Introduction to Social Choice

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Outline

What Is Social Choice Theory

- Study of decision problems in which a group has to make the decision
- The decision affects all members of the group
  - Their opinions should count!
- Applications
  - Political elections
  - Other elections
  - Allocations problems (e.g. allocation of money among agents, alocation of goods, tasks, resources....)
  - . . .

Formal Model

- Set of agents $N = \{1, 2, \cdots, n\}$
- Set of outcomes O
- Set of strict total orders on O, L
- Social choice function: $f : L^n \rightarrow O$
- Social welfare function: $f : L^n \rightarrow L^-$ where L^- is the set of weak total orders on O

Assumptions

- Agents have preferences over alternatives
  - Agents can rank order outcomes
- Voters are sincere
  - They truthfully tell the center their preferences
- Outcome is enforced on all agents

Assume that there are only two alternatives, x and y. We can represent the family of preferences by

$$(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$$

where α_i is 1, 0, or -1 according to whether agent i preferes x to y, is ambivalent between them, or prefers y to x.

Definition (Paretian)

A social choice function is **paretian** *if it respects unanimity of strict preferences on the part of the agents.*

Majority Voting

$$f(\alpha_1, \ldots, \alpha_n) = \text{sign} \sum_i \alpha_i$$

$f(\alpha) = 1$ if and only if more agents prefer x to y and -1 if and only if more agents prefer y to x. Clearly majority voting is paretian.

Additional Properties

- Symmetric among agents
- Neutral between alternatives
- Positively responsive

Theorem (May's Theorem)

A social choice function f is a majority voting rule if and only if it is symmetric among agents, neutral between alternatives, and positively responsive.

Plurality Voting

The rules of plurality voting are probably familiar to you (e.g. the Canadian election system)

- One name is ticked on a ballot
- One round of voting
- One candidate is chosen
  - Candidate with the most votes

Is this a "good" voting system?

Plurality Example

- 3 candidates
  - Lib, NDP, C
- 21 voters with the following preferences
  - 10 C>NDP>Lib
  - 6 NDP>Lib>C
  - 5 Lib>NDP>C
- Result: C 10, NDP 6, Lib 5

The Conservative candidate wins, but a majority of voters (11) prefer all other parties more than the Conservatives.

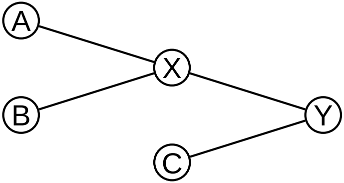
What Can We Do?

Majority system works well when there are two alternatives, but has problems when there are more alternatives.

Proposal: Organize a series of votes between 2 alternatives at a time

Agendas

- 3 alternatives $\{A, B, C\}$
- Agenda: $\langle A, B, C\rangle$

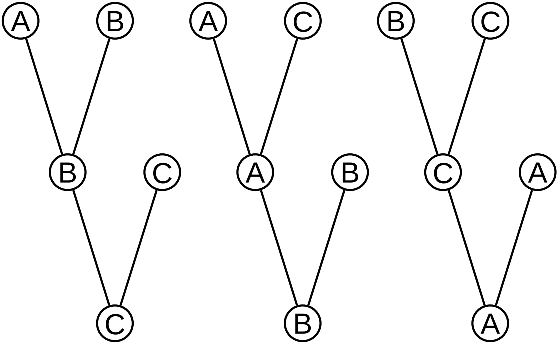


where X is the outcome of majority vote between A and B, and Y is the outcome of majority vote between X and C.

Agenda Paradox: Power of the Agenda Setter

3 types of agents: $A > C > B$ (35%), $B > A > C$ (33%), $C > B > A$ (32%).

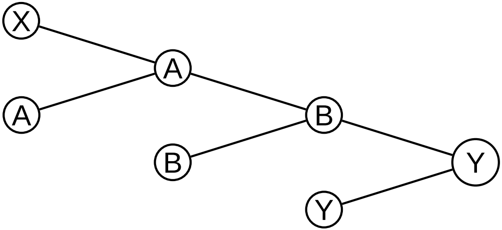
3 different agendas:



Pareto Dominated Winner Paradox

4 alternatives and 3 agents

- $X > Y > B > A$
- $A > X > Y > B$
- $B > A > X > Y$



BUT Everyone prefers X to Y

Pareto Dominated Winner Paradox

4 alternatives and 3 agents

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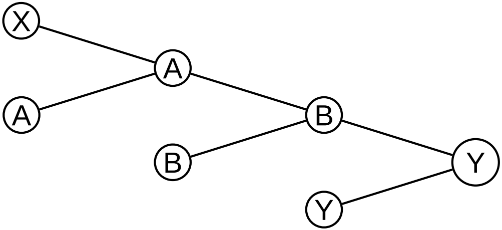

BUT Everyone prefers X to Y

Maybe the problem is with the ballots

Now have agents reveal their entire preference ordering. Condorcet proposed the following

- Compare each pair of alternatives
- Declare "A" is socially preferred to "B" if more voters strictly prefer A to B

Condorcet Principle: If one alternative is preferred to *all other* candidates, then it should be selected.

Definition (Condorcet Winner)

An outcome $o \in O$ is a Condorcet Winner if $\forall o' \in O$, $\#(o > o') \geq \#(o' > o)$.

Condorcet Example

- 3 candidates
  - Lib, NDP, C
- 21 voters with the following preferences
  - 10 C>NDP>Lib
  - 6 NDP>Lib>C
  - 5 Lib>NDP>C

Result: NDP win since 11/21 prefer them to the Conservatives and 16/21 prefer them to the Liberals.

There Are Other Problems With Condorcet Winners

- 3 candidates: Liberal, NDP, Conservative
- 3 voters with preferences
  - Liberal > NDP>Conservative
  - NDP>Conservative>Liberal
  - Conservative>Liberal>NDP

Result: Condorcet winners do not always exist.

Borda Count

- Each ballot is a list of ordered alternatives
- On each ballot, compute the rank of each alternative
- Rank order alternatives based on decreasing sum of their ranks

$$\begin{array}{ccc} A > B > C & & A : 4 \\ A > C > B & \Rightarrow & B : 8 \\ C > A > B & & C : 6 \end{array}$$

Borda Count

- The Borda Count is simple
- There is always a Borda winner
- BUT the Borda winner is not always the Condorcet winner

3 voters: 2 with preferences B>A>C>D and one with A>C>D>B
Borda scores: A:5, B:6, C:8, D:11
Therefore A wins, but B is the Condorcet winner.

Other Borda Count Issues: Inverted-Order Paradox

Agents

- X>C>B>A
- A>X>C>B
- B>A>X>C
- X>C>B>A
- A>X>C>B
- B>A>X>C
- X>C>B>A

Borda Scores

- X:13, A:18, B:19, C:20

Remove X

- C:13, B:14, A:15

Other Borda Count Issues: Inverted-Order Paradox

Agents

- X>C>B>A
- A>X>C>B
- B>A>X>C
- X>C>B>A
- A>X>C>B
- B>A>X>C
- X>C>B>A

Borda Scores

- X:13, A:18, B:19, C:20

Remove X

- C:13, B:14, A:15

Vulnerability to Irrelevant Alternatives

3 types of agents

- X>Z>Y (35%)
- Y>X>Z (33%)
- Z>Y>X (32%)

The Borda winner is X.

Remove alternative Z. Then the Borda winner is Y.

Vulnerability to Irrelevant Alternatives

3 types of agents

- X>Z>Y (35%)
- Y>X>Z (33%)
- Z>Y>X (32%)

The Borda winner is X.
Remove alternative Z. Then the Borda winner is Y.

Other Examples of Rules

- Copeland
  - Do pairwise comparisons of outcomes.
  - Assign 1 point if an outcome wins, 0 if it loses, $\frac{1}{2}$ if it ties
  - Winner is the outcome with the highest summed score
- Kemeny
  - Given outcomes a and b, ranking r and vote v, define $\delta_{a,b}(r,v) = 1$ if r and v agree on relative ranking of a and b
  - *Kemeny ranking* r' maximises $\sum_v \sum_{a,b} \delta_{a,b}(r,v)$

Properties for Voting Protocols

Property (Universality)

A voting protocol should work with any set of preferences.

Property (Transitivity)

A voting protocol should produce an ordered list of alternatives (social welfare function).

Property (Pareto efficiency)

If all agents prefer X to Y, then in the outcome X should be prefered to Y. That is, SWF f is pareto efficient if for any $o_1, o_2 \in O$, $\forall i \in N, o_1 >_i o_2$ then $o_1 >_f o_2$.

More Properties

Property (Independence of Irrelevant Alternatives (IIA))

Comparison of two alternatives depends only on their standings among agents' preferences, and not on the ranking of other alternatives.

Property (No Dictators)

A SWF f has no dictator if $\neg\exists i \forall o_1, o_2 \in O$, $o_1 >_i o_2 \Rightarrow o_1 >_f o_2$

Arrow's Theorem

Theorem (Arrow's Theorem)

If there are 3 or more alternatives and a finite number of agents, then there is no SWF which satisfies the 5 desired properties.

Is There Anything That Can Be Done?

Can we relax the properties?

- No dictator?
  - Fundamental for a voting protocol
- Paretian?
  - Also pretty fundamental
- Transitivity?
  - Maybe you only need to know the top ranked alternative?
    - Stronger form of Arrow's theorem says that you are still in trouble
- IIA?
- Universality
  - Some hope here (1 dimensional preferences, spacial preferences...)

Take-home Message

- Despair?
  - No ideal voting method
  - That would be boring!
- A group of more complex that an individual
- Weigh the pro's and cons of each system and understand the setting they will be used in
- Do not believe anyone who says they have the best voting system out there!

Voting as MLE

There is an alternative view of voting.

- There is some sense that some outcomes are *better* than others.
  - This ranking is not merely based on idiosyncratic preferences of voters.
- Voters preferences are *noisy* estimates of the quality.
- Voting is used to infer outcomes true ranking based on the voters' noisy signals (votes)