_{\substack{u<v \\ uv \in E}} (-g(u,v)) + \sum_{\substack{w>v \\ wv \in E}} (g(v,w)) = \sum_{u: uv \in E} g(u,v) = b_{st}(v)$.

Therefore, $B\vec{c} = B(\vec{f} - \vec{g}) = 0$, which implies that $\sum_{u: uv \in E} c(v,u) = 0$ for all v.

So, $$\mathcal{E}(\vec{g}) = \sum_{uv \in E} r_{uv}\, g(u,v)^2 = \sum_{uv \in E} r_{u,v} \left(f(u,v) + c(u,v) \right)^2$$

$$= \sum_{uv \in E} r_{uv} f(u,v)^2 + 2 \sum_{uv \in E} r_{uv} f(u,v) c(u,v) + \sum_{uv \in E} r_{u,v} c(u,v)^2 .$$

Observe that the first term is $\mathcal{E}(\vec{f})$, and the last term is positive if $\vec{f} \neq \vec{g}$.

Hence, we will complete the proof that $\mathcal{E}(\vec{g}) \geq \mathcal{E}(\vec{f})$ by showing that the middle term is zero.

To see this, $$\sum_{uv \in E} r_{u,v} f(u,v) c(u,v) = \sum_{uv \in E} (\phi(u) - \phi(v)) c(u,v) \quad \text{by Ohm's law}$$

$$= \sum_{uv \in E} \left(\phi(u) c(u,v) + \phi(v) c(v,u) \right)$$

$$= \sum_{u \in V} \phi(u) \sum_{v: uv \in E} c(u,v) = 0 . \quad \square$$

Remark: This proof is elementary but not very insightful.

There is an alternative proof based on convex optimization.

The energy function is convex and it is minimized when the gradient of the Lagrangian is zero.

This gives a short proof of the Thompson's principle.

Effective resistance as distance

Let us try to get more intuition about effective resistances.

The Rayleigh's monotonicity principle says that the effective resistance cannot decrease if we increase the resistance of some edge.

<u>Theorem</u> (Rayleigh's monotonicity principle) Let $\vec{r}' \geq \vec{r}$ be the resistances.

Then, $Reff_{\vec{r}'}(s,t) \geq Reff_{\vec{r}}(s,t)$, where $Reff_{\vec{r}}(s,t)$ denotes the effective resistance given resistances $\vec{r}$.

<u>Proof</u> Recall that $Reff_{\vec{r}}(s,t)$ is equal to the energy of the electrical flow $\mathcal{E}_{\vec{r}}(f)$.

So, to prove the theorem, we need to prove that $\mathcal{E}_{\vec{r}'}(\vec{f}') \geq \mathcal{E}_{\vec{r}}(\vec{f})$, where $\vec{f}$ and $\vec{f}'$ are the electrical flow under resistances $\vec{r}$ and $\vec{r}'$.

To see this, note that $\mathcal{E}_{\vec{r}'}(f') \geq \mathcal{E}_{\vec{r}}(f')$ as $\vec{r}' \geq \vec{r}$

$\geq \mathcal{E}_{\vec{r}}(f)$ as $\vec{f}$ minimizes energy by the Thompson's principle. □

Intuitively, if there is a short path between s and t, then the effective resistance between s and t is small.

Also, if there are more disjoint paths between s and t, then the effective resistance is smaller.

We can use the Rayleigh's monotonicity principle to give a bound on the effective resistance.

<u>Claim</u> If there are k edge-disjoint paths from s to t, each of length at most ℓ.

Then $Reff(s,t) \leq \ell/k$, assuming that the resistance on each edge is one.

<u>Proof</u> Increase the resistances of all other edges to infinity (or equivalently, delete all other edges).

By the monotonicity principle, the effective resistance of the resulting network could not be smaller, and it is at most k/ℓ by direct calculation. □

Effective resistances provide an alternative way to measure the distance of two nodes in a graph, sometimes more useful than the traditional shortest path distance.

For instance, one could use the effective resistance as distances to identify clusters in a social network.

As a sanity check, effective resistances satisfy the triangle inequality.

<u>Claim</u> $Reff(a,b) + Reff(b,c) \geq Reff(a,c)$ for any $a,b,c \in V$.

<u>Proof</u> Let $\vec{\phi}_{a,b}, \vec{\phi}_{b,c}, \vec{\phi}_{a,c}$ be the voltage vectors of the one-unit electrical flow from a to b, b to c, and a to c respectively.

Then $\vec{\phi}_{a,b} = L^{\dagger}(\chi_a - \chi_b)$, $\vec{\phi}_{a,c} = L^{\dagger}(\chi_a - \chi_c)$, and $\vec{\phi}_{b,c} = L^{\dagger}(\chi_b - \chi_c)$, where $\chi_i \in \mathbb{R}^n$ is the vector with 1 in the i-th entry and zero otherwise.

So, $\vec{\phi}_{a,b} + \vec{\phi}_{b,c} = L^{\dagger}(\chi_a - \chi_b) + L^{\dagger}(\chi_b - \chi_c) = L^{\dagger}(\chi_a - \chi_c) = \vec{\phi}_{a,c}$.

Thus, $Reff(a,c) = (\chi_a - \chi_c)^T \vec{\phi}_{a,c} = (\chi_a - \chi_c)^T \vec{\phi}_{a,b} + (\chi_a - \chi_c)^T \vec{\phi}_{b,c}$.

Note that $(\chi_a - \chi_c)^T \vec{\phi}_{a,b} = \vec{\phi}_{a,b}(a) - \vec{\phi}_{a,b}(c) \le \vec{\phi}_{a,b}(a) - \vec{\phi}_{a,b}(b)$ as $\vec{\phi}_{a,b}(a) \ge \vec{\phi}_{a,b}(c) \ge \vec{\phi}_{a,b}(b) \ \forall c$.

$$= (\chi_a - \chi_b)^T \vec{\phi}_{a,b} = \text{Reff}(a,b).$$

Similarly, the second term is at most $R_{eff}(b,c)$, and hence the claim follows. □

In the following, we will talk about the connection between effective resistances and hitting times, which will give even more intuitions about using effective resistances as distances.

Random walks on undirected graphs

In an undirected graph, the transition probability $P_{uv} = 1/d(u) \quad \forall u,v \in V$.

The Markov chain is irreducible if the graph is connected.

Also, it can be checked that the Markov chain is aperiodic if and only if the graph is non-bipartite, as an odd cycle can be used to show that $(P_{u,v})^t > 0$ for a large enough t for any $u,v \in V$.

Using the same proof as in the Eulerian directed graphs, it is easy to check that $\pi_v = d(v)/2m$ is a stationary distribution, so we have the following result from the fundamental theorem of Markov chains.

Theorem For any connected non-bipartite undirected graph, a random walk will converge to the distribution $\pi_v = d(v)/2m$ regardless of the initial distribution, where $m = |E(G)|$.

Next, we are interested in studying the following quantities:

① Hitting time $h_{u,v}$; ② Commute time $C_{u,v} = h_{u,v} + h_{v,u}$;

③ Cover time : the first time when all vertices are visited at least once.

④ Mixing time : how fast the random walk converges to the unique limiting distribution.

Interestingly, there are close connections between the quantities ①-③ and the concepts in electrical networks.

Theorem For any two vertices s and t, the commute time $C_{s,t} = 2m\, \text{Reff}(s,t)$ where $m = |E(G)|$.

Proof Roughly speaking, the proof goes by showing that the two quantities satisfy the same set of equations.

First, let's work out the equations for hitting times.

Note that $h_{vt} = \frac{1}{d(v)} \sum_{w: vw \in E} (1 + h_{wt})$ for any $v \in V - t$, with $h_{tt} = 0$.

This is equivalent to $d(v) = d(v)\, h_{vt} - \sum_{w: vw \in E} h_{wt} = \sum_{w: vw \in E} (h_{vt} - h_{wt})$ for $v \in V - t$.

Observe that this is essentially a Laplacian system of linear equations.

To see this, consider the electrical flow problem where we inject $d(v)$ units of currents to each $v \in V-t$ and remove $2m-d(t)$ units of currents from t.

Let ϕ_{vt} be the voltage at v in this electrical flow with $\phi_{tt}=0$.

Then, we would like to claim that ϕ_{vt} and h_{vt} would satisfy the same equations.

In this electrical flow, we have $d(v) = \sum_{w: vw \in E} f(vw) = \sum_{w: vw \in E} (\phi_{vt} - \phi_{wt})$ by Ohm's law, for $v \in V-t$.

Let $\vec{b_t}$ be the demand vector with $b_t(v) = d(v)$ for $v \in V-t$ and $b_t(t) = -2m + d(t)$.

Let $\vec{\phi_t}$ be the vector with $\vec{\phi_t}(v) = \phi_{vt}$.

Then, the values ϕ_{vt} satisfy the Laplacian system $L\vec{\phi_t} = \vec{b_t}$ with $\vec{\phi_t}(t) = 0$.

Since G is connected, we know that the set of solutions is $\{ L^{+}\vec{b_t} + c\vec{1} \mid c \in \mathbb{R} \}$.

So, there is a unique solution with $\phi_{tt}=0$.

Since h_{vt} also satisfy the same set of equations with $h_{tt}=0$, we must have $\phi_{vt} = h_{vt}$.

So, we also have $L\vec{h_t} = \vec{b_t}$, where $\vec{h_t}$ is a vector with $\vec{h_t}(v) = h_{vt}$ $\forall v$.

Similarly, let $\vec{b_s}$ be the demand vector with $b_s(v) = d(v)$ for $v \in V-t$ and $b_s(s) = -2m + d(s)$.

Then, as above, let $\vec{h_s}$ be the hitting time vector with $\vec{h_s}(v) = h_{vs}$ and $\vec{h_s}(s) = h_{ss} = 0$.

Then, $\vec{h_s}$ is the unique solution to $L\vec{h_s} = \vec{b_s}$ with $h_{s,s}=0$.

Now, $L(\vec{h_t} - \vec{h_s}) = \vec{b_t} - \vec{b_s} = 2m(\chi_s - \chi_t)$, and so $(\vec{h_t} - \vec{h_s})/2m = L^{+}(\chi_s - \chi_t)$.

So, $\frac{1}{2m}(\vec{h_t} - \vec{h_s})$ is the voltage vector when one unit of electrical flow is sent from s to t.

Then, $R_{eff}(s,t) = (\chi_s - \chi_t)^T \left(\frac{1}{2m}(\vec{h_t} - \vec{h_s})\right) = \frac{1}{2m}\left(\vec{h_t}(s) - \vec{h_t}(t) - \vec{h_s}(s) + \vec{h_s}(t)\right) = \frac{1}{2m}(h_{ts} + h_{st}) = \frac{C_{st}}{2m}$. □

Once we have established the connection to electrical networks, we can use intuitions from physics to derive bounds.

<u>Corollary</u> For any edge $uv \in E$, $C_{uv} \leq 2m$.

<u>Proof</u> The voltage difference between u and v is at most one in a one-unit flow (by Ohm's law). □

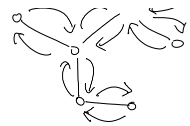
<u>Theorem</u> The cover time of a connected graph is at most $2m(n-1)$.

<u>Proof</u> Let T be a spanning tree of G.

Consider a walk that goes through T where each edge in T is traversed once in each direction.

Then this is a walk that visits every vertex at least once.

So, the cover time is bounded by the expected length of this walk.



Then this is a walk that visits every vertex at least once.

So, the cover time is bounded by the expected length of this walk,

which is at most $\sum_{uv \in T} (h_{uv} + h_{vu}) = \sum_{uv \in T} C_{uv} \leq (n-1) \cdot 2m$,

where the last inequality follows from the corollary. $\square$

For the complete graph with n vertices, the cover time is $O(n \log n)$ (why?), but the above gives $O(n^3)$.

The following is a much better estimate of the cover time.

<u>Theorem</u> Let $R(G) = \max_{uv} R_{uv}$ be the resistance diameter and $C(G)$ be the cover time of G.

Then, $m \cdot R(G) \leq C(G) \leq 2e^3 m R(G) \ln n + n$.

<u>Proof</u> Let $R(G) = R_{uv}$. Then we know that $2m R_{uv} = C_{uv} = h_{uv} + h_{vu}$.

Therefore, $C(G) \geq \max\{h_{uv}, h_{vu}\} \geq C_{uv}/2 = m R_{uv}$, hence the lower bound.

For the upper bound, note that the maximum hitting time is at most $2mR(G)$, regardless of which vertex.

So, if the random walk runs for $2e^3 m R(G)$ steps, by Markov's inequality, a vertex is not covered with

probability at most $1/e^3$.

If the random walk runs for $2e^3 m R(G) \ln n$ steps, then this probability is at most $1/n^3$.

By union bound, some vertex is not visited in $2e^3 m R(G) \ln n$ steps is at most $1/n^2$.

When this happens, we just use the pessimistic bound that $C(G) \leq n^3$.

Combining, we have $C(G) \leq (1 - \frac{1}{n^2}) \cdot 2e^3 m R(G) \ln n + \frac{1}{n^2}(n^3) \leq 2e^3 m R(G) \ln n + n$. $\square$

<u>Graph connectivity</u>

If we want to test whether there is a path from s to t in an undirected graph, we can simply run

a random with $2n^3$ steps.

Since the cover time is at most n^3, it succeeds with probability at least $1/2$ by Markov's inequality.

This algorithm only needs to use $O(\log n)$ space, and still runs in polynomial time.

<u>Plan</u>

In our original plan, we will talk about spectral sparsification (by effective resistances) and

near linear-time Laplacian solvers.

Due to time limit, I will talk about the multiplicative weight update method next time, and that will be the last lecture included in the final exam.

Then, I will talk about spectral sparsification, solving maximum flow using electrical flow, and fast Laplacian solvers in the remaining optional lectures.

References - Lecture notes on "graphs and networks" by Dan Spielman.

- Chapter 6 of "randomized algorithms" by Motwani and Raghavan.