Uncertainty

CS 486/686: Introduction to Artificial Intelligence

Introduction

- Logical agents make epistemological commitments that propositions are true, false, or unknown
  - Once an agent has enough facts it can derive plans that are guaranteed to work

Introduction

- **But**
  - Agents rarely have access to the full truth about their environment

Introduction

- The logical approach breaks down when dealing with uncertainty
- Example: Diagnosis
  - $\forall$ p Symptom(p, Toothache)$\Rightarrow$Disease(p,Cavity)
  - $\forall$ p Symptom(p, Toothache)$\Rightarrow$Disease(p,Cavity)$\vee$ Disease(p,HitInTheJaw) $\vee$ Disease(p,GumDisease)$\vee$
  - $\forall$ p Disease(p, Cavity)$\Rightarrow$Symptom(p,Toothache)

First Order Logic Fails Because

- We are lazy
  - Too much work to write down all antecedents and consequences
- Theoretical ignorance
  - Sometimes there is no complete theory
- Practical ignorance
  - Even if we knew all the rules, we might be uncertain about a particular instance (not enough information yet)

Probability to the Rescue

- Allows us to deal with uncertainty that comes from laziness or ignorance
- Clear semantics
- Provides principled answers for
  - combining evidence, predictive and diagnostic reasoning, incorporation of new evidence
- Can be learned from data

Discrete Random Variables

- Random variable A describes an outcome that can not be determined in advance (ie. roll of a dice)
- Discrete random variable: possible values come from a countable domain (sample space)
  - If X is the outcome of a dice throw then $X \in \{1,2,3,4,5,6\}$
- **Boolean random variable:** $A \in \{\text{True, False}\}$
  - A=The Canadian PM in 2040 will be male
  - A=You have Ebola
  - A=You wake up tomorrow with a headache

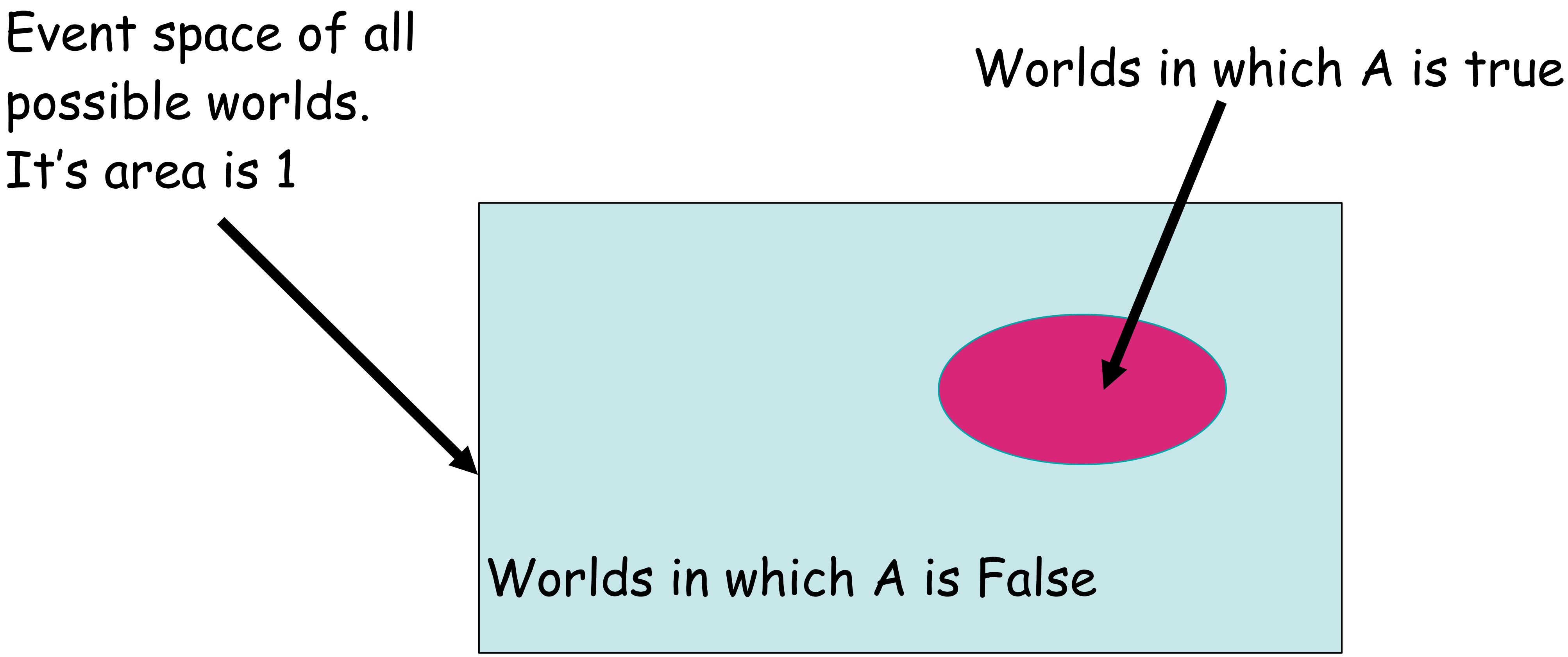
Events

- An event is a complete specification of the state of the world in which an agent is uncertain
  - Subset of the sample space
- Example
  - (Cavity=True)$\wedge$(Toothache=True)
  - Dice=2
- Events must be
  - Mutually exclusive
  - Exhaustive

Probabilities

- We let P(A) denote the "degree of belief" we have that statement A is true
  - "The fraction of possible worlds in which A is true"
- Note: P(A) DOES NOT correspond to a degree of truth

Visualizing A



P(A) = Area of oval

Axioms of Probability

- $0 \leq P(A) \leq 1$
- $P(True) = 1$
- $P(False) = 0$
- $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

- These axioms limit the class of functions that can be considered as probability functions

Interpreting the Axioms

- **$0 \leq P(A) \leq 1$**
- $P(True)=1$
- **$P(False)=0$**
- $P(A \vee B)=P(A)+P(B)-P(A \wedge B)$

The area of A can't be smaller than 0

A zero area would mean no world could ever have A as true

Interpreting the Axioms

- **$0 \leq P(A) \leq 1$**
- **$P(True)=1$**
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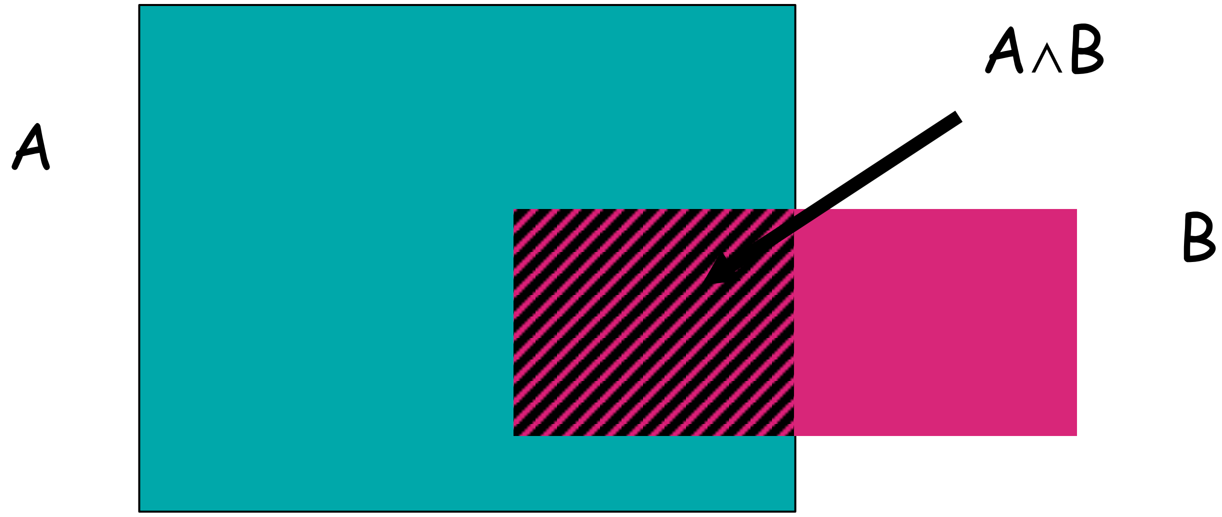
The area of A can't be larger than 1

An area of 1 would mean no world could ever have A as false

Interpreting the Axioms

- $0 \leq P(A) \leq 1$
- $P(True)=1$
- $P(False)=0$
- **$P(A \vee B)=P(A)+P(B)-P(A \wedge B)$**



Take the Axioms Seriously

- There have been attempts to use different methodologies for uncertainty
  - Fuzzy logic
  - Three-valued logic
  - Dempster-Shafer
  - ...
- But if you follow the axioms of probability then no one can take advantage of you :)

Theorems from the Axioms

- **Thm**: P(~A)=1-P(A)
- **Proof**: P(AV~A)=P(A)+P(~A)-P(A^~A)

  P(True)=P(A)+P(~A)-P(False)

  1 = P(A)+P(~A)-0

  P(~A)=1-P(A)

Multivalued Random Variables

- Assume domain of A (sample space) is $\{v_1, v_2, \ldots, v_k\}$
- A can take on exactly one value out of this set
  - $P(A=v_i, A=v_j)=0$ if i not equal to j
  - $P(A=v_1 \text{ or } A=v_2 \text{ or } \ldots \text{ or } A=v_k)=1$

Useful Fact

- Given axioms of probability and $P(A=v_i, A=v_j)=0$ for $i \neq j$, and $P(A=v_1$ or $A=v_2$ or ... or $A=v_k)=1$ then
  - $P(A=v_1$ or $A=v_2$ or ... or $A=v_i) = \sum_{j=1}^{i} P(A=v_j)$
  - $\sum_{j=1}^{k} P(A=v_j) = 1$

Terminology

- **Probability Distribution**
  - A specification of a probability for each event in the sample space
- Assume the world is described by two or more random variables
  - **Joint probability distribution**
    - Specification of probabilities for all combinations of events

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  - $P(B, (A=v_1$ or $A=v_2$ or ... or $A=v_i)) = \sum_{j=1}^{i} P(B, A=v_j)$
  - $\sum_{j=1}^{k} P(B, A=v_j) = 1$

Marginalization

Example: Joint Distribution

sunny

	cold	~cold
headache	0.108	0.012
~headache	0.016	0.064

~sunny

	cold	~cold
headache	0.072	0.008
~headache	0.144	0.576

P(headache^sunny^cold)=0.108 P(~headache^sunny^~cold)=0.064

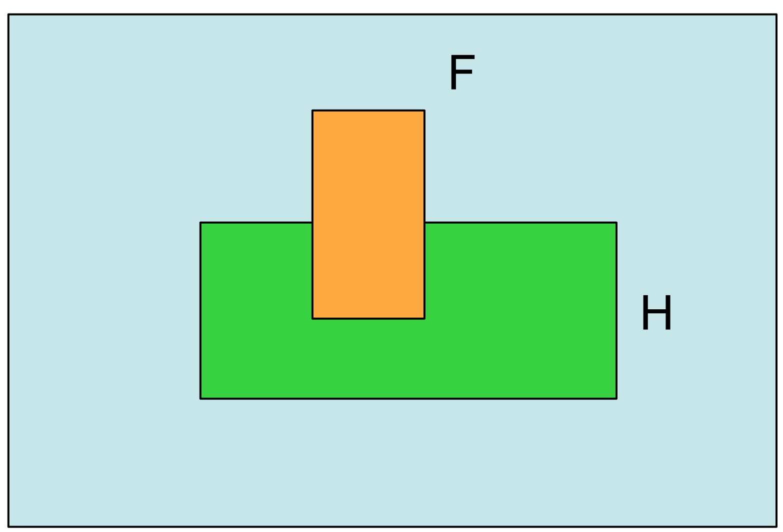
P(headache V sunny) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28

P(headache)=0.108+0.012+0.072+0.008=0.2

marginalization

Conditional Probability

- P(A|B): fraction of worlds in which B is true that also have A true

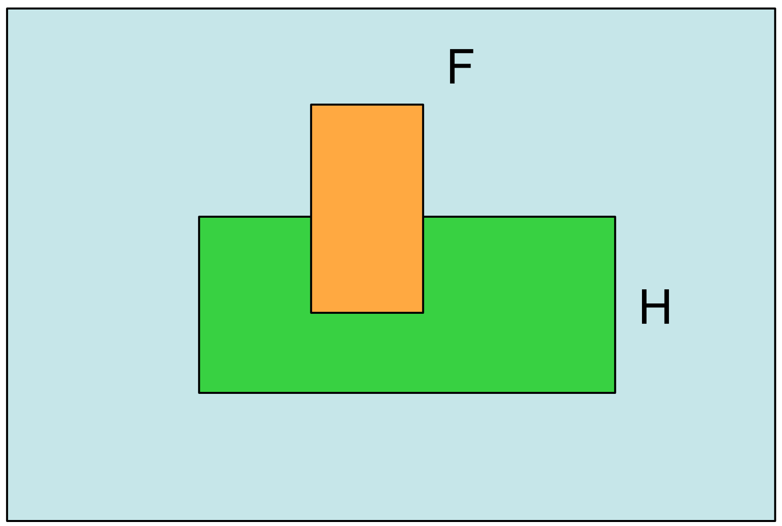


H="Have headache"
F="Have Flu"

P(H)=1/10
P(F)=1/40
P(H|F)=1/2

Headaches are rare and flu is rarer, but if you have the flu that there is a 50-50 chance you will have a headache

Conditional Probability



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Headaches are rare and flu is rarer, but if you have the flu that there is a 50-50 chance you will have a headache

P(H|F)= Fraction of flu inflicted worlds in which you have a headache

=(# worlds with flu and headache)/ (# worlds with flu)

= (Area of "H and F" region)/ (Area of "F" region)

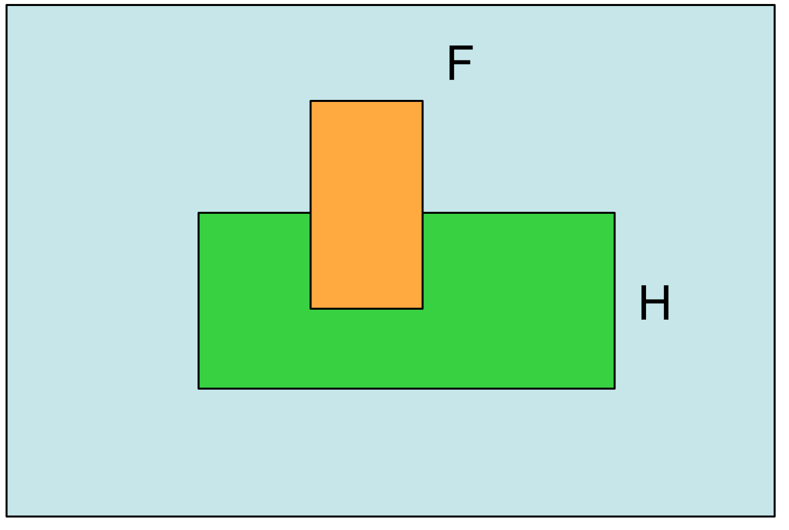
= P(H ^ F)/ P(F)

Conditional Probability

- $P(A|B)=P(A \wedge B)/P(B)$
- Chain Rule:
  - $P(A \wedge B)=P(A|B)P(B)$

Memorize these!

Conditional Probability



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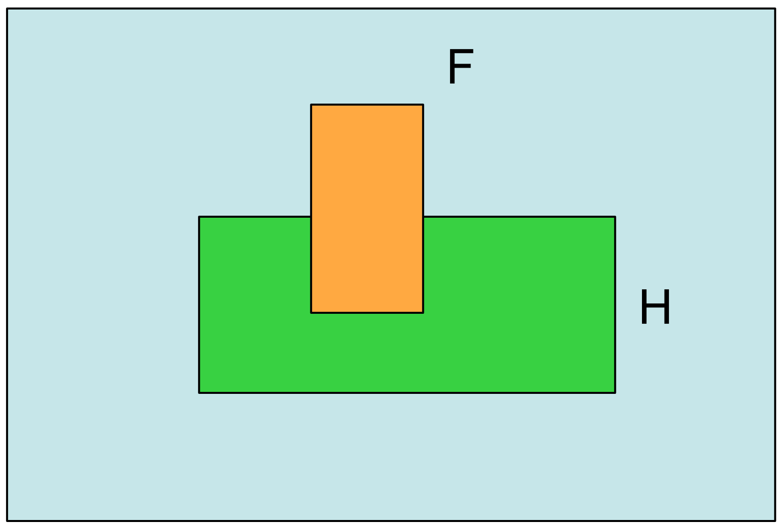
One day you wake up with a headache. You think "Drat! 50% of flues are associated with headaches so I must have a 50-50 chance of coming down with the flu"

Is your reasoning correct?

Conditional Probability



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$P(F \wedge H) =$

$P(F|H) =$

Example: Joint Distribution

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~sunny

	cold	~cold
headache	0.072	0.008
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P(headache ^ cold| sunny)= P(headache ^ cold ^ sunny)/P(sunny)

= 0.108/(0.108+0.012+0.016+0.064)

= 0. 54

P(headache ^ cold| ~sunny)= P(headache ^ cold ^ ~sunny)/P(~sunny)

= 0.072/(0.072+0.008+0.144+0.576)

= 0.09

Bayes Rule

- Note:
  - $P(A|B)P(B)=P(A \wedge B)=P(B \wedge A)=P(B|A)P(A)$
- Bayes Rule:
  - $P(B|A)=[P(A|B)P(B)]/P(A)$

Memorize this!

General Forms of Bayes Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\neg A)P(\neg A)}$$

$$P(A|B \wedge X) = \frac{P(B|A \wedge X)P(A \wedge X)}{P(B \wedge X)}$$

$$P(A = v_i|B) = \frac{P(B|A = v_i)P(A = v_i)}{\sum_{k=1}^{n} P(B|A = v_k)P(A = v_k)}$$

Using Bayes Rule for Inference

- Often we want to form a hypothesis about the world based on what we have observed
- Bayes rule is vitally important when viewed in terms of stating the belief given to hypothesis **H,** given evidence **e**

Prior probability

Likelihood

$$P(H|e) = \frac{P(e|H)P(H)}{P(e)}$$

Posterior probability

Normalizing constant

Example

- A doctor knows that H1N1 causes a fever 95% of the time. She knows that if a person is selected at random from the population, they have a 10^{-7} chance of having H1N1. 1 in 100 people suffer from a fever.
- You go to the doctor complaining about a fever. What is the probability that H1N1 is the cause of the fever?

Computing Conditional Probabilities

- Often we are interested in the posterior joint distribution of some **query variable** Y given specific evidence e for **evidence variables** E
  - Hidden variables: X-Y-E
- If we had the joint prob. distribution then could marginalize
  - $P(Y|E=e)=\alpha\sum_{h}P(Y\wedge(E=e)\wedge(H=h))$

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Problem: Joint distribution is usually too big to handle

Independence

- Two variables A and B are **independent** if knowledge of A does not change uncertainty of B (and vice versa)
  - $P(A|B)=P(A)$
  - $P(B|A)=P(B)$
  - $P(A \wedge B)=P(A)P(B)$
  - In general: $P(X_1,X_2,\ldots,X_n)=\prod_i P(X_i)$

Conditional Independence

- Full independence is often too strong a requirement
- Two variables A and B are **conditionally independent** given C if
  - $P(a|b,c)=P(a|c)$ for all a,b,c
  - i.e. knowing the value of B does not change the prediction of A ***if the value of C is known***

Conditional Independence

- Diagnosis problem
  - Fl=Flu, Fv=Fever, C=Cough
- Full joint dist. has 2^3-1=7 independent entries
- If someone has the flu, we can assume that the probability of a cough does not depend on having a fever (P(C I Fl,Fv)=P(C I Fl))
- If the same condition holds if the patient does not have the Flu then C and Fv are **conditionally independent** given FL (P(C I ~Fl, Fv)=P(C I ~Fl))

Conditional Independence

- Full distribution can be written as

$$
\begin{aligned}
P(C, Fl, FC) &= P(C, Fv|Fl)P(Fl) \\
&= P(C|Fl)P(Fv|Fl)P(Fl)
\end{aligned}
$$

- We only need 5 numbers!
- Huge savings if there are lots of variables

Conditional Independence

- Such a probability distribution is sometimes called a **Naive Bayes model**

- In practice they work well - even when the independence assumption is not true

Summary

- What you should know
  - Basic definitions and axioms
  - Marginalization
  - Conditional Probabilities
  - Chain Rule and Bayes Rule
  - Independence and Conditional Independence