You are allowed to discuss with others but are not allowed to use any references other than the course notes. Please list your collaborators for each question. You must write your own solutions.

There are totally 55 marks (including the bonus). The full mark is 50. This homework is counted 5% of the course grade.

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1. **Programming Problem: Finding Shortest Path** (20 marks)

   There are N cites named with numbers $1, \ldots, N$. One day you want to travel from city s to city t. There are two types of one-way road to connect between two cities. Public roads - which are free to use, and private roads - which cost one dollar per use. It takes one hour for you to pass through each road. Initially you have K dollars and you want to reach your destination as fast as possible.

   So given all the information of the roads, you need to find the shortest path from city s to city t.

   INPUT: The first line with have three integers N, K, M, where N is the number of cities, K is the amount of money you have, M is the number of roads. It is guaranteed that $1 \leq N \leq 100,000$, $0 \leq M \leq 300,000$, and $0 \leq K \leq 10$.

   The next M lines each contains three integers a, b and c to describe one road:

   - a is the starting city of the road, $1 \leq a \leq N$
   - b is the ending city of the road, $1 \leq b \leq N$
   - c represents the type of the road. If $c = 0$ then the road is public, if $c = 1$ then the road is private.

   The last line contains two integer s, t, $1 \leq s, t \leq N$, which represents the city you travel from and the city you want to go.

   OUTPUT: If it is impossible to travel from city s to city t using K dollars, output -1. Otherwise, output the minimum number of hours X you have to spent in the first line. Then next line of the output should contain $X + 1$ integers, the list of the cities you travel in order. The first integer should be s and the last integer should be t. If there are multiple shortest paths, output the one which costs the least amount of money. If there are still multiples of them, you can output any one of them.

   SAMPLE INPUT:
   ```
   5 2 6
   1 2 1
   1 3 1
   1 4 0
   2 5 0
   3 5 1
   4 5 1
   1 5
   ```
   SAMPLE OUTPUT:
   ```
   2
   1 2 5
   ```

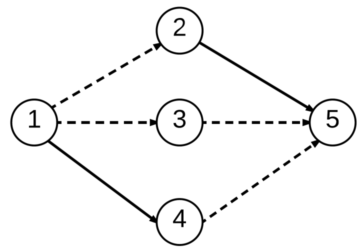
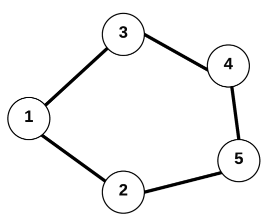


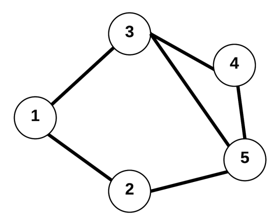
Figure 1: This is the directed graph of the sample input. The solid lines represents public roads and the dashed lines represent private roads. All three possible paths takes two hours, while paths $(1 \to 2 \to 5)$ and $(1 \to 4 \to 5)$ cost only one dollar. So the output can also be $(1 \to 4 \to 5)$.

2. **Unique Shortest Path** (10 marks)

   Given an undirected graph $G = (V, E)$ and two vertices $s, t \in V$, design an algorithm to determine if there is a unique shortest path from s to t. You will get full marks if the time complexity is $O(|V|+|E|)$ and the proofs are correct.



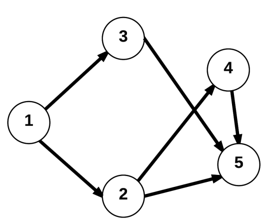
(a) The unique shortest path from 1 to 5 is $1 - 2 - 5$.



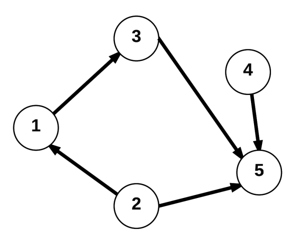
(b) There are two shortest paths from 1 to 5: $1-2-5$ and $1-3-5$.

3. **Reachability** (10 marks)

   Given a directed graph $G = (V, E)$, design an algorithm to find a vertex $s \in V$ from which all other vertices are reachable (i.e. there is a directed path from s to v for all $v \in V$) or report that none exists. You will get full marks if the time complexity is $O(|V| + |E|)$ and the proofs are correct.



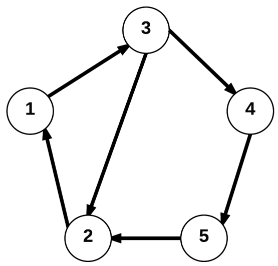
(a) Vertex 1 can reach all other vertices.



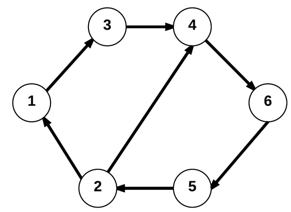
(b) No vertex can reach all other vertices.

4. **Odd Cycle** (10 marks)

Given a strongly connected directed graph $G = (V, E)$, design an algorithm to determine if there is a directed odd cycle in G or not. You will get full marks if the time complexity is $O(|V| + |E|)$ and the proofs are correct.



(a) A graph with a directed odd cycle $1 \to 3 \to 2 \to 1$.



(b) A graph without directed odd cycles.

Bonus: (5 marks) You will get up to 5 bonus marks if your algorithm can work for any directed graph and also print out a simple directed odd cycle when it exists.