CS 341 - Algorithms, Winter 2016. University of Waterloo

Lecture 19: Hard graph problems

We show that two classical graph problems, the Hamiltonian cycle problem and the graph coloring problem, are NP-complete by reductions from 3SAT.

Both reductions are nontrivial and require clever design of some gadgets.

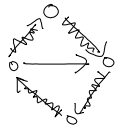
Hamiltonian cycle [KT 8.5]

We will introduce an intermediate problem for our reduction.

Directed Hamiltonian cycle (DHC)

Input: A directed graph $G = (V, E)$

Output: Does G has a directed cycle that touches every vertex exactly once?



We will show that $3SAT \leq_p DHC \leq_p HC$.

3SAT and DHC are very different, so it requires some creativity to connect the two problems.

Theorem Directed Hamiltonian Cycle is NP-complete.

Proof It is easy to see that DHC is in NP. To prove it is NPc, we will prove $3SAT \leq_p DHC$.

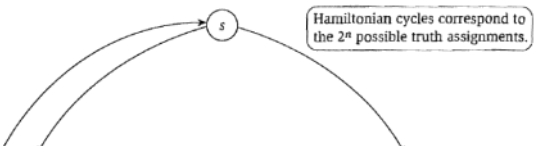
Given a 3SAT instance with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$, we would like to construct a directed graph G so that the formula is satisfiable iff the graph has a Hamiltonian cycle.

We need to create some graph structures for the variables and the truth assignment.

The idea is to associate a long "two-way" path to a variable and going the path in one way corresponds to setting the variable true, while going the other way corresponds to setting false.

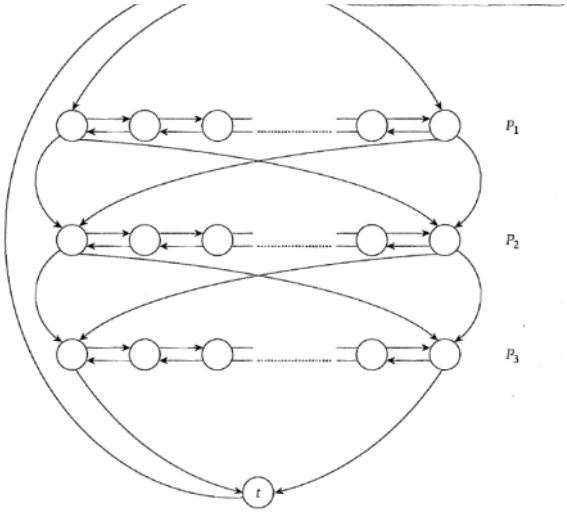
The intention is that if we go from the left-to-right if x_i is true and otherwise from right-to-left.

In the following graph, there is a one-to-one correspondence between the 2^n truth assignments of the n variables and the directed Hamiltonian cycles of the graph.



There are totally n two-way paths, one for each variable.

[KT]



one for each variable.

Each path is of length $3m$.

The two endpoints of the path P_i connect to the two endpoints of the path P_{i+1}.

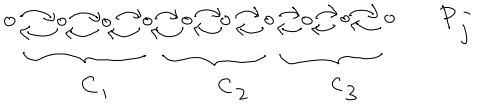
There is a source s connects to the two ends of P_1, and the two ends of P_n connect to the sink t, and t connects to s.

It should be clear that there are only 2^n possible Hamiltonian cycles in this graph, since we must use each path P_i either from left-to-right or from right-to-left, as the intermediate vertices on the paths are not connected to anything else.

That's a good start, with one-to-one correspondence between truth assignments and Hamiltonian cycles

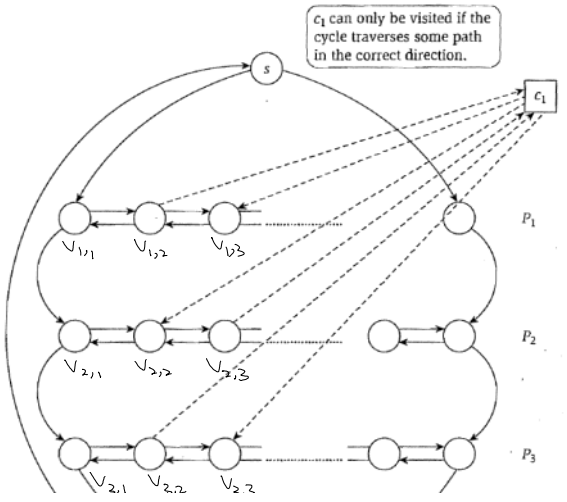
Next, we would like to add some clause structures to "kill" all the Hamiltonian cycles that don't correspond to satisfying assignment.

Recall that the two-way paths are of length $3m$. The first three edges belong to the first clause, and in general vertices $3i-2, 3i-1, 3i$ belong to the i-th clause.



Suppose there is a clause, say $X_1 \vee \bar{X}_2 \vee X_3$, then we want that the Hamiltonian cycle to either (1) go from left-to-right in P_1, or (2) go from right-to-left in P_2 or (3) go from left-to-right in P_3 to satisfy the clause.

[KT]



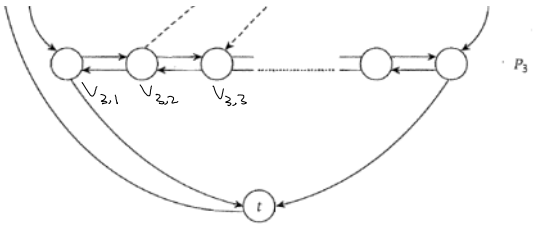
To this end, we create one vertex c_j for each clause C_j.

Call the vertices on P_i be $v_{i,1}, \ldots, v_{i,3m+1}$

If literal x_i appears in C_j, then we add the edges $v_{i,3j-1}\, c_j$ and $c_j v_{i,3j}$.

Otherwise, if literal $\bar{x}_i$ appears in C_j, then we add edges $c_j v_{i,3j-1}$ and $v_{i,3j}\, c_j$.

We do this for every clause C_j.



edges $c_j v_{i,3j-1}$ and $v_{i,3j} c_j$.

We do this for every clause C_j.

Note that the edges for C_j and C_k don't share vertices, and that's why we created long paths for.

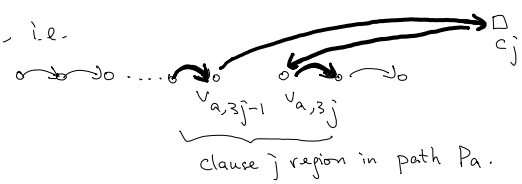
That's the whole construction. Clearly, it can be done in polynomial time.

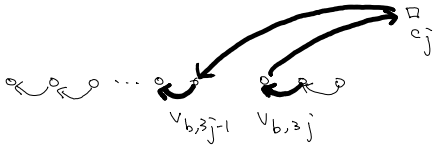
It remains to prove that there is a satisfying assignment for the formula if and only if there is a Hamiltonian cycle in the graph.

Suppose there is a satisfying assignment. If $x_i = T$, we visit P_i from the left end to the right end; otherwise if $x_i = F$, we visit P_i from right to left.

For a clause say $C_j = (x_a \vee \bar{x}_b \vee x_c)$, at least one literal is true in the satisfying assignment, say x_a.

Then, when we transverse P_a from left-to-right, we "detour" to visit vertex c_j during the clause j region in path P_a, i.e.





Similarly, if $x_b = F$, then we can also detour to visit c_j when we transverse P_b from right-to-left.

Since each clause is satisfied in a satisfying assignment, we can visit all the clause vertices following these directions and detours.

Suppose there is a Hamiltonian cycle. We would like to argue that it must look like the Hamiltonian cycle above that corresponds to a satisfying assignment, but why must it be the case?

The crucial observation is that if we use the directed edge $v_{a,3j-1} c_j$, then we must use the edge $c_j v_{a,3j}$ immediately after it, as otherwise the vertex will become a "dead-end" and we can't complete the cycle.

Since each clause vertex is visited in a Hamiltonian cycle, at least one variable path is going in the correct direction, and all the paths must go from left-to-right or right-to-left (as it must come back immediately after each "detour" to clause vertices), and so this corresponds to a satisfying assignment as we intended. □

It is quite straightforward to show that $HC \leq_p DHC$ (exercise), but we want the other direction.

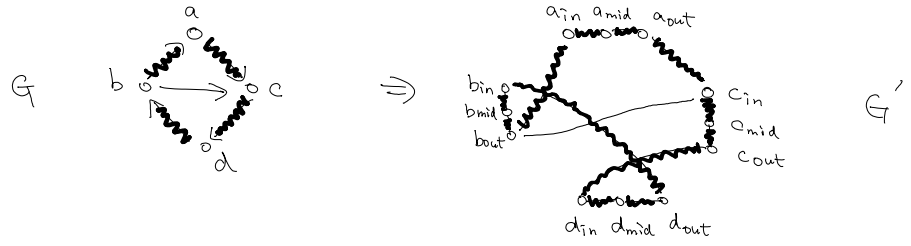
The proof is not too difficult either.

The idea is to use a path of length 3 to simulate a directed edge.

Claim $DHC \leq_p HC$.

proof Given a directed graph $G=(V,E)$ for DHC, we construct an undirected graph G' in which we create three vertices v_{in}, v_{mid}, v_{out} for each vertex $v \in V(G)$.

Then, we create the edges in G' as follows: there is an edge between v_{in} and v_{mid} and an edge between v_{mid} and v_{out} for every $v \in V(G)$. Also, for each directed edge $uv \in E(G)$, we add an undirected edge $u_{out} v_{in}$ in G'.



That's the whole construction. Clearly it can be done in polynomial time.

Now we prove that G has a directed Hamiltonian cycle iff G' has an undirected Hamiltonian cycle.

One direction is immediate: if G has a Hamiltonian cycle, by following the cycle and replacing each directed edge uv by $u_{out} v_{in}$ and using the paths $v_{in} - v_{mid} - v_{out}$ for all v, it is a Hamiltonian cycle in G'.

The other direction is slightly more interesting: in an undirected Hamiltonian cycle, start with a vertex v_{in}, then v_{mid} must be a neighbor of v_{in} in the HC, as otherwise v_{mid} will be a "dead-end" since it is of degree two, and then v_{out} must be a neighbor of v_{mid}.

Then, from v_{out}, by construction, the cycle must go to w_{in} for some w, and then w_{mid} and w_{out} as argued above.

So, following the undirected Hamiltonian cycle of G', it must be of the form in the previous direction, and hence it corresponds to a directed Hamiltonian cycle in G. □

As we have done before in L17, since HC is a special case of TSP, we have $HC \leq_p TSP$, and it follows that TSP is NP-complete.

Graph Coloring [KT 8.7]

Graph coloring [KT 8.7]

Graph coloring is one of the most classical problem in graph theory, e.g. 4-color theorem.

Input: An undirected graph $G=(V,E)$, and a positive integer k.

Output: Is it possible to use k colors to color all the vertices so that every vertex receives one color and any two adjacent vertices receive different colors?

When $k=1$, it is possible if and only if the graph has no edges.

When $k=2$, it is possible if and only if the graph is bipartite (why?).

When $k=3$, the problem becomes NP-Complete.

I should mention that graph coloring is very useful in modeling resource allocation problems, like the interval coloring problem that we have seen.

Theorem 3-coloring is NP-complete.

Proof It is clear that 3-coloring is in NP (easy exercise). We prove 3-coloring $\leq_p$ 3SAT.

Given a 3SAT formula with n variables $x_1, \ldots, x_n$, and m clauses $C_1, \ldots, C_m$, we want to construct a graph G so that the formula is satisfiable iff G is 3-colorable.

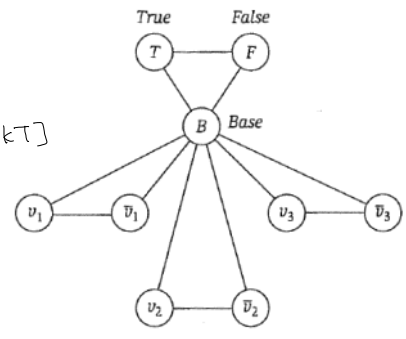
Quite naturally, we would like to associate one color to True and one color to False.

We would like to make sure that the literals are colored consistently.

This is not difficult to do: we just create two vertices x_i and $\bar{x}_i$ for each variable, and add an edge between them so that they get different colors, i.e.  .

Since there are three colors, to make sure that the two literals get T/F colors, we connect every literal vertex to a common vertex, called the base vertex.

[KT]



Call the three colors, T, F, B.

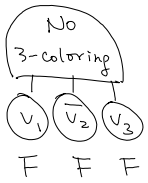
We create two vertices v_i and $\bar{v}_i$ for each variable x_i.

These edges make sure that either (v_i=T and $\bar{v}_i$=F) or (v_i=F and $\bar{v}_i$=T).

So, there is a one-to-one correspondence between truth assignments and the 3-colorings.

As before, we would like to add some clause structures to "kill" the colorings that don't correspond to satisfying assignment.

Say we have a clause $(x_1 \vee \bar{x}_2 \vee x_3)$. it would be good if there is a "gadget" to connect to the three vertices $v_1, \bar{v}_2, v_3$ so that there is a 3-coloring for the gadget iff at least one of $v_1, \bar{v}_2, v_3$ has color T, i.e.



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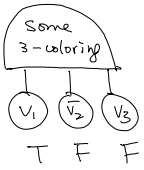
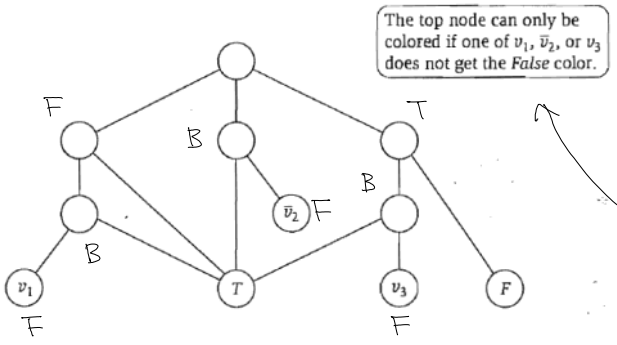


With some trial-and-error, we can construct a gadget with exactly this "functionality".



We create one such gadget for each clause.

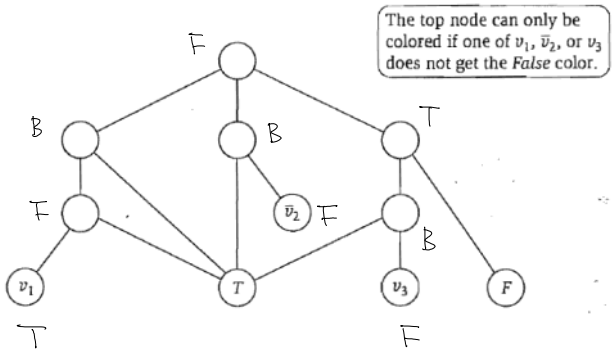
All the unlabeled vertices are new vertices only for the clause.

We need to verify this claim.

First, suppose all $v_1, \bar{v}_2, v_3$ are colored F, then the top node in the gadget cannot be colored.

See the picture; all the colors are forced, and there is no way to color the top node.

On the other hand, if one of $v_1, \bar{v}_2, v_3$ is colored T, then it is possible to color all nodes in gadget.



If v_1 is colored T, then we can color all the nodes in the gadget using 3 colors.

We leave checking all other cases as (coloring) exercises.

It is clear that the reduction can be done in polynomial time.

With the claim, it is not difficult to prove that there is a satisfying assignment iff there is a 3-coloring in the graph.

If there is a satisfying assignment, for each variable x_i, if x_i = True, we color v_i = T and $\bar{v}_i$ = F; otherwise if x_i = False, we color v_i = F and $\bar{v}_i$ = T.

Clearly, it is a valid coloring for the initial graph.

Since it is a satisfying assignment, each clause gadget has one of the three "inputs" set to be T, and thus it can be extended to a 3-coloring in the gadget by the claim.

On the other hand, if there is a 3-coloring of the graph, then by the claim for each gadget, there is one "input" with color T.

By the initial graph, the coloring must be consistent with a truth assignment, and thus following the coloring, we can define a truth assignment that satisfies all the clauses. □
