Lecture 25: Electrical networks

We prove some basic results about electrical flows and effective resistances, and show a connection to hitting times of random walks in undirected graphs.

**Electric Flow**

We think of an undirected graph as an electrical network, where each edge $e$ is a resistor with resistance $r_e$.

The flow of electric current is governed by two rules:

1. **Kirchhoff's law (flow conservation law):** The sum of the currents entering a node is equal to the sum of currents leaving it.

2. **Ohm's law (potential flow law):** The potential drop across a resistor is equal to the current flowing over the resistor times the resistance.

For example, consider this network:

![Diagram of electrical network]

If one ampere is injected into $s$ and one ampere is removed from $t$, then the voltages at the nodes and the currents on the resistors are shown in the figure on the right.

**Notation**

Let's write a matrix formulation of the problem.

Let $G=(V,E)$ be the underlying undirected graph.

Let $V \in \mathbb{R}^M$ be the vector of potentials at vertices.

Let $i(a,b)$ be the current flowing from vertex $a$ to vertex $b$ for an edge $(a,b)$. As this is a directed quantity, we define $i(b,a) = -i(a,b)$.
Let \( i \in \mathbb{R} \) be the vector of currents flowing over the edges, where each edge \( e=(a,b) \) appears once as \( i(a,b) \) where \( a \neq b \).

Let \( \omega_e = \frac{1}{r_e} \) be the "conductance" of the edge \( e \).

**Matrix Formulation**

The Ohm's law states that

\[
i(a,b) = \frac{v(a)-v(b)}{r_{a,b}} = \omega_{a,b} (v(a)-v(b)).
\]

The Kirchhoff's law states that

\[
\sum_{b \in \text{neigh}(a), b \neq a} i(a,b) = i_{\text{ext}}(a),
\]

where \( i_{\text{ext}}(a) \) denotes the external current entering the network through the node \( a \), so it is a positive number if \( a \) is a source and a negative number if \( a \) is a sink and zero otherwise.

By Ohm's law,

\[
\sum_{b \in \text{neigh}(a)} i(a,b) = \sum_{b \in \text{neigh}(a)} \omega_{a,b} (v(a)-v(b)) = d(a) v(a) - \sum_{b \in \text{neigh}(a)} \omega_{a,b} v(b),
\]

where

\[
d(a) = \sum_{b \in \text{neigh}(a)} \omega_{a,b}
\]

is the weighted degree of \( a \).

Then this is just equivalent to \( L_G v = i_{\text{ext}} \), where \( L_G \) is the weighted Laplacian of \( G \) and

\( i_{\text{ext}} \) is the vector of external currents at the vertices.

**Computing Voltages**

Therefore, if we can solve a Laplacian system quickly, then we can compute the voltages

(and thus the currents) quickly.

Notice that \( L_G \) is not of full rank. Assume without loss of generality that \( G \) is connected.

Then we know that nullspace (\( L_G \)) = \( \mathbb{R}^{n-1} \). Let \( x = \sum_{i=1}^n \lambda_i v_i \) where \( v_i \) is an orthonormal basis with \( v_1 \perp \mathbb{R} \).

Then \( L_G x = \sum_{i=1}^n \lambda_i v_i \), and \( \lambda_1 = 0 \), and so \( L_G x \) is perpendicular to \( \mathbb{R} \).

Therefore, there is a solution to \( L_G v = i_{\text{ext}} \) if and only if \( i_{\text{ext}} \perp \mathbb{R} \), which should be clear to our problem as the total external currents injecting into the network should be equal to the total external currents removing from the network.

Let \( \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \) be the eigenvalues of \( L_G \) with corresponding eigenvectors \( v_1, v_2, \ldots, v_n \).

Then \( L_G^\dagger = \sum_{i=1}^n \frac{1}{\lambda_i} v_i v_i^T \). We define the pseudo-inverse as

\[
L_G^\dagger = \sum_{i=2}^n \frac{1}{\lambda_i} v_i v_i^T.
\]
For \( x \in \mathbb{R}^n \), \( y = L_q^\dagger x \) is the unique solution to \( L_q y = x \) with \( y \perp 1 \), and the set of all solutions is \( \{ y + c 1^T | c \in \mathbb{R} \} \).

**Computing Currents**

Once we have computed the voltages, then it is easy to compute the currents.

Let us write down a matrix formulation for our discussions later.

Let \( B \) be an \( m \times n \) matrix whose the rows are indexed by the edges and the columns are indexed by the vertices, and the row corresponding to the edge \( e = (a,b) \) with \( a < b \) is \( (x_a - x_b)^T \), where \( x_a \) is the characteristic vector with one in the \( a \)-th entry and zero otherwise.

Let \( W \) be the \( m \times m \) diagonal matrix where \( W_{ee} = w_e \) is the weight of edge \( e \).

Then \( \vec{i} = WB \vec{v} \).

Notice that \( L_q = \sum_{e \in V} W_e (x_a - x_b)(x_a - x_b)^T = B^TWB \). So, \( \vec{i}_{ext} = L_q \vec{v} = B^TWB \vec{v} = B^T \vec{v} \), which can also be checked directly from the definition.

**Effective Resistance**

The effective resistance between vertices \( a \) and \( b \) is defined as \( V(a) - V(b) \) when one ampere is injected into \( a \) and removed from \( b \). You can think of it as the resistance between \( a \) and \( b \) given by the whole network. We denote it by \( R_{eff}(a,b) \).

To compute \( R_{eff}(a,b) \), first we compute the voltages when one ampere is injected into \( a \) and removed from \( b \). By the matrix formulation, this is the solution of \( L_q \vec{v} = (x_a - x_b) \), which is given by \( \vec{v} = L_q^\dagger (x_a - x_b) \). Then \( R_{eff}(a,b) \) is just \( (x_a - x_b)^T L_q^\dagger (x_a - x_b) \).

So, once we have \( L_q^\dagger \), we can compute \( R_{eff}(a,b) \) for all \( a, b \) easily.

**Energy**

Recall from physics that the energy dissipated in a resistor network with currents \( \vec{i}_{ab} \) \( V(a,b) \) is:

\[
E(\vec{I}) = \vec{I}^T \begin{bmatrix} R \end{bmatrix} \vec{I} = \sum_{(a,b) \in E} \frac{1}{R_{ab}} (V(a) - V(b))^2 = \sum_{(a,b) \in E} W_{ab}(V(a) - V(b))^2 = V^T L_q \vec{v},
\]
where \( R \) is the sym diagonal matrix where \( R(i,i) = \rho_i \).

Intuitively, if we think of the whole network as one resistor from \( a \) to \( b \), then
\[
\text{Reff}(a,b) = (v(a) - v(b)) / (i(a)) = E(i) \text{ if one unit of current is sent from } a \text{ to } b.
\]
This can be proved formally as \( \text{Reff}(a,b) = (x_a \cdot x_b)^T L_G (x_a \cdot x_b) = (L_G v)^T (L_G v) = v^T L_G L_G v = \sum L_G v = E(i) \), as it is easy to verify that \( L_G L_G L_G = L_G \).

**Energy Minimization**

The electric flow from \( s \) to \( t \) is the one that minimizes the energy.

Let \( j \) be one unit of flow from \( s \) to \( t \), satisfying the flow conservation rule at every vertex.

Define its energy to be \( E(j) = \sum_{v} v^T R_j v = \sum a \cdot J_e^2 \).

**Theorem (Thompson's Principle)** \( \text{Reff}(s,t) \leq E(j) \).

**Proof** Let \( j \) be the electrical flow of one unit from \( s \) to \( t \), and \( v \) be the corresponding voltages.

Consider \( \zeta = j - i \).

As both \( i \) and \( j \) satisfy flow conservation constraints, we have \( B_i^T \zeta = B_j^T \zeta = (x_i - x_j) \) as the

\[ \text{v-th entry of } B_i^T \zeta \text{ is } \sum_{w \in E} (-i(w)) + \sum_{w \in E} (v(w)) = \sum_{w \in E} i(w) - i_e - j_e = i_e - j_e. \]

Therefore, \( B_i^T \zeta = B_j^T \zeta = 0 \), and hence \( \sum_{w \in E} v(w) = 0 \) for all \( v \).

\[ E(j) = \sum_{a,b} j(a,b) \cdot r_{a,b} = \sum_{a,b} (i(a,b) + c(a,b)) \cdot r_{a,b} = \sum_{a,b} (i(a,b) \cdot r_{a,b} + 2 \sum_{a,b} i(a,b) \cdot c(a,b) \cdot r_{a,b} + \sum_{a,b} c(a,b) \cdot r_{a,b}) \]

Observe that the first term is \( E(i) \), and the last term is positive if \( \zeta = j \).

Hence we will complete the proof once we show that \( \sum_{a,b} i(a,b) \cdot c(a,b) \cdot r_{a,b} = 0 \).

To see this, \( \sum_{a,b} i(a,b) \cdot c(a,b) \cdot r_{a,b} = \sum_{a,b} (v(a) - v(b)) \cdot c(a,b) \) (by Ohm's Law)

\[ = \sum_{a,b} (v(a) \cdot c(a,b) + v(b) \cdot c(b,a)) \]

\[ = \sum_{a \in V} v(a) \sum_{b \in b(a,b)} c(a,b) = 0. \]
Effective Resistance as Distance

Let us try to get some intuition about the effective resistances.

The Rayleigh's monotonicity principle says that the effective resistance cannot decrease if we increase the resistance of some edge.

**Theorem** (Rayleigh's Monotonicity Principle) Let $r, r'$ be the resistances. Then $E_r(i) \geq E_{r'}(i)$, where $E_r(i)$ denotes the energy of flow $i$ under the resistances $r$.

**Proof.** Let $i$ and $i'$ be the electric flow under resistances $r$ and $r'$ respectively. Then $E_r(i') \geq E_r(i)$ as $r \geq r'$.

$E_r(i) \geq E_{r'}(i)$ by the Thompson's principle. □

Intuitively, if there is a short path between $s$ and $t$, then the effective resistance between $s$ and $t$ is small. Also, if there are many paths between $s$ and $t$, then the effective resistance between $s$ and $t$ is smaller. One can use the Rayleigh's monotonicity principle to give a bound on the effective resistance.

**Claim.** If there are $k$ edge-disjoint paths from $s$ to $t$, each of length at most $l$, then $\text{Reff}(s,t) \leq l/k$, assuming the graph is unweighted.

**Proof.** Increase the resistance of all other edges to infinity. Then the effective resistance of the resulting graph is at most $l/k$ by direct calculation. By monotonicity, the effective resistance in the original network could not be larger than that. □

Effective resistances provide an alternative way to measure the distance of two nodes in a graph, sometimes more useful than the traditional shortest path distance. For instance, one could use the effective resistances as distances to identify clusters in a social network.

Actually, effective resistances satisfy the triangle inequality.

**Claim** \[
\text{Reff}(a,b) + \text{Reff}(b,c) \geq \text{Reff}(a,c).\]
Claim \( \text{Reff}(a,b) + \text{Reff}(b,c) \geq \text{Reff}(a,c) \).

Proof Let \( V_{a,b}, V_{a,c}, V_{b,c} \) be the voltages when one unit of current is sent from \( a \) to \( b \), \( a \) to \( c \), \( b \) to \( c \), respectively.

Then \( V_{a,b} = L_q (X_a - X_b) \), \( V_{a,c} = L_q (X_a - X_c) \) and \( V_{b,c} = L_q (X_b - X_c) \).

So \( V_{a,b} + V_{b,c} = V_{a,c} \).

\[ \text{Reff}(a,c) = (X_a - X_c)^T V_{a,c} = (X_a - X_c)^T V_{a,b} + (X_a - X_c)^T V_{b,c} \]

Note that \( (X_a - X_c)^T V_{a,b} = V_{a,b}(a) - V_{a,b}(c) \leq V_{a,b}(a) - V_{a,b}(b) \) as \( V_{a,b}(a) \geq V_{a,b}(c) \geq V_{a,b}(b) \) for all \( c \in V \)

Similarly, the second term is at most \( \text{Reff}(b,c) \), and hence the claim follows. \( \Box \)

In the following we will talk about the connection between effective resistances and hitting times, which will give even more intuitions about using effective resistances as distances.

Random Walk in Undirected Graphs

Recall that the hitting time from \( a \) to \( b \) is the expected number of steps to reach \( b \) if the random walk starts from \( a \), denoted by \( h_{ab} \).

The cover time is the expected number of steps to reach every vertex at least once.

The commute time, denoted by \( C_{ab} \), is defined as \( h_{ab} + h_{ba} \).

Theorem \( C_{a,t} = \text{Reff}(a,t) \).

Proof Let \( v \in V \setminus T \). Then \( h_{vt} = \sum_{w \in \text{vert}} \frac{1}{\text{d}(w)} (1 + h_{wt}) \), with \( h_{tt} = 0 \).

This is equivalent to \( \text{d}(v) = \text{d}(v) h_{vt} = \sum_{w \in \text{vert}} h_{vt} = \sum_{w \in \text{vert}} (h_{vt} - h_{wt}) \) for \( v \in V \setminus T \).

Observe that this is very similar to a Laplacian system of linear equations.

Let \( \phi_{vt} \) be the voltage at \( v \) with \( \phi_{tt} = 0 \), when \( \text{d}(v) \) units of currents are injected from \( v \in V \setminus T \) and \( 2m - \text{d}(t) \) units of current are removed from \( t \).

Then the values \( \phi_{vt} \) and \( h_{vt} \) would satisfy the same equations.
This is because, for $v \in \mathbb{V}$, $\nabla \Phi = \nabla \Phi(v) \cdot \nabla (\Phi(v) - \Phi(w))$ by Ohm’s law.

Let $i^\Phi_\mathbb{I}(t)$ be the vector of the external currents with $i^\Phi_\mathbb{I}(w) = c(w)$ for $v \in \mathbb{V}$ and $i^\Phi_\mathbb{I}(t) = -2m\mathcal{I}d(t)$. And let $\Phi^\mathbb{I}_\mathbb{I}$ be the vector with $\Phi^\mathbb{I}_\mathbb{I}(w) = \Phi^\mathbb{I}_\mathbb{I}(w)$.

Then the values $\Phi^\mathbb{I}_\mathbb{I}$ satisfy the Laplacian system $L \Phi^\mathbb{I}_\mathbb{I} = \Phi^\mathbb{I}_\mathbb{I}$ with $\Phi^\mathbb{I}_\mathbb{I}(w) = 0$.

We know that the set of solutions to this Laplacian system is $\{ -L^\dagger \Phi^\mathbb{I}_\mathbb{I} + C \}$. C $\in \mathbb{R}$.

There is a unique solution with $\Phi^\mathbb{I}_\mathbb{I}(w) = 0$, hence we must have $h^\mathbb{I}_\mathbb{I} = \Phi^\mathbb{I}_\mathbb{I}$ for all $v$.

Let $i^\mathbb{I}_\mathbb{I}(w)$ be the vector of external currents with $i^\mathbb{I}_\mathbb{I}(w) = c(w)$ if $v \in \mathbb{V}$ and $i^\mathbb{I}_\mathbb{I}(w) = -2m\mathcal{I}d(s)$.

Then, as above, let $h^\mathbb{I}_\mathbb{I}$ be the hitting time vector with $h^\mathbb{I}_\mathbb{I}(w) = h^\mathbb{I}_\mathbb{I}$ and $h^\mathbb{I}_\mathbb{I}(s) = h^\mathbb{I}_\mathbb{I}(s) = 0$.

Then $h^\mathbb{I}_\mathbb{I}$ is the unique solution to $L \Phi h^\mathbb{I}_\mathbb{I} = \Phi^\mathbb{I}_\mathbb{I}$ with $h^\mathbb{I}_\mathbb{I}(w) = 0$.

Now, $L \Phi (h^\mathbb{I}_\mathbb{I} - h^\mathbb{I}_\mathbb{I}) = i^\mathbb{I}_\mathbb{I} - i^\mathbb{I}_\mathbb{I} = 2m(\mathcal{I}_t - \mathcal{I}_t)$, and so $\langle h^\mathbb{I}_\mathbb{I} - h^\mathbb{I}_\mathbb{I} \rangle = 2m = L \Phi (\mathcal{I}_t - \mathcal{I}_t)$.

So, $\frac{1}{2m}(h^\mathbb{I}_\mathbb{I} - h^\mathbb{I}_\mathbb{I})$ is a voltage vector when $2m$ amperes are sent from $s$ to $t$.

Reff ($s, t$) = $(\mathcal{I}_t - \mathcal{I}_t) \cdot (\frac{1}{2m}(h^\mathbb{I}_\mathbb{I} - h^\mathbb{I}_\mathbb{I})) = \frac{1}{2m}(h^\mathbb{I}_\mathbb{I} + h^\mathbb{I}_\mathbb{I}(s)) = \frac{1}{2m}C_{st}$.

Using this connection, we can use it to give bounds on the commute time and cover time.

**Corollary** For any edge $uv$, $C_{uv} \leq 2m$.

**Proof** The effective resistance between $u$ and $v$ is at most one, by Ohm’s law.

**Theorem** The cover time of an undirected graph is at most $2m(n-1)$.

**Proof** Let $T$ be a spanning tree of $G$.

Consider a walk that goes through $T$ where each edge in $T$ is traversed once in each direction.

Then this is a walk that visits every vertex at least once.

So the cover time of $G$ is bounded by the expected length of this walk, which is at most

$$\sum_{w \in T} (h_{uw} + h_{wu}) = \sum_{w \in T} C_{uw} \leq 2m(n-1).$$

For the complete graph with $n$ vertices, the cover time is $\Theta(n \log n)$ (coupon collector problem), but
the above bound gives only $O(n^3)$.

The following is a tighter estimate of the cover time.

**Theorem** Let $R(G) = \max_{u,v} R_{\text{eff}}(u,v)$. Then $mR(G) \leq \text{cover time} \leq 2e^3mR(G)\log n + n$.

**Proof** Let $R(G) = R_{\text{eff}}(u,v)$. Then $2mR_{\text{eff}}(u,v) = C_{uv} = h_{uv} + h_{vu}$.

So the cover time is at least $\max \{ h_{uv}, h_{vu} \} \geq C_{uv}/2 = mR_{\text{eff}}(u,v)$, hence the lower bound.

For the upper bound, since the maximum hitting time is at most $2mR(G)$, regardless of the starting vertex. So, a vertex is not covered after $2e^3mR(G)$ steps is at most $\frac{1}{e^3}$ by the Markov's inequality.

If the random walk runs for $2e^3mR(G)\ln n$ steps, then a vertex is not covered with probability $\leq \frac{1}{n^3}$. By the union bound, some vertex is not covered after $2e^3mR(G)\ln n$ is of probability at most $\frac{1}{n^2}$.

When this happens, we just use the bound that the cover time is at most $n^3$.

Then the cover time is at most $2e^3mR(G)\ln n + (\frac{1}{n^2})n^3 = 2e^3mR(G)\ln n + n$. \qed

**References** I follow the presentations of the course notes of Spielman on "graphs and networks", and also Chapter 6 of "Randomized algorithms" by Motwani-Raghavan about random walks.