Task 2(a)

Given $f = [1\ 2\ 3\ 4\ 1\ 2\ 3\ 4]$, so $\boxed{N=8}$.

Define $W_8 = e^{i\frac{2\pi}{8}} = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

and therefore $\overline{W}_8 = W_8^{-1} = e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

The Forward FFT is defined by

$$F_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n \overline{W}_N^{nk}, \quad \text{for } k=0,\dots,N-1.$$

For the FFT algorithm described in Section 7.7, we define the following two vectors of size $\frac{N}{2}=4$:

$$g_n = \frac{1}{2}(f_n + f_{n+4}), \quad n=0,1,2,3$$

$$h_n = \frac{1}{2}(f_n - f_{n+4})\overline{W}_8^{n}, \quad n=0,1,2,3$$

I.e., $g = \frac{1}{2}[(1+1)\ (2+2)\ (3+3)\ (4+4)]$

$$\Rightarrow \boxed{g = [1\ 2\ 3\ 4]}$$

$h = \frac{1}{2}[(1-1)\overline{W}_8^{0}\ (2-2)\overline{W}_8^{1}\ (3-3)\overline{W}_8^{2}\ (4-4)\overline{W}_8^{3}]$

$$\Rightarrow \boxed{h = [0\ 0\ 0\ 0]}$$

Task 2(b)

DFT(g): $\quad G_k = \frac{1}{4} \sum_{n=0}^{3} g_n \overline{W}_4^{nk}$ where $\overline{W}_4 = -i$

So,

$$G_0 = \frac{1}{4}[g_0 + g_1 + g_2 + g_3] = \frac{1}{4}[1+2+3+4] = \boxed{\frac{5}{2}}$$

$$G_1 = \frac{1}{4}[g_0 + g_1(-i) + g_2(-i)^2 + g_3(-i)^3]$$

$$= \frac{1}{4}[1 - 2i - 3 + 4i] = \boxed{-\frac{1}{2} + \frac{1}{2}i}$$

Actually, just the same as Task 1(a)

$$G_2 = \frac{1}{4}[g_0 + g_1(-i)^2 + g_2(-i)^4 + g_3(-i)^6]$$

$$= \frac{1}{4}[1 - 2 + 3 - 4] = \boxed{-\frac{1}{2}}$$

$$G_3 = \frac{1}{4}[g_0 + g_1(-i)^3 + g_2(-i)^6 + g_3(-i)^9]$$

$$= \frac{1}{4}[1 + 2i - 3 - 4i] = \boxed{-\frac{1}{2} - \frac{1}{2}i}$$

Therefore, $\boxed{G = [\tfrac{5}{2} \quad (-\tfrac{1}{2}+\tfrac{1}{2}i) \quad -\tfrac{1}{2} \quad (-\tfrac{1}{2}-\tfrac{1}{2}i)]}$

Similarly, for DFT(h): $\quad H_k = \frac{1}{4} \sum_{n=0}^{3} h_n \overline{W}_4^{nk}$.

So, $\quad H_0 = \frac{1}{4}[h_0 + h_1 + h_2 + h_3] = 0$

and since all $h_n = 0$, we have

$$\boxed{H = [0 \quad 0 \quad 0 \quad 0]}.$$

Task 2(c)

The development in Section 7.7 of the Course Notes shows that the even coefficients of $F = DFT(f)$ are precisely given by G, and the odd coefficients are given by H.

That is,

$$F = [G_0, H_0, G_1, H_1, G_2, H_2, G_3, H_3]$$

I.e.,

$$\boxed{F = \left[\tfrac{5}{2} \quad 0 \quad \left(-\tfrac{1}{2}+\tfrac{1}{2}i\right) \quad 0 \quad -\tfrac{1}{2} \quad 0 \quad \left(-\tfrac{1}{2}-\tfrac{1}{2}i\right) \quad 0\right]}.$$