**CS 341: Algorithms @ Waterloo** **Last Updated:** July 19, 2021
**Homework 5** **Due Date: August 4, 2021, 11:00pm**

You are allowed to discuss with others but are not allowed to use any references other than the course notes and the three reference books. Please list your collaborators for each question. You must write your own solutions. See the course outline for the homework policy.

The full mark is 50. This homework is counted 10% of the course grade.

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1. **Written Problem: Facility Scheduling** (10 marks)

   There are $m$ facilities and $n$ people. Each person requests to use a subset of facilities. Each facility can only be used by one person in one day. Suppose each person requests to use at most $d$ facilities, and each facility is requested by at most $d$ people. Prove that there is a schedule (which person to use which facility on which day) that uses at most $d$ days so that every person can use all the facilities that they requested. Design an efficient algorithm to output such a schedule. You will get full marks if the time complexity is polynomial and the proofs are correct.

   Table 1: Input: Facility Requests

   | | Adam | Mary | John | Peter |
   |---|---|---|---|---|
   | Soccer | ✓ | | ✓ | |
   | Basketball | ✓ | ✓ | | |
   | Tennis | | | ✓ | ✓ |

   Table 2: Output: Final Schedule

   | | Day 1 | Day 2 |
   |---|---|---|
   | Soccer | Adam | John |
   | Basketball | Mary | Adam |
   | Tennis | John | Peter |

2. **Written Problem: Generalized Assignment Problem** (10 marks)

   This is a generalization of the assignment problem. There are $m$ machines and $n$ jobs. Each machine is capable of doing a subset of jobs. Each machine $i$ has a capacity $C_i$, meaning that it has $C_j$ units of resource (e.g. memory, disk space, or processing time). Each job $j$ has a demand $D_j$, meaning that it requires $D_j$ units of resource to complete. Now, we would like to assign all the jobs to the machines, so that each job is assigned to only one machine, and no machine is overloaded (i.e. the total demands assigned to machine $i$ does not exceed its capacity $C_i$).

   Input: $m$ positive numbers $C_1, \ldots, C_m$, $n$ positive numbers $D_1, \ldots, D_n$, for each $1 \leq i \leq m$ and $1 \leq j \leq n$ whether machine $i$ is capable of doing job $j$.

   Output: Does there exist an assignment such that all the jobs are assigned to machines, so that each job is assigned to only one machine and no machine is overloaded?

   Either design a polynomial time algorithm to solve this problem (and provide a proof of correctness and analyze its time complexity as usual), or prove that this problem is NP-complete.

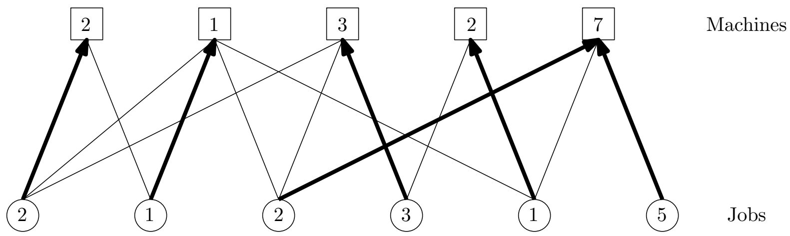


Figure 1: The assignment of jobs are indicated by the highlighted arrows. The numbers in the machines are their capacity, and the numbers in the jobs are their demand.

3. **Written Problem: SAT Solver** (10 marks)

   Suppose there is a black box algorithm $B$ that given any 3SAT formula with $n$ variables and $m$ clauses it will determine whether the formula is satisfiable or not (just return YES/NO) with time complexity polynomial in $n$ and $m$. Show how to use $B$ (possibly multiple times) to find a satisfying assignment (a truth assignment to the variables that satisfies all the clauses) when it exists. Describe your algorithm clearly, justify its correctness, and analyze its time complexity. You will get full marks if your algorithm is correct and the time complexity is polynomial in $n$ and $m$.

4. **Written Problem: Minimum Covering Set** (10 marks)

   Prove that the following problem is NP-complete.

   Input: An undirected graph $G = (V, E)$ and a positive integer $k$.

   Output: Does there exist a covering set with at most $k$ vertices?

   (Recall from HW4 that a subset of vertices $S \subseteq V$ is a covering set if for every vertex $u \in V - S$ there exists a vertex $v \in S$ such that $uv \in E$.)

5. **Written Problem: Directed Acyclic Subgraph** (10 marks)

   Prove that the following problem is NP-complete.

   Input: A directed graph $G = (V, E)$ and a positive integer $k$.

   Output: Does there exist a subset $F \subseteq E$ with at most $k$ edges such that $G - F$ is directed acyclic?