CS 270 Combinatorial Algorithms and Data Structures, Spring 2015

Lecture 25: Electrical networks

We prove some basic results about electrical flows and effective resistances, and show a connection to hitting times of random walks in undirected graphs.

Electric Flow

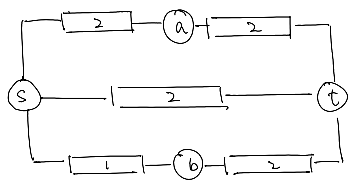
We think of an undirected graph as an electrical network, where each edge e is a resistor with resistance r_e.

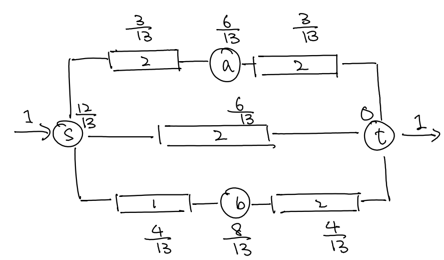
The flow of electric current is governed by two rules:

① Kirchhoff's law (flow conservation law): The sum of the currents entering a node is equal to the sum of currents leaving it.

② Ohm's law (potential flow law): The potential drop across a resistor is equal to the current flowing over the resistor times the resistance.

For example, consider this network:





If one ampere is injected into s and one ampere is removed from t, then the voltages at the nodes and the currents on the resistors are shown in the figure on the right.

Notation

Let's write a matrix formulation of the problem.

Let $G=(V,E)$ be the underlying undirected graph.

Let $v \in \mathbb{R}^{|V|}$ be the vector of potentials at vertices.

Let $i(a,b)$ be the current flowing from vertex a to vertex b for an edge (a,b). As this is a directed quantity, we define $i(b,a) = -i(a,b)$.

Let $i \in \mathbb{R}$ be the vector of currents flowing over the edges, where each edge $e=(a,b)$ appears once as $i(a,b)$ where $a<b$.

Let $w_e = 1/r_e$ be the "conductance" of the edge e.

Matrix Formulation

The Ohm's law states that $i(a,b) = \dfrac{v(a)-v(b)}{r_{a,b}} = w_{a,b}(v(a)-v(b))$.

The Kirchhoff's law states that $\sum_{b:(a,b)\in E} i(a,b) = i_{ext}(a)$, where $i_{ext}(a)$ denotes the external current entering the network through the node a, so it is a positive number if a is a source and a negative number if a is a sink and zero otherwise.

By Ohm's law, $\sum_{b:(a,b)\in E} i(a,b) = \sum_{b:(a,b)\in E} w_{a,b}(v(a)-v(b)) = d(a)v(a) - \sum_{b:(a,b)\in E} w_{a,b}\, v(b)$, where $d(a) = \sum_{b:(a,b)\in E} w_{a,b}$ is the weighted degree of a.

Then this is just equivalent to $L_G v = i_{ext}$, where L_G is the weighted Laplacian of G and i_{ext} is the vector of external currents at the vertices.

Computing Voltages

Therefore, if we can solve a Laplacian system quickly, then we can compute the voltages (and thus the currents) quickly.

Notice that L_G is not of full rank. Assume without loss of generality that G is connected.

Then we know that $\text{nullspace}(L_G) = \vec{1}$. Let $x = \sum_{i=1}^{n} c_i v_i$ where v_i is an orthonormal basis with $v_1 = \vec{1}$.

Then $L_G \cdot x = \sum_{i=1}^{n} c_i \lambda_i v_i = \sum_{i=2}^{n} c_i \lambda_i v_i$ as $\lambda_1 = 0$, and so $L_G \cdot x$ is perpendicular to $\vec{1}$.

Therefore, there is a solution to $L_G v = i_{ext}$ if and only if $i_{ext} \perp \vec{1}$, which should be clear to our problem as the total external currents injecting into the network should be equal to the total external currents removing from the network.

Let $0 = \lambda_1 < \lambda_2 \le \dots \le \lambda_n$ be the eigenvalues of L_G with corresponding eigenvectors $v_1, v_2, \dots, v_n$.

Then $L_G = \sum_{i=1}^{n} \lambda_i v_i v_i^T$. We define the pseudo-inverse as $L_G^{\dagger} = \sum_{i=2}^{n} \frac{1}{\lambda_i} v_i v_i^T$.

For $x \perp \vec{1}$, $y = L_G^{\dagger} x$ is the unique solution to $L_G y = x$ with $y \perp \vec{1}$, and the set of all solutions is $\{ y + c\vec{1} \mid c \in \mathbb{R} \}$.

Computing Currents

Once we have computed the voltages, then it is easy to compute the currents.

Let us write down a matrix formulation for our discussions later.

Let B be an $m \times n$ matrix where the rows are indexed by the edges and the columns are indexed by the vertices, and the row corresponding to the edge $e = (a,b)$ with $a < b$ is $(\chi_a - \chi_b)^T$, where χ_a is the characteristic vector with one in the a-th entry and zero otherwise.

Let W be the $m \times m$ diagonal matrix where $W_{e,e} = w_e$ is the weight of edge e.

Then $\vec{i} = WBv$.

Notice that $L_G = \sum_{e=(a,b)} w_e (\chi_a - \chi_b)(\chi_a - \chi_b)^T = B^T W B$. So, $i_{ext} = L_G v = B^T W B v = B^T \vec{i}$, which can also be checked directly from the definition.

Effective Resistance

The effective resistance between vertices a and b is defined as $v(a) - v(b)$ when one ampere is injected into a and removed from b. You can think of it as the resistance between a and b given by the whole network. We denote it by $R_{eff}(a,b)$.

To compute $R_{eff}(a,b)$, first we compute the voltages when one ampere is injected into a and removed from b. By the matrix formulation, this is the solution of $L_G v = (\chi_a - \chi_b)$, which is given by $v = L_G^{\dagger}(\chi_a - \chi_b)$. Then $R_{eff}(a,b)$ is just $(\chi_a - \chi_b)^T L_G^{\dagger} (\chi_a - \chi_b)$.

So, once we have $L_G^{\dagger}$, we can compute $R_{eff}(a,b)$ for all a,b easily.

Energy

Recall from physics that the energy dissipated in a resistor network with currents $i(a,b)\ \forall a,b$ is:

$$\mathcal{E}(\vec{i}) \overset{def}{=} \vec{i}^T R \vec{i} = \sum_{(a,b) \in E} i^2(a,b)\, r_{a,b} = \sum_{(a,b) \in E} \frac{1}{r_{a,b}} (v(a) - v(b))^2 = \sum_{(a,b) \in E} w_{a,b} (v(a) - v(b))^2 = v^T L_G v,$$

where R is the $m \times m$ diagonal matrix where $R(e,e) = r_e$.

Intuitively, if we think of the whole network as one resistor from a to b, then

$R_{eff}(a,b) = (v(a)-v(b))/i(a,b) = \mathcal{E}(\vec{i})$ if one unit of current is sent from a to b.

This can be proved formally as $R_{eff}(a,b) = (\chi_a-\chi_b)^T L_G^{\dagger}(\chi_a-\chi_b) = (L_G v)^T L_G^{\dagger}(L_G v) = v^T L_G L_G^{\dagger} L_G v = v^T L_G v = \mathcal{E}(\vec{i})$,

as it is easy to verify that $L_G L_G^{\dagger} L_G = L_G$.

Energy Minimization

The electric flow from s to t is the one that minimizes the energy.

Let $\vec{j}$ be one unit of flow from s to t, satisfying the flow conservation rule at every vertex.

Define its energy to be $\mathcal{E}(j) = \vec{j}^T R \vec{j} = \sum_e r_e \cdot j^2(e)$.

Theorem (Thompson's Principle) $R_{eff}(s,t) \leq \mathcal{E}(j)$.

Proof Let $\vec{i}$ be the electrical flow of one unit from s to t, and $\vec{v}$ be the corresponding voltages.

Consider $\vec{c} = \vec{j} - \vec{i}$.

As both $\vec{i}$ and $\vec{j}$ satisfy flow conservation constraints, we have $B^T\vec{i} = B^T\vec{j} = (\chi_s - \chi_t)$ as the

v-th entry of $B^T\vec{i}$ is $\sum_{\substack{u<v \\ uv\in E}} (-i(u,v)) + \sum_{\substack{w>v \\ wv\in E}} (i(v,w)) = \sum_{u:uv\in E} i(v,u) = i_{ext}(v)$.

Therefore, $B^T\vec{c} = B^T(\vec{i}-\vec{j}) = 0$, and hence $\sum_{u:uv\in E} c(v,u) = 0$ for all v.

$$\mathcal{E}(j) = \sum_{ab\in E} j^2(a,b)\, r_{a,b} = \sum_{ab\in E} (i(a,b)+c(a,b))^2 \cdot r_{a,b}$$

$$= \sum_{ab\in E} i^2(a,b)\cdot r_{a,b} + 2\sum_{ab\in E} i(a,b)\cdot c(a,b)\cdot r_{a,b} + \sum_{ab\in E} c^2(a,b)\cdot r_{a,b}.$$

Observe that the first term is $\mathcal{E}(\vec{i})$, and the last term is positive if $\vec{i} \neq \vec{j}$.

Hence we will complete the proof once we show that $\sum_{ab\in E} i(a,b)\cdot c(a,b)\cdot r_{a,b} = 0$.

To see this, $\sum_{ab\in E} i(a,b)\cdot c(a,b)\cdot r_{a,b} = \sum_{ab\in E} (v(a)-v(b))\, c(a,b)$ (by Ohm's law)

$$= \sum_{ab\in E} (v(a)\cdot c(a,b) + v(b)\, c(b,a))$$

$$= \sum_{a\in V} v(a) \sum_{b:ab\in E} c(a,b) = 0. \quad \square$$

Effective Resistance as Distance

Let us try to get some intuition about the effective resistances

The Rayleigh's monotonicity principle says that the effective resistance cannot decrease if we increase the resistance of some edge.

Theorem (Rayleigh's Monotonicity Principle) Let $\vec{r}' \geq \vec{r}$ be the resistances. Then $\mathcal{E}_{r'}(\vec{i}') \geq \mathcal{E}_r(\vec{i})$, where $\mathcal{E}_r(\vec{i})$ denotes the energy of flow $\vec{i}$ under the resistances $\vec{r}$.

proof Let $\vec{i}$ and $\vec{i}'$ be the electric flow under resistances $\vec{r}$ and $\vec{r}'$ respectively.

$$\text{Then} \quad \mathcal{E}_{r'}(\vec{i}') \geq \mathcal{E}_r(\vec{i}') \qquad \text{as } r' \geq r$$

$$\geq \mathcal{E}_r(\vec{i}) \qquad \text{by the Thompson's principle.} \ \square$$

Intuitively, if there is a short path between s and t, then the effective resistance between s and t is small. Also, if there are many paths between s and t, then the effective resistance between s and t is smaller. One can use the Rayleigh's monotonicity principle to give a bound on the effective resistance.

Claim If there are k edge-disjoint paths from s to t, each of length at most l. Then $R_{eff}(s,t) \leq l/k$, assuming the graph is unweighted.

proof Increase the resistances of all other edges to infinity. Then the effective resistance of the resulting graph is at most l/k by direct calculation. By monotonicity the effective resistance in the original network could not be larger than that. $\square$

Effective resistances provide an alternative way to measure the distance of two nodes in a graph, sometimes more useful than the traditional shortest path distance. For instance, one could use the effective resistances as distances to identify clusters in a social network.

Actually, effective resistances satisfy the triangle inequality.

Claim $R_{eff}(a,b) + R_{eff}(b,c) \geq R_{eff}(a,c)$.

Claim $\quad R_{eff}(a,b) + R_{eff}(b,c) \geq R_{eff}(a,c)$.

proof $\quad$ Let $\vec{v}_{a,b}, \vec{v}_{a,c}, \vec{v}_{b,c}$ be the voltages when one unit of current is sent from a to b, a to c, b to c, resp.

Then $\vec{v}_{a,b} = L_G^{\dagger}(\chi_a - \chi_b)$, $\vec{v}_{a,c} = L_G^{\dagger}(\chi_a - \chi_c)$ and $\vec{v}_{b,c} = L_G^{\dagger}(\chi_b - \chi_c)$.

So $\vec{v}_{a,b} + \vec{v}_{b,c} = \vec{v}_{a,c}$.

$R_{eff}(a,c) = (\chi_a - \chi_c)^T \vec{v}_{a,c} = (\chi_a - \chi_c)^T \vec{v}_{a,b} + (\chi_a - \chi_c)^T \vec{v}_{b,c}$.

Note that $(\chi_a - \chi_c)^T \vec{v}_{a,b} = \vec{v}_{a,b}(a) - \vec{v}_{a,b}(c) \leq \vec{v}_{a,b}(a) - \vec{v}_{a,b}(b)$ as $v_{a,b}(a) \geq v_{a,b}(c) \geq v_{a,b}(b)$ for all $c \in V$

$= R_{eff}(a,b)$

Similarly, the second term is at most $R_{eff}(b,c)$, and hence the claim follows. $\square$

In the following we will talk about the connection between effective resistances and hitting times, which will give even more intuitions about using effective resistances as distances.

Random Walk in Undirected Graphs

Recall that the hitting time from a to b is the expected number of steps to reach b if the random walk starts from a, denoted by h_{ab}.

The cover time is the expected number of steps to reach every vertex at least once.

The commute time, denoted by C_{ab}, is defined as $h_{ab} + h_{ba}$.

Theorem $\quad C_{s,t} = R_{eff}(s,t)$.

proof $\quad$ Let $v \in V - t$. Then $h_{vt} = \sum_{w: vw \in E} \frac{1}{d(v)}(1 + h_{wt})$, with $h_{tt} = 0$.

This is equivalent to $d(v) = d(v) \cdot h_{vt} - \sum_{w: vw \in E} h_{wt} = \sum_{w: vw \in E} (h_{vt} - h_{wt})$ for $v \in V - t$.

Observe that this is very similar to a Laplacian system of linear equations.

Let ϕ_{vt} be the voltage at v with $\phi_{tt} = 0$, when $d(v)$ units of currents are injected from $v \in V - t$ and $2m - d(t)$ units of current are removed from t.

Then the values ϕ_{vt} and h_{vt} would satisfy the same equations.

This is because, for $v \in V-t$, $d(v) = \sum_{w: vw \in E} (\phi_{vt} - \phi_{wt})$ by Ohm's law

Let $\vec{i_t}$ be the vector of the external currents with $\vec{i_t}(v) = d(v)$ for $v \in V-t$ and

$\vec{i_t}(t) = -2m + d(t)$. And let $\vec{\phi_t}$ be the vector with $\vec{\phi_t}(v) = \phi_{vt}$.

Then the values ϕ_{vt} satisfy the Laplacian system $L_G \vec{\phi_t} = \vec{i_t}$ with $\vec{\phi_t}(t) = 0$.

We know that the set of solution to this Laplacian system is $\{ L_G^{\dagger} \vec{i_t} + c\vec{1} \mid c \in \mathbb{R} \}$.

There is a unique solution with $\phi_{tt} = 0$, hence we must have $h_{vt} = \phi_{vt}$ for all v.

Let $\vec{i_s}$ be the vector of external currents with $\vec{i_s}(v) = d(v)$ if $v \in V-s$ and $\vec{i_s}(s) = -2m + d(s)$.

Then, as above, let $\vec{h_s}$ be the hitting time vector with $\vec{h_s}(v) = h_{vs}$ and $\vec{h_s}(s) = h_{ss} = 0$.

Then $\vec{h_s}$ is the unique solution to $L_G \vec{h_s} = \vec{i_s}$ with $h_{ss} = 0$.

Now, $L_G(\vec{h_t} - \vec{h_s}) = \vec{i_t} - \vec{i_s} = 2m(\chi_s - \chi_t)$, and so $(\vec{h_t} - \vec{h_s})/2m = L_G^{\dagger}(\chi_s - \chi_t)$.

So, $\frac{1}{2m}(\vec{h_t} - \vec{h_s})$ is a voltage vector when $2m$ amperes are sent from s to t.

$R_{eff}(s,t) = (\chi_s - \chi_t)^T \left(\frac{1}{2m} (\vec{h_t} - \vec{h_s}) \right) = \frac{1}{2m} \left(\vec{h_t}(s) + \vec{h_s}(t) \right) = \frac{1}{2m} (h_{st} + h_{ts}) = \frac{1}{2m} C_{st}$. □

Using this connection, we can use it to give bounds on the commute time and cover time.

Corollary For any edge uv, $C_{uv} \le 2m$.

proof The effective resistance between u and v is at most one, by Ohm's law. □

Theorem The cover time of an undirected graph is at most $2m(n-1)$.

proof Let T be a spanning tree of G.

Consider a walk that goes through T where each edge in T is transversed once in each direction.

Then this is a walk that visits every vertex at least once.

So the cover time of G is bounded by the expected length of this walk, which is at most

$\sum_{uv \in T} (h_{uv} + h_{vu}) = \sum_{uv \in T} C_{uv} \le 2m(n-1)$. □

For the complete graph with n vertices, the cover time is $\theta(n \log n)$ (coupon collector problem), but the above bound gives only $O(n^3)$.

The following is a tighter estimate of the cover time.

Theorem Let $R(G) = \max_{u,v} R_{eff}(u,v)$ Then $mR(G) \leq$ cover time $\leq 2e^3 mR(n)\log n + n$.

proof Let $R(G) = R_{eff}(u,v)$. Then $2m R_{eff}(u,v) = C_{uv} = h_{uv} + h_{vu}$.

So the cover time is at least $\max\{h_{uv}, h_{vu}\} \geq C_{uv}/2 = mR_{eff}(u,v)$, hence the lower bound.

For the upper bound, since the maximum hitting time is at most $2mR(G)$, regardless the starting vertex.

So, a vertex is not covered after $2e^3 mR(G)$ steps is at most $1/e^3$ by the Markov's inequality.

If the random walk runs for $2e^3 mR(G)\ln n$ steps, then a vertex is not covered with probability $\leq 1/n^3$.

By the union bound, some vertex is not covered after $2e^3 mR(G)\ln n$ is of probability at most $1/n^2$.

When this happens, we just use the bound that the cover time is at most n^3.

Then the cover time is at most $2e^3 mR(G)\ln n + (\frac{1}{n^2})n^3 = 2e^3 mR(G)\ln n + n$. □

References I follow the presentations of the course notes of Spielman on "graphs and networks", and also chapter 6 of "randomized algorithms" by Motwani-Raghavan about random walks.