CS 341: Algorithms @ Waterloo **Last Updated:** Mar 17, 2016
Homework 5 **Due Date: Apr 4, 2016, 11:59pm**

There are 60 marks. The full mark is 50. This homework is counted 5% of the course grade.

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1. **Facility Scheduling** (12 marks)

   There are m facilities and n people. Each person requests to use a subset of facilities. Each facility can only be used by one person in one day. Suppose each person requests to use at most d facilities, and each facility is requested by at most d people. Model this as a graph problem and prove that there is a schedule (which person to use which facility on which day) that uses at most d days so that every person can use all the facilities that he/she requested. Give a polynomial time algorithm to output such a schedule.

   Table 1: Input: Facility Requests

	Adam	Mary	John	Peter
Soccer	✓		✓	
Basketball	✓	✓		
Tennis			✓	✓

   Table 2: Output: Final Schedule

	Day 1	Day 2
Soccer	Adam	John
Basketball	Mary	Adam
Tennis	John	Peter

   (Hint: First consider the case when every person requests exactly d facilities, and each facility is requested by exactly d people.)

2. **Hamiltonian Cycle** (8 marks)

   Prove that the Hamiltonian Cycle problem is NP-complete even when restricted to undirected bipartite graphs.

3. **Degree Bounded Spanning Tree** (8 marks)

   Prove that the following problem is NP-complete.

   Input: An undirected graph $G = (V, E)$ and an positive integer k.

   Output: Does there exist a spanning tree $T \subseteq E$ with maximum degree at most k?

   (Hint: Hamiltonian Path)

4. **Shortest Simple Path** (10 marks)

   Prove that the following problem is NP-complete.

   Input: A directed graph $G = (V, E)$ where each edge $e \in E$ has a length l_e, an integer L, and two vertices $s, t \in V$. Note that both l_e and L could be negative numbers.

   Output: Does there exist a *simple* path from s to t with total length at most L? A path is simple if it visits every vertex at most once.

5. **Intersecting Set** (10 marks)

   Prove that the following problem is NP-complete.

   Input: m sets $S_1, S_2, \ldots, S_m$ where each $S_i \subseteq \{1, \ldots, n\}$, and an positive integer k.

   Output: Does there exist a subset $T \subset \{1, \ldots, n\}$ with $|T| \leq k$ such that $T \cap S_i \neq \emptyset$? In words, does there exists a subset T with at most k elements that intersects every set S_i?

6. **Acyclic Subgraph** (12 marks)

   Prove that the following problem is NP-complete.

   Input: A directed graph $G = (V, E)$ and a positive integer k.

   Output: Does there exist a subset $F \subseteq E$ with at most k edges such that $G - F$ is directed acyclic?