1. Consider the following balls-and-bin game. We start with one black ball and one white ball in a bin. We repeatedly do the following: choose one ball from the bin uniformly at random, and then put the ball back in the bin with another ball of the same color. We repeat until there are n balls in the bin. Show that the number of white balls is equally likely to be any number between 1 and $n-1$.

2. The following problem is known as the Monty Hall problem, after the host of the game show "Let's Make a Deal". There are three curtains. Behind one curtain is a new car, and behind the other two are goats. The game is played as follows. The contestant chooses the curtain that she thinks the car is behind. Monty then opens one of the other curtains to show a goat. (Monty may have more than one goat to choose from: in this case, assume he chooses which goat to show uniformly at random.) The contestant can then stay with the curtain she originally chose or switch to the other unopened curtain. After that, the location of the car is revealed, and the contestant wins the car or the remaining goat. Should the contestant switch curtains or not, or does it make no difference?

3. (a) Consider the set $\{1, 2, \ldots, n\}$. We generate subset X of this set as follows: a fair coin is flipped independently for each element of this set: if the coin lands heads then the element is added to X, and otherwise it is not. Argue that the resulting set X is equally likely to be any one of the 2^n possible subsets.

   (b) Suppose that two sets X and Y are chosen independently and uniformly at random from all the 2^n subsets of $\{1, 2, \ldots, n\}$. Determine $\Pr(X \subseteq Y)$ and $\Pr(X \cup Y = \{1, \ldots, n\})$.

4. At a party n men take off their hats. The hats are then mixed up, and each man randomly selects one. We say that a match occurs if a man selects his own hat. What is the probability of

   (a) no matches;

   (b) exactly k matches?

5. Two gamblers, A and B, bet on the outcome of successive flips of a coin. On each flip, if the coin comes up heads, A collects 1 unit from B, whereas if it comes up tails, A pays 1 unit to B. They continue to do this until one of them runs out of money. If it is assumed that the successive flips of the coin are independent and each flip results in a head with probability p, what is the probability that A ends up with all the money if he starts with i units and B starts with $N - i$ units?

6. There are $k + 1$ coins in a box. The ith coin will, when flipped, turn up heads with probability $i/k, i = 0, 1, \ldots, k$. A coin is randomly selected from the box and is then repeatedly flipped. If the first n flips all result in heads, what is the conditional probability that the $(n + 1)$st flip will do likewise?

7. Perform n tosses of a biased coin which gives tails with probability p. What is the

   (a) expected number of tails?

   (b) variance of the number of tails?

8. Suppose you are given a coin for which the probability of HEADS, say p, is *unknown*. How can you use this coin to generate unbiased coin-flips? Give a scheme for which the expected number of flips of the biased coin for extracting one unbiased coin-flip is no more than $1/(p(1 - p))$.